Applications of SDP's

Quadratic program:

$$\min/\max \quad \frac{1}{2} x^T Q x + c^T x$$
$$\text{s.t} \quad Ax \le b$$

In general NP hard to find solutions.

We use the ideas developed in Goemans-Williamson algorithm to approximate solutions of QP

Recall that for Goemans-Williamson algorithm we rounded the solution by choosing random hyperplane, we can use the same rounding for other problems such as following,

$$(1) \qquad \max \sum_{1 \le i,j \le n} a_{ij} x_i x_j = x^T A x \qquad \text{where } A = (a_{ij})$$
$$\text{s.t} \quad x_i \in \{-1, +1\}$$

We run the VP relaxation as ussual (like the example of Max Cut

$$\max \sum_{ij} a_{ij} \langle v_i, v_j \rangle$$

(2)

$$\langle v_i, v_i \rangle = 1$$
$$v_i \in \mathbb{R}^n$$

We need Opt solution of the original (1) to be positive to be able to talk about approximations so $A = (a_{ij}) \succeq 0$ psd.

We can solve (2) in poly time (such psd can be solved + ε error)

Use the same relaxation as the max-cut example

i.e $r = (r_1, \dots, r_n)$ $r_i \sim N(0,1)$

set $\bar{x}_i = 1$ if $\langle v_i, r \rangle \geq 0$

$\bar{x}_i = -1$ if $\langle v_i, r \rangle < 0$

(Rounding Method to ~~solve~~ (1) approximate)

Note $Alg = \sum_{ij} a_{ij} \mathbb{E}[\bar{x}_i \bar{x}_j]$

$$\mathbb{E}[\bar{x}_i \bar{x}_j] = 1 \cdot \Pr[\bar{x}_i \bar{x}_j = 1] + (-1) \cdot \Pr[\bar{x}_i \bar{x}_j = -1]$$

$$= \left(1 - \frac{1}{\pi} \arccos(\langle v_i, v_j \rangle)\right) - \frac{1}{\pi} \arccos(\langle v_i, v_j \rangle)$$

$$= 1 - \frac{2}{\pi} \arccos(\langle v_i, v_j \rangle)$$

$\arcsin(x) + \arccos(x) = \frac{\pi}{2}$

$$\mathbb{E}[\bar{x}_i \bar{x}_j] = \frac{2}{\pi} \arcsin(\langle v_i, v_j \rangle)$$

Fact: Matrix $Z = (z_{ij})$ $\quad z_{ij} := \arcsin(x_{ij}) - x_{ij}$

$|x_{ij}| \leq 1$ $\quad X = (x_{ij}) \succeq 0 \quad \Rightarrow Z \succeq 0$ $\quad$ (*

<u>Thm</u> Choosing random hyperplane method is a $\frac{2}{\pi}$-approximation algorithm for the quadratic program (1)

pf: $\text{Alg} = \sum_{ij} a_{ij} \mathbb{E}[\bar{x}_i \bar{x}_j]$

$$= \sum_{ij} a_{ij} \frac{2}{\pi} \arcsin(\langle v_i, v_j \rangle)$$

We use the fact (*)

$$\Rightarrow \frac{2}{\pi} \sum_{ij} a_{ij} [\arcsin(\langle v_i, v_j \rangle) - \langle v_i, v_j \rangle] \geq 0$$

If $A \succeq 0, B \succeq 0$
<u>Then</u> $A \circ B \succeq 0$

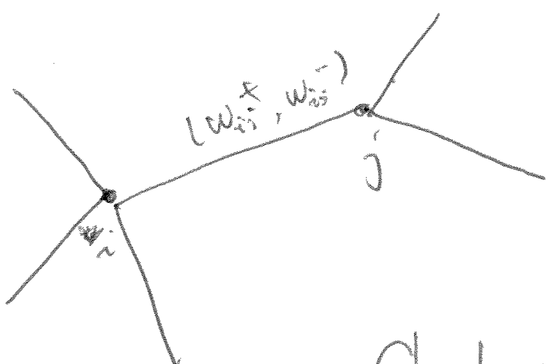
$$\Longrightarrow \quad \text{Alg} = \frac{2}{\pi} \sum_{ij} a_{ij} \arcsin(\langle v_i, v_j \rangle)$$

$$\geq \frac{2}{\pi} \sum_{ij} a_{ij} \langle v_i, v_j \rangle = \frac{2}{\pi} \text{Opt}_{VP} \geq \frac{2}{\pi} \text{Opt}_{(1)}$$

Hence the random hyperplane rounding can approximate the quadratic programs solution to factor $\frac{2}{\pi} \approx 63\%$

Application to Correlation Clustering

<u>Problem</u> $G = (V, E)$.



