Of course this book would not have been possible without the authors and I would like to thank all of them for their hard work and patience, and most of them for their respect for deadlines.

Finally, I would like to thank my coeditor Jorge Ramírez Alfonsín for doing most of the work.

Jorge Ramírez Alfonsín

In summer 1997, in Marseille, Bruce Reed first mentioned to me his intention to prepare a volume on perfect graphs and asked me to work on it with him. It has been a pleasure to do so. And I would like to thank Bruce not only for involving me in this project but also for his continuous support and generosity.

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1

Origins and Genesis

C. Berge[a] and J.L. Ramírez Alfonsín[b]

[a] *CNRS, CAMS, Paris, France*
[b] *Forschungsinstitut für Diskrete Mathematik, Universität Bonn, Germany*

1.1 Perfection

Perfection is freedom from fault or defect, an exemplification of supreme excellence, an unsurpassable degree of accuracy. Can perfection be achieved?

C. Berge introduced, to some extent, perfection to graph theory when investigating new combinatorial properties of graphs, involving some well-known invariants, and defining *perfect graphs*. Perfect graphs were very much motivated by Shannon's notion of the *zero-error capacity* of a graph [22].

In the next section, we briefly discuss Shannon's work on communication theory. In the rest of the chapter, we reproduce some of the earliest published works concerning perfect graphs, as well as the page of Shannon's paper where he defines the zero-error capacity. Rather than presenting a history of the early days of perfect graph theory, we reproduce C. Berge's paper, 'Motivations and history of some of my conjectures' [9], in Section 1.3. We also reproduce (from [4]) what are, as far as we are aware, the first explicit written statements of the *strong and weak perfect graph conjectures*, and translate the abstract of the talk in which Berge first defined perfect graphs [2].

Despite many important results on the subject, the perfect graph conjecture remains unsolved. Does perfection exist? In any case, characterizing perfection turned out to be very hard! Various researchers' failed attempts to do so have generated an important body of work which will still be very interesting even if the perfect graph conjecture is solved.

1.2 Communication Theory

Information is often, passed from a sender to a receiver via a physical transmission channel. This transfer of information (say, signals) may be subject to the

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uncontrollable ambient noise and imperfections of the physical signalling process itself. Thus each input symbol may give rise to a variety of output symbols. We say that two input symbols are *confoundable* if they can give rise to the same output symbol. More generally, we say that two string of inputs are *confoundable* if there is some string of output symbols which may be caused by both of them.

In [22], Shannon discussed restricting the messages sent through a channel to a set no two of which are confoundable, thereby ensuring that no errors will be introduced during transmission.

One way of doing so is to restrict ourselves to a set of symbols no two of which are confoundable. For if P is a transmission channel with input symbols set Σ, and Σ' is a subset of Σ no two of whose elements are confoundable, then no two elements of $(\Sigma')^n$ (the set of ordered n-tuples with elements in Σ') are confoundable, indeed we can determine the input by considering the output, one symbol at a time.

Thus, if we let $t(P)$ be the size of the largest subset of Σ, no two of whose elements are confoundable, then there are $t(P)^m$ m-symbol messages which can be transmitted through P without risk of error.

However, by being clever we can often find larger sets of m-symbol messages no two of which are confoundable. Consider, for example, a transmission channel P for which $\Sigma = \{a, b, c, d, e\}$ such that the pairs of confoundable symbols are $\{ab, bc, cd, de, ea\}$. Then clearly $t(P) = 2$. On the other hand the messages $\{ab, bd, ca, dc, ee\}$ are clearly unconfoundable (for example, bd cannot be confused with ca because although b can be confused with c, d cannot be confused with a). So, using these five *codewords*, we can construct $5^{\lfloor m/2\rfloor}$ m-symbol messages no two of which are confoundable. This is a significant improvement on the 2^m messages obtained using two nonconfoundable symbols.

More generally, using any set C of codewords of length n, no two of which are confoundable, we can construct $|C|^{\lfloor m/n\rfloor}$, that is, essentially $2^{m\log|C|/n}$, m-symbol messages no two of which are confoundable.

