Introduction to unitary t-designs

Artem Kaznatcheev

McGill University

January 7, 2010

Preliminaries: $U(d)$

- $U(d)$ is the topologically compact and connected group of norm preserving (unitary) operators on $\mathbb{C}^d$.

Preliminaries: $U(d)$

- $U(d)$ is the topologically compact and connected group of norm preserving (unitary) operators on $\mathbb{C}^d$.
- We can introduce the Haar measure and use it to integrate functions f of $U \in U(d)$ to find their averages:

$$\langle f \rangle = \int_{U(d)} f(U)\, dU.$$

- For convenience we normalize integration by assuming that $\int_{U(d)} dU = 1$.

Preliminaries: $U(d)$

- $U(d)$ is the topologically compact and connected group of norm preserving (unitary) operators on $\mathbb{C}^d$.
- We can introduce the Haar measure and use it to integrate functions f of $U \in U(d)$ to find their averages:

$$\langle f \rangle = \int_{U(d)} f(U)\, dU.$$

- For convenience we normalize integration by assuming that $\int_{U(d)} dU = 1$.
- The goal of unitary t-designs is to evaluate averages of polynomials via a finite sum.

Preliminaries: $\mathrm{Hom}(r, s)$

Definition

$\mathrm{Hom}(r, s)$ is the set of polynomials homogeneous of degree r in entries of $U \in U(d)$ and homogeneous of degree s in U^*.

Preliminaries: $\mathrm{Hom}(r, s)$

Definition

$\mathrm{Hom}(r, s)$ is the set of polynomials homogeneous of degree r in entries of $U \in U(d)$ and homogeneous of degree s in U^*.

Examples

$$
\begin{array}{rclr}
U, V & \mapsto & U^* V^* UV & \in Hom(2,2) \\
U & \mapsto & U^* V^* UV & \in Hom(1,1)
\end{array}
$$

Preliminaries: $\mathrm{Hom}(r, s)$

Definition

$\mathrm{Hom}(r, s)$ is the set of polynomials homogeneous of degree r in entries of $U \in U(d)$ and homogeneous of degree s in U^*.

Examples

$$\begin{array}{rcll} U, V & \mapsto & U^* V^* UV & \in Hom(2,2) \\ U & \mapsto & U^* V^* UV & \in Hom(1,1) \\ U & \mapsto & \dfrac{tr(U^* U)}{d} & \in Hom(1,1) \end{array}$$

Preliminaries: $\mathrm{Hom}(r, s)$

Definition

$\mathrm{Hom}(r, s)$ is the set of polynomials homogeneous of degree r in entries of $U \in U(d)$ and homogeneous of degree s in U^*.

Examples

$$
\begin{array}{rclr}
U, V & \mapsto & U^* V^* UV & \in Hom(2,2) \\
U & \mapsto & U^* V^* UV & \in Hom(1,1) \\
U & \mapsto & \dfrac{tr(U^* U)}{d} & \in Hom(1,1) \\
U, V & \mapsto & tr(U^* V) U^2 + VU^* VU & \in Hom(3,1) \\
U & \mapsto & \underbrace{tr(U^* V) U^2}_{Hom(2,1)} + \underbrace{VU^* VU}_{Hom(1,1)} & \notin Hom(2,1)
\end{array}
$$

Functional definition of unitary t-designs

Definition

A function $w : X \to (0,1]$ is a weight function on X if for all $U \in X$ we have $w(U) > 0$ and $\sum_{U \in X} w(U) = 1$

Functional definition of unitary t-designs

Definition

A function $w : X \to (0,1]$ is a weight function on X if for all $U \in X$ we have $w(U) > 0$ and $\sum_{U\in X} w(U) = 1$

Definition

A tuple (X,w) with finite $X \subset U(d)$ and weight function w on X is a unitary t-design if

$$\sum_{U\in X} w(U)f(U) = \int_{U(d)} f(U)\, dU$$

for all $f \in \mathrm{Hom}(t,t)$.

Functional definition of unitary t-designs

Definition

A function $w : X \to (0,1]$ is a weight function on X if for all $U \in X$ we have $w(U) > 0$ and $\sum_{U \in X} w(U) = 1$

Definition

A tuple (X,w) with finite $X \subset U(d)$ and weight function w on X is a unitary t-design if

$$\sum_{U \in X} w(U) f(U) = \int_{U(d)} f(U)\, dU$$

for all $f \in \mathrm{Hom}(t,t)$.

Definition

A finite $X \subset U(d)$ is an unweighted t-design if it is a unitary t-design with a uniform weight function (i.e. $w(U) = \frac{1}{|X|}$ for all $U \in X$).

Functional definition is general enough

Proposition

Every t-design is a $(t-1)$*-design.*

Functional definition is general enough

Proposition

Every t-design is a $(t-1)$-design.