w_{ij}^{+} - How similar are two data points i, j are

w_{ij}^{-} - How different i, j are

Clustering the vertices into clusters S' such that

S' ← partition of V to $|S'|$ diff sets

(P) $$\max_{S'} \sum_{(i,j) \in E[S']} w_{ij}^{+} + \sum_{(i,j) \in \delta(S')} w_{ij}^{-}$$

where $E[S']$ = Set of edges that have both end point on a same cluster

$\delta(S')$ = Set of edges that have endpoint in diff clusters

Easy $\frac{1}{2}$ Aprx alg

Let Opt be optimal solution to problem (P)

then clearly

$$\text{Opt} \le \sum_{(i,j)\in E} w_{ij}^+ + \sum_{(i,j)\in E} w_{ij}^-$$

$$\le 2 \max\left(\sum_{(i,j)\in E} w_{ij}^+ , \sum_{(i,j)\in E} w_{ij}^- \right)$$

Note choosing $S' = \{V\}$ makes $E[S']$ = Set of all edges in G

$$\delta(S') = \phi$$

$$\Rightarrow \sum_{(i,j)\in E} w_{ij}^+$$

Choosing $S' = \{ \{i\} : i \in V \}$ makes $E[S'] = \phi$

$$\delta(S') = \cancel{\phi}\ \underset{\text{all edges}}{E}$$

$$\Rightarrow \sum_{(i,j)\in E} w_{ij}^-$$

Better of the two Clustering gives $\max\left(\sum_{(i,j)\in E} w_{ij}^+ , \sum_{(i,j)\in E} w_{ij}^- \right)$

$$\Rightarrow \frac{1}{2}\text{-approx.}$$

Using SDP's we can improve our apx guarantee to $\frac{3}{4}$! (Swamy)

The Correlation Clustering can be written as

$$\max \sum_{(i,j)\in E} \left(w_{ij}^{+} \langle x_i, x_j \rangle + w_{ij}^{-} (1 - \langle x_i, x_j \rangle) \right)$$

s.t $\quad x_i \in \{e_1, \dots, e_n\} \quad$ e_i base vector in $\mathbb{R}^n$

Vector programming relaxation which is semidefinite program

$$\max \sum_{(i,j)\in E} \left(w_{ij}^{+} \langle v_i, v_j \rangle + w_{ij}^{-} (1 - \langle v_i, v_j \rangle) \right)$$

(VP)
$$\langle v_i, v_i \rangle = 1$$
$$\langle v_i, v_j \rangle \geq 0$$
$$v_i \in \mathbb{R}^n \quad \forall i$$

As usual $Opt_{VP} \geq Opt$ since any feasable solution to the original is also feasable in the vector program

We can solve (VP) to optimality but how do we get the clusters we want?

We use the random hyperplane method but two hyperplanes r_1, r_2

The clusters are defined as follows

$$R_1 = \{ i \in V : \langle r_1, v_i \rangle \geq 0 , \langle r_2, v_i \rangle \geq 0 \}$$

$$R_2 = \{ i \in V : \langle r_1, v_i \rangle \geq 0 , \langle r_2, v_i \rangle < 0 \}$$

$$R_3 = \{ i \in V : \langle r_1, v_i \rangle < 0 , \langle r_2, v_i \rangle \geq 0 \}$$

$$R_4 = \{ i \in V : \langle r_1, v_i \rangle < 0 , \langle r_2, v_i \rangle < 0 \}$$

$S = \{ R_1, R_2, R_3, R_4 \}$ so we only use four clusters.

What is the expected value of this cluster?

Let X_{ij} r.v 1 if vertex i & j end up on the same cluster and zero otherwise.

$$\mathbb{E}[X_{ij}] = \left(1 - \frac{1}{\pi} \arccos(\langle v_i, v_j \rangle)\right)^2$$

both v_i & v_j has to lie on a same side of random hyperplane. Two hyperplanes are chosen indep

$$\text{Alg} = \mathbb{E}\Big[\sum_{(i,j)\in E} w_{ij}^{+} X_{ij} + w_{ij}^{-} (1 - X_{ij}) \Big]$$

$$= \sum_{E} w_{ij}^{+} \mathbb{E}[X_{ij}] + w_{ij}^{-} (1 - \mathbb{E}[X_{ij}])$$

$$= \sum_{E} w_{ij}^{+} \left(1 - \frac{1}{\pi} \arccos(\langle v_i, v_j \rangle)\right)^2 + w_{ij}^{-} \Big(1 - \Big(1 - \frac{1}{\pi} \arccos(\langle v_i, v_j$$

$$\geq 0.75 \sum_{E} w_{ij}^{+} \langle v_i, v_j \rangle + w_{ij}^{-} (1 - \langle v_i, v_j \rangle)$$

$$= 0.75\, \text{Opt}_{vp} \geq \frac{3}{4} \text{Opt} \qquad \blacksquare \qquad \text{Yay!}$$

Basic Calculus fact (*)

$$\frac{\left(1 - \frac{1}{\pi} \arccos(x)\right)^2}{x} \geq 0.75$$

$$\frac{1 - \left(1 - \frac{1}{\pi} \arccos(x)\right)^2}{(1 - x)} \geq 0.75$$

$$\forall x \in [0, 1]$$