Thus, letting $N(n, P)$ be the maximum size of a set of codewords over Σ, no two of which are confoundable, we see that there are $2^{\lfloor m/n\rfloor \log N(n,P)}$ m-symbol messages which can be transmitted through P without danger of error.

This motivated Shannon to define the *zero-error capacity* of a transmission channel P as

$$C_0 = \lim_{n\to\infty} \frac{1}{n} \log N(n, P).$$

Remark 1.1 *It is not difficult to see that this limit exists because concatenating codewords implies* $\frac{1}{n}\log N(n, P) \leq \frac{1}{kn}\log N(kn, P)$ *and* $\log|\Sigma|$ *is an upper bound on* $C_0(P)$.

As the above remark shows, $C_0(P)$ is the theoretical limit of the bit-per-symbol error-free transmission rate of the channel P.

Formally, Shannon considered a *discrete memoryless channel* given by input alphabet $\Sigma_1 = \{x_1, \ldots, x_n\}$ and output alphabet $\Sigma_2 = \{y_1, \ldots, y_m\}$ and a transmission probability matrix P, where P_{ij} represent the probability that output y_j is obtained on input x_i. Then two symbols x_i and x_l are confoundable if, for some j, both P_{ij} and P_{lj} are positive. Such channels are distinct from those which have infinite continuous input alphabets, such as the reals. They are also distinct from those in which the occurrence of noise of some type during the transmission of one symbol suggests that similar noise may occur during the transmission of the next symbol (for then the matrix P should vary according to the strings of previous output and input). However, it was not necessary for us to introduce this level of precision.

Indeed, as we now show, the capacity of a channel P depends only on the graph $G(P)$ that has as vertex set Σ and in which two symbols are adjacent precisely if they are confoundable. To see this, note first that $t(P)$ is the size of a largest set of vertices in $G(P)$ no two of which are adjacent. Such a set is called *stable* and the size of a largest stable set in G is denoted $\alpha(G)$. Thus $t(P) = \alpha(G(P))$.

The product $G_1 \times G_2$ of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ has vertex set $V_1 \times V_2$ and two distinct pairs (a, b) and (c, d) are adjacent if and only if (i) either $a = c$ or $ac \in E_1$, or (ii) either $b = d$ or $bd \in E_2$. We let $G^1 = G$ and $G^i = G^{i-1} \times G$ for $i \geq 2$. Then it is easy to see that $N(n, P) = \alpha(G(P)^n)$ and $C_0(P) = \lim_{n\to\infty} \frac{1}{n} \log \alpha(G(P)^n)$. Thus we can speak of the Shannon capacity of a graph G as

$$C_0(G) = \lim_{n\to\infty} \frac{1}{n} \log \alpha(G^n).$$

Determining the above limit for a general graph is wide open. Even the simplest nontrivial case, the pentagon, had withstood attack for more than twenty years, until L. Lovász [18] proved that $C_0(C_5) = \frac{1}{2}\log 5$. That $C_0(C_5) \geq \frac{1}{2}\log 5$ was proved above.

Lovász's proof that $C_0(C_5) \leq \frac{1}{2}\log 5$ involved an important new invariant of a graph theory, called the *theta function*; see Chapters 11 and 12 for further details.

It is easy to verify that $\alpha(G^n) \geq (\alpha(G))^n$ as for any stable set S, in G, S^n is a stable set in G^n. Thus, $C_0(G) \geq \log(\alpha(G))$. In [22], Shannon asked which graphs satisfy $C_0(G) = \log(\alpha(G))$.

In attempting to answer this question, we need to bound $\alpha(G^n)$ from above. We can obtain an upper bound on α by considering the *cliques* in G—those subgraphs in which all pairs of vertices of G are adjacent. The clique cover number of G, denoted $\theta(G)$, is the minimum number of cliques into which the vertices of G can be partitioned. Clearly $\alpha \leq \theta$. Furthermore, $\theta(G^2) \leq (\theta(G))^2$ as is easy to verify. So, $C_0(G) \leq \log\theta(G)$.