Proposition

For any $f \in \mathrm{Hom}(r, s)$ *with* $r \neq s$

$$\int_{U(d)} f(U)\, dU = 0$$

Lemma

For any $f \in \mathrm{Hom}(r, s)$, $U \in U(d)$, *and* $c \in \mathbb{C}$ *we have* $f(cU) = c^r \bar{c}^s f(U)$

Strengths and shortcomings of the functional definition

Strengths:

- Average of any polynomial with degrees in U and U^* less than t can be evaluated one summand at a time.
- Multi-variable polynomials can be evaluated:

$$\int_{U(d)} \cdots \int f(U_1, ..., U_n) dU_1 ... dU_n$$

$$= \sum_{U_1 \in X} ... \sum_{U_n \in X} w(U_1)...w(U_n) f(U_1, ..., U_n).$$

Strengths and shortcomings of the functional definition

Strengths:

- Average of any polynomial with degrees in U and U^* less than t can be evaluated one summand at a time.
- Multi-variable polynomials can be evaluated:

$$\int_{U(d)} \cdots \int f(U_1, ..., U_n) dU_1 ... dU_n$$

$$= \sum_{U_1 \in X} ... \sum_{U_n \in X} w(U_1)...w(U_n) f(U_1, ..., U_n).$$

Shortcomings:

- Not clear how to test if a given (X, w) is a t-design.
- If (X, w) is not a design, then how far away is it?

Tensor product definition of unitary t-designs

Definition

A tuple (X,w) with finite $X \subset U(d)$ and weight function w on X is a unitary t-design if

$$\sum_{U \in X} w(U) U^{\otimes t} \otimes (U^*)^{\otimes t} = \int_{U(d)} U^{\otimes t} \otimes (U^*)^{\otimes t} dU$$

Tensor product definition of unitary t-designs

Definition

A tuple (X,w) with finite $X \subset U(d)$ and weight function w on X is a unitary t-design if

$$\sum_{U \in X} w(U) U^{\otimes t} \otimes (U^*)^{\otimes t} = \int_{U(d)} U^{\otimes t} \otimes (U^*)^{\otimes t} dU$$

- More tractable for checking if an arbitrary (X, w) is a t-design.
- Literature has explicit formula for the RHS for many choices of d and t [Col03, CS06].
- Still not metric.

Metric definition of unitary t-designs

Definition

A weight function w is a proper weight function on X if for all other choices of weight function w' on X, we have:

$$\sum_{U,V \in X} w(U)w(V)|tr(U^* V)|^{2t} \leq \sum_{U,V \in X} w'(U)w'(V)|tr(U^* V)|^{2t}.$$

The trace double sum is a function Σ defined for finite $X \subset U(d)$ as:

$$\Sigma(X) = \sum_{U,V \in X} w(U)w(V)|tr(U^* V)|^{2t},$$

Definition

A finite $X \subset U(d)$ is a unitary t-design if

$$\Sigma(X) = \langle |tr(U)|^{2t} \rangle$$

Strengths and shortcomings of the metric definition

Strengths:

- $\Sigma(X) > \langle |tr(U)|^{2t} \rangle$ if X is not a t-design. This gives us a useful metric to say how far a set with proper weight function is from being a design.

Strengths and shortcomings of the metric definition

Strengths:

- $\Sigma(X) > \langle |tr(U)|^{2t} \rangle$ if X is not a t-design. This gives us a useful metric to say how far a set with proper weight function is from being a design.
- $\langle |tr(U)|^{2t} \rangle$ has a nice combinatorial interpertation: the number of permutations of $\{1, ..., t\}$ with no increasing subsequences of order greater than d [DS94, Rai98].
- If $d \geq t$ then RHS is $t!$.

Strengths and shortcomings of the metric definition

Strengths:

- $\Sigma(X) > \langle |tr(U)|^{2t} \rangle$ if X is not a t-design. This gives us a useful metric to say how far a set with proper weight function is from being a design.
- $\langle |tr(U)|^{2t} \rangle$ has a nice combinatorial interpertation: the number of permutations of $\{1, ..., t\}$ with no increasing subsequences of order greater than d [DS94, Rai98].
- If $d \geq t$ then RHS is $t!$.
- One of the easiest way to test if X is a t-design

Strengths and shortcomings of the metric definition

Strengths:

- $\Sigma(X) > \langle |tr(U)|^{2t} \rangle$ if X is not a t-design. This gives us a useful metric to say how far a set with proper weight function is from being a design.
- $\langle |tr(U)|^{2t} \rangle$ has a nice combinatorial interpertation: the number of permutations of $\{1, ..., t\}$ with no increasing subsequences of order greater than d [DS94, Rai98].
- If $d \geq t$ then RHS is $t!$.
- One of the easiest way to test if X is a t-design

Shortcomings:

- Does not give any insight into what t-designs are useful for.