Thus if $\alpha = \theta$ then $C_0(G) = \log\alpha(G)$. This was the motivation for Berge to study the graphs satisfying $\alpha = \theta$, which led to the birth of perfect graph theory.

1.3 The Perfect Graph Conjecture[1]

In 1958–1959, I started to investigate new combinatorial properties of a graph G with an emphasis on three invariants: $\alpha(G)$ (called first, with von Neumann, the 'coefficient of internal stability', then the 'stability number', or also the 'independence number'), $\theta(G)$ (the 'partition number', the minimum number of cliques needed to cover the vertex set), and $\phi(G)$ (the *zero-error capacity* introduced by Claude Shannon for a noisy channel). These investigations concerned the four following classes:

(1) a graph G is in *Class 1* if $\phi(G) = \alpha(G)$;[2]

(2) a graph G is in *Class 2* if $\alpha(G') = \theta(G')$ (the 'Beautiful Property') for all induced subgraphs G' of G;

[1] Reproduced from Section 2 in [9] with kind permission of Elsevier Science.

[2] This should be $\phi(G) = \log\alpha(G)$ (Ed.).

(3) a graph G is in *Class 3* if $\gamma(G') = \omega(G')$ (the chromatic number[3] is equal to the clique number[4]) for every induced subgraph G';
(4) a graph G is in *Class 4* if G contains no induced C_{2k+1}, the odd cycle of length $2k+1 \geq 5$ (called *odd hole*) and no induced complement of an odd hole (called *odd antihole*).

It is easy to see that every graph in Class 2 is also in Class 4; the Perfect Graph Conjecture says that *Classes 2 and 4 are equivalent.*

To trace back the history of perfect graphs, we shall distinguish different steps:

June 1957: When he heard that I was writing a book on graph theory, my friend M.P. Schutzenberger drew my attention to an interesting paper of Shannon [22] which was presented at a meeting for engineers and statisticians, but which could have been missed by mathematicians working in algebra or combinatorics. In this paper, Shannon posed two problems:

(1) what are the minimal graphs which do not belong to Class 1? (He knew that C_5 was the smallest one.)
(2) what is the zero-error capacity of the graph C_5?

The second problem was solved by Lovász [18] several years later. The first problem, completed by my young student Alain Ghouila-Houri (Shannon overlooked the antiholes), was discussed in January 1960 at the seminar of Professor Fortet where I asked:

Is it true that every graph in Class 4 is also in Class 1?

(see Ghouila-Houri [13]). This conjecture, somewhat weaker than the Perfect Graph Conjecture, was motivated by the remark that for the most usual channels, the graphs representing the possible confusions between a set of signals (in particular the interval graphs) have no odd holes and no odd antiholes, and are *optimal in the sense of Shannon.* I developed this idea at the General Assembly of U.R.S.I. (Information Theory) in Tokyo [3] where my research paper [4] was distributed to all the participants; this paper appeared much later in a book edited by Caianello [5], but at the time I had the possibility to add in the galley proofs new references and an appendix with some results proved in [6], in order to make the conjecture more plausible and more interesting. In fact, at that time, no one really cared about such a problem except Ghouila-Houri: unfortunately, in 1966, this remarkable young mathematician committed suicide, and all the notes concerning his results about the zero-error capacity of the antiholes were definitively lost.

October 1959: Invited by T. Gallai, I attended the first international meeting on Graph Theory at Dobogókő (Hungary), with A. Stone, W. Tutte, A. Rényi, P. Erdős, G. Dirac, G. Hajós, H. Sachs, and others. A this meeting, Hajnal and Surányi presented an elegant result [14] which could be rephrased as follows. *Every triangulated graph belongs to Class 2 (a graph is 'triangulated', or 'chordal' if every cycle of length larger than 3 has a chord).*

[3] The chromatic number of a graph G, nowadays denoted by $\chi(G)$, is the smallest cardinality of a set K for which there is a function $f : V(G) \to K$ such that two adjacent vertices have different elements from K.

[4] The clique number $\omega(G)$ of a graph G is the largest cardinality of a set of vertices every two of which are adjacent.

April 1960: Invited by H. Sachs, I attended the second international meeting on Graph Theory at the Martin Luther University, Halle-Wittenberg, with G. Hajós, G. Dirac, A. Kotzig, and G. Ringel and I presented a new result:

Every triangulated graph belongs to Class 3.

At that time, I was trying to find all the minimal counterexamples to Class 3 (because *I suspected that the only ones were the holes and the antiholes*, conviction which appeared later to be equivalent to the Perfect Graph Conjecture). In the extended abstract of my talk in Halle published in German [2], as in a more developed text published simultaneously in French [1], I stressed the importance of holes and the antiholes for this problem.

A first remark was that all the graphs which were known to belong to Class 3 are *without odd holes.* In honour of T. Gallai, I proposed to call 'semi-Gallai' a graph which has no odd holes. However, a terminology change was imposed by the editor of the Proceedings of Halle-Wittenberg, who added to my paper the following footnote: 'The original title of the presentation given in Halle was *Colouring of Gallai, resp. semi-Gallai, graphs.*[5] Gallai informed us, however, that this was an oversight since he had not concerned himself closely with these graphs. Therefore the title was changed with the author's agreement following a suggestion by Dirac.'

It is not true that every graph without holes belongs to Class 3, and the smallest counterexample, published in [1, 2], is the antihole of size 7. Clearly, no antihole belongs to Class 3, but we had also to check that the antiholes of size ≥ 7 *do not contain a hole*, and are *minimal* with respect to the nonmembership in Class 3. At that time, we were pretty sure that there were no other minimal obstructions; for that reason, at the end of my talk in Halle, I proposed the following open problem: *If a graph G and its complement are semi-Gallai graphs, is it true that* $\gamma(G) = \omega(G)$*?*

Clearly, this statement is equivalent to the Perfect Graph Conjecture.

July–August 1961: During a long symposium on Combinatorial Theory at Rand Corporation (with R.C. Bose, G. Dantzig, J. Edmonds, L. Ford, R. Fulkerson, A. Hoffman, N. Mendelsohn, Ph. Wolfe, and others), I presented a new result: *Every unimodular graph is in Class 2 and in Class 3.*

(I call *'totally unimodular'* a matrix which was called at that time a *'matrix with the unimodularity property'*, and a *'unimodular graph'* is a graph with a totally unimodular clique-incident matrix). At this meeting, I met for the first time Alan Hoffman, who mentioned to me interesting new problems about comparability graphs. Also, the fruitful discussions we had together encouraged me to write a paper in English about all the graphs for which I could prove their membership in either Class 2 or Class 3 (with the obvious conclusion that each graph in Class 2 seems to belong to Class 3 and vice versa). When I came back to France, I sent my manuscript to Alan, at Yorktown Heights, for comments and hopefully for submission to some US journal.

March–April 1964: I attended a NATO Advanced Study Institute on Graph Theory organized by Dr. E. Aparo at the beautiful Villa Monastero in Frascati, Italy, with L.W. Beineke, P.R. Bryant, A.L. Dulmage, J. Groenveld, P.W. Kastelyn, N.S. Mendelsohn, J.W. Moon, R.C. Read, W.T. Tutte and W.T. Youngs. For one week I had the opportunity to present this new concept. I had received a few months

[5] At that time triangulated graphs were known as Gallai graphs.

earlier an answer from Alan who discussed the problem at the I.B.M. Research Center of Yorktown Heights with Paul Gilmore and Harry McAndrew and suggested some improvements to my paper: so, before the end of the meeting, I could hand over to 'il Direttore' (Frank Harary) a final version, with proper acknowledgements to 'Dr. A.J. Hoffman and Dr. P. Gilmore for suggestions and helpful discussions' and to 'Dr. M.H. McAndrew for the proof of Theorem 5 which is shorter than our original version'. Unfortunately, my paper did not came out until three years later [6].