Characterization of minimal t-designs

Definition

A minimal (unweighted) t-design X is a t-design such that all $Y \subset X$ are not (unweighted) t-designs.

Characterization of minimal t-designs

Definition

A minimal (unweighted) t-design X is a t-design such that all $Y \subset X$ are not (unweighted) t-designs.

Theorem

A t-design X is minimal if and only if it has a unique proper weight function w.

Characterization of minimal t-designs

Definition

A minimal (unweighted) t-design X is a t-design such that all $Y \subset X$ are not (unweighted) t-designs.

Theorem

A t-design X is minimal if and only if it has a unique proper weight function w.

- Useful tool for proving minimality.

Characterization of minimal t-designs

Definition

A minimal (unweighted) t-design X is a t-design such that all $Y \subset X$ are not (unweighted) t-designs.

Theorem

A t-design X is minimal if and only if it has a unique proper weight function w.

- Useful tool for proving minimality.
- Sadly, minimal designs are not necessarily minimum.
- Currently working on finding correspondences between minimal and minimum designs.

A lower bound on the size of t-designs

Proposition

If $X \subset U(d)$ is a t-design then $|X| \geq \frac{d^{2t}}{\langle |tr(U)|^{2t} \rangle}$.

A lower bound on the size of t-designs

Proposition

If $X \subset U(d)$ is a t-design then $|X| \geq \frac{d^{2t}}{\langle |tr(U)|^{2t} \rangle}$.

- Best known bounds are by Roy and Scott [RS08]: $|X| \geq \binom{d^2+t-1}{t}$
- Asymptotically, for large d and fixed t, both bounds are $\Theta(d^{2t})$

1-design construction

- Let $|e_1\rangle ... |e_d\rangle$ be an orthonormal basis of $\mathbb{C}^d$ that is mutually unbiased with the standard basis.
- Define $I_i = \sqrt{d}\mathrm{diag}(|e_i\rangle)$ for $1 \leq i \leq d$.

1-design construction

- Let $|e_1\rangle ... |e_d\rangle$ be an orthonormal basis of $\mathbb{C}^d$ that is mutually unbiased with the standard basis.
- Define $I_i = \sqrt{d}\mathrm{diag}(|e_i\rangle)$ for $1 \leq i \leq d$.
- Consider the cyclic permutation group of order d, represented as d-by-d matrices: $C^1 ... C^d$ where $C^d = C^0 = I$.
- Define $C_i^m = C^m I_i$

1-design construction

- Let $|e_1\rangle ... |e_d\rangle$ be an orthonormal basis of $\mathbb{C}^d$ that is mutually unbiased with the standard basis.
- Define $I_i = \sqrt{d}\mathrm{diag}(|e_i\rangle)$ for $1 \leq i \leq d$.
- Consider the cyclic permutation group of order d, represented as d-by-d matrices: $C^1 ... C^d$ where $C^d = C^0 = I$.
- Define $C_i^m = C^m I_i$

For any tuple $1 \leq i, j, m, n \leq d$ we have:

$$tr((C_i^m)^* C_j^n) = tr(I_i^* C^{d-m+n} I_j) = \begin{cases} d & \text{if } i = j \text{ and } m = n \\ 0 & \text{otherwise} \end{cases}$$

Evaluating the average commutator over $U(d)$

Theorem

For any $V \in U(d)$ and $[U, V] = U^* V^* U V$ we have:

$$\langle[\,\cdot\,, V]\rangle = \frac{tr(V^*)}{d} V$$

Proof of EAC

Consider the diagonalization of V^*, i.e. $V^* = P^*DP$, with $D = \mathrm{diag}(\lambda_1, ..., \lambda_d)$.

$$\int_{U(d)} U^*V^*UV\, dU = \Big[\int_{U(d)} U^*V^*U\, dU\Big]V = \Big[\int_{U(d)} U^*P^*DPU\, dU\Big]V$$

Proof of EAC

Consider the diagonalization of V^*, i.e. $V^* = P^* DP$, with $D = \mathrm{diag}(\lambda_1, ..., \lambda_d)$.

$$\int_{U(d)} U^* V^* UV \, dU = \Big[\int_{U(d)} U^* V^* U \, dU \Big] V = \Big[\int_{U(d)} U^* P^* DPU \, dU \Big] V$$

But we know a symmetry that allows substituting $PU \to U$ without changing the average.

$$\int_{U(d)} U^* P^* DPU \, dU = \int_{U(d)} U^* DU \, dU$$

- Let $f(U) = U^* DU$.