In fact, this approach led Gilmore to an attempt to axiomatize the relevant properties of cliques in graphs and to a rediscovery of the Halle open problem. This strengthened my conviction that the conjecture in *its strongest version* was valid, even if I was more interested in trying to prove that the graphs of Class 2 (the 'α-perfect' graphs) are the same as the graphs of Class 3 (the 'γ-perfect' graphs). This became the 'weak' conjecture, which seemed easier to settle than the 'stronger' conjecture. The weak conjecture was proved in 1971 by Lovász [16] who made this terminology obsolete: since ('α-perfect') and ('γ-perfect') are synonymous, both of them may be replaced by 'perfect', and the 'strong conjecture' became the 'Perfect Graph Conjecture'.

1965–1969: During that period, I did not do much research in combinatorics: I was in Rome as elected Director of the International Computation Center, and I was obliged to postpone the 'Seminar on Combinatorial Problems' of the University of Paris which we founded with M.P. Schutzenberger in 1961.

In July 1966, I organized in Rome an international symposium on graph theory with Andrásfai, Balas, Behzad, Dantzig, Dénes, Edmonds, Erdős, Hajós, Jewell, Kastelyn, Kotzig, Lawler, Minty, Motzkin, Mycielski, Nash-Williams, Nivat, Raynaud, Rosa, Rosenstiehl, Sabidussi, Sachs, and others, and I invited Gilmore to be the Chairman. During the meeting, I worked with Hajós on some properties of the Gallai graphs that we presented together to the symposium. Gallai [12] had a generalization of the Hajnal–Surányi theorem: *If in a graph each of the odd cycles of length at least 5 has two noncrossing chords, then the graph belongs to Class 2.*

In fact, he proved more, but his proof was complicated and for that reason, Surányi published separately a shorter proof [23]. In a letter, Gallai told me that he knew also that his graphs belong to Class 3, but here again, he did not produce a shorter proof. Our proof was simple, but not as short and elegant as the proof produced by Meyniel [20] in 1972 for a stronger result (rediscovered nearly simultaneously in Armenia by Markosian and Karapetian [19]), which can be restated as follows: *If each odd cycle of length at least 5 has at least two chords, then the graph belongs to Class 3.*

In 1967, I gave several talks on perfect graphs, in particular at the Bose symposium at Chapel Hill where Mark Watkins published a short report (an addendum to [7]) which contributed to making the Perfect Graph Conjecture popular. It was not always so, and the first symposium lecture about perfect graphs from other mathematicians was delivered by Horst Sachs [21] at the Calgary conference in 1969. We learned from him that E. Olaru defended his doctoral dissertation on perfect graphs at Ilmenau in 1969; his thesis was the first one on this topic.

At the Waterloo conference in 1968, I proposed for the Perfect Graph Conjecture a completely different approach. A new idea at that time was to treat a general family of nonempty sets (called 'edges') the same way as the family of edges of a graph in order to obtain a theorem which reduces to a *graph theory* theorem when the 'edges' are 2-element subsets. In a paper of Lovász [15], this point of view was used to extend the

concept of chromatic number, and this family was called a 'set-system'. In my paper [8], it was called a 'graphoid', and this led me to discover a new class of perfect graphs, the 'balanced graphs', which generalize the line-graphs of bipartite graphs. We must add that it is because of the simplicity of this new point of view that L. Lovász found in 1971 a proof of the Weak Perfect Graph Conjecture, published simultaneously in the context of hypergraph theory [16] and in the context of graph theory [17]. He gave later another equivalent formulation with the polyhedral point of view, and this was followed by several nice results (of e.g. R.G. Bland, V. Chvátal, R. Giles, R.L. Graham, H.C. Huang, M.W. Padberg, A.F. Perold, L.E. Trotter, A. Tucker, S.H. Whitesides).

Lovász's proof of the Weak Perfect Graph Conjecture was closely related to an earlier work of Ray Fulkerson on antiblocking pairs of polyhedral (especially his 'max-min inequality'). Ray proved that the conjecture was equivalent to another statement, which he found too strong to be true; when I sent him a postcard from Waterloo to inform him that the validity of the conjecture had just been established by Lovász, he was able to supply the missing link in only a few hours. Later, Ray invited me to publish the whole story in a volume that he was editing [11].