Proof of EAC

Consider the diagonalization of V^*, i.e. $V^* = P^* D P$, with $D = \mathrm{diag}(\lambda_1, ..., \lambda_d)$.

$$\int_{U(d)} U^* V^* U V \, dU = \Big[\int_{U(d)} U^* V^* U \, dU \Big] V = \Big[\int_{U(d)} U^* P^* D P U \, dU \Big] V$$

But we know a symmetry that allows substituting $PU \to U$ without changing the average.

$$\int_{U(d)} U^* P^* D P U \, dU = \int_{U(d)} U^* D U \, dU$$

- Let $f(U) = U^* D U$.
- Look at the elements of the design: $f(C_i^m) = I_i^* (C^m)^* D C^m I_i$.

Proof of EAC

Consider the diagonalization of V^*, i.e. $V^* = P^* D P$, with $D = \mathrm{diag}(\lambda_1, ..., \lambda_d)$.

$$\int_{U(d)} U^* V^* U V \, dU = \Big[\int_{U(d)} U^* V^* U \, dU \Big] V = \Big[\int_{U(d)} U^* P^* D P U \, dU \Big] V$$

But we know a symmetry that allows substituting $PU \to U$ without changing the average.

$$\int_{U(d)} U^* P^* D P U \, dU = \int_{U(d)} U^* D U \, dU$$

- Let $f(U) = U^* D U$.
- Look at the elements of the design: $f(C_i^m) = I_i^* (C^m)^* D C^m I_i$.
- $(C^m)^* D C^m = \mathrm{diag}(\lambda_{c^m(1)}, ..., \lambda_{c^m(d)})$

Proof of EAC

Consider the diagonalization of V^*, i.e. $V^* = P^*DP$, with $D = \mathrm{diag}(\lambda_1, ..., \lambda_d)$.

$$\int_{U(d)} U^*V^*UV\, dU = \Big[\int_{U(d)} U^*V^*U\, dU \Big] V = \Big[\int_{U(d)} U^*P^*DPU\, dU \Big] V$$

But we know a symmetry that allows substituting $PU \to U$ without changing the average.

$$\int_{U(d)} U^*P^*DPU\, dU = \int_{U(d)} U^*DU\, dU$$

- Let $f(U) = U^*DU$.
- Look at the elements of the design: $f(C_i^m) = I_i^*(C^m)^*DC^mI_i$.
- $(C^m)^*DC^m = \mathrm{diag}(\lambda_{c^m(1)}, ..., \lambda_{c^m(d)})$

Thus, $\langle f \rangle = \frac{\lambda_1 + ... + \lambda_d}{d} I$ $\square$

t-designs are non-commuting

Definition

$X \subset U(d)$ is a *non-commuting* if there is some $U, V \in X$ such that $[U, V] \neq I$.

Theorem

For all $d \geq 2$ if $X \subset U(d)$ is a t-design then X is non-commuting.

Supports our intuition that designs must be well ‘spread out’.

Concluding remarks

- Introduced 3 definitions of unitary t-designs

Concluding remarks

- Introduced 3 definitions of unitary t-designs
- Classified minimal designs: a t-design is minimal if and only if it has a unique proper weight function.

Concluding remarks

- Introduced 3 definitions of unitary t-designs
- Classified minimal designs: a t-design is minimal if and only if it has a unique proper weight function.
- Used an orthonormal basis of $\mathbb{C}^{d \times d}$ as a 1-design.
- Evaluated the average commutator on $U(d)$: $\langle [\cdot , V] \rangle = \frac{tr(V^*)}{d} V$
- Showed that t-designs are non-commuting

Concluding remarks

- Introduced 3 definitions of unitary t-designs
- Classified minimal designs: a t-design is minimal if and only if it has a unique proper weight function.
- Used an orthonormal basis of $\mathbb{C}^{d \times d}$ as a 1-design.
- Evaluated the average commutator on $U(d)$: $\langle [\,\cdot\,, V] \rangle = \frac{tr(V^*)}{d} V$
- Showed that t-designs are non-commuting

Thank you for listening!

References I

B. Collins.
Moments and cumulants of polynomial random variables on unitary groups, the Itzykson-Zuber integral, and free probability.
International Mathematics Research Notices, pages 953–982, 2003.

B. Collins and P. Śniady.
Integration with respect to the haar measure on unitary, orthogonal and symplectic group.
Communications in Mathematical Physics, 264:773–795, 2006.

P. Diaconis and M. Shahshahani.
On the eigenvalues of random matrices.
Journal of Applied Probability, 31A:49–62, 1994.

E. M. Rains.
Increasing subsequences and the classical groups.
Electronic Journal of Combinatorics, 5:Research Paper 12, 9 pp., 1998.

References II

A. Roy and A. J. Scott.
Unitary designs and codes.
2008.