The most significant results obtained before 1980 have been assembled in [10], but many classes of perfect graphs have been introduced since then by different authors, using completely different arguments. Other papers deal with recognition algorithms for specific classes, complexity of optimization problems in perfect graphs (of e.g. Grötschel, Lovász and Schrijver). Other important results have been found in the last decade but, after more than 30 years, the Perfect Graph Conjecture remains open.

1.4 Shannon's Capacity

Figure 1.1 is a reproduction of the page of Shannon's paper where he defines the zero-error capacity.

1.5 Translation of the Halle-Wittenberg Proceedings[6]: Colouring of graphs, all cycles or all odd cycles of which are rigid[7]

Let G be a graph, X be the set of its nodes and U be the set of its edges. A complete subgraph of G is called a *clique*. The chromatic number of G is denoted by $\gamma(G)$ (this is the smallest number of colours allowing a colouring of the nodes of G such that two nodes joined by an edge never have the same colour). $\omega(G)$ is the number of nodes of a largest clique of G. $\omega(G) \leq \gamma(G)$ does always hold, and we want to study certain classes of graphs for which $\omega(G) = \gamma(G)$ holds.

A graph G is called a *Gallai graph* if every elementary cycle of size greater than 3 has a chord, i.e. an edge connecting two nonconsecutive nodes of the cycle. For example, the Husimi trees, which are studied in physics (and which are connected graphs without elementary cycles of length different from 3) are Gallai graphs. Similarly,

[6] Translated from [2] with kind permission of *Wiss. Z. Martin-Luther Univ. Halle-Wittenberg.*
[7] The title of the talk presented in Halle, was originally: Colouring of Gallai and semi-Gallai graphs respectively. Mr. T. Gallai communicated that there was a misunderstanding, since he had not studied this type of graphs more closely. Therefore, the title has been changed according to a suggestion of Mr. Dirac with the author's agreement. The corresponding changes in the text could not be made for typographical reasons (subeditor's note).

The sum of two channels is the channel formed by using inputs from either of the two given channels with the same transition probabilities to the set of output letters consisting of the logical sum of the two alphabets. Thus the sum channel is defined by a transition matrix formed by placing the matrix of one channel below and to the right of that for the other channel and filling the remaining two rectangles with zeros. If $p_i(j)$ and $\|p_i'(j)\|$ are the individual matrices, the sum has the following matrix:

$$\begin{array}{cccccc} p_1(1) & \cdots & p_1(r) & 0 & \cdots & 0 \\ \vdots & & \vdots & & & \vdots \\ p_t(1) & \cdots & p_t(r) & 0 & \cdots & 0 \\ 0 & \cdots & 0 & p_1'(1) & \cdots & p_1'(r) \\ \vdots & & \vdots & & & \vdots \\ 0 & \cdots & 0 & p_t'(1) & \cdots & p_t'(r) \end{array}$$

The product of the two channels is the channel whose input alphabet consists of all ordered pairs (i, i') where i is a letter from the first channel alphabet and i' from the second, whose output alphabet is the similar set of ordered pairs of letters from the two individual output alphabets and whose transition probability from (i, i') to (j, j') is $p_i(j)p_i'(j)$.

The sum of channels corresponds physically to a situation where either of the two channels may be used (but not both), a new choice being made for each transmitted letter. The product channel corresponds to a situation where both channels are used each unit of time. It is interesting to note that multiplication and addition of channels are both associative and commutative, and that the product distributes over a sum. Thus one can develop a kind of algebra for channels in which it is possible to write, for example, a polynomial $\sum a_n K^n$, where the a_n are non-negative integers and K is a channel. We shall not, however, investigate here the algebraic properties of this system.

The Zero Error Capacity

In a discrete channel we will say that two input letters are adjacent if there is an output letter which can be caused by either of these two. Thus, i and j are adjacent if there exists a t such that both $p_i(t)$ and $p_j(t)$ do not vanish....

If all input letters are adjacent to each other, any code with more than one word has probability of error at the receiving point greater than zero. In fact, the probability of error in decoding words satisfies

$$P_0 \geq \frac{M-1}{M} p_{\min}^n$$

where $p_{\min}$ is the smallest (non-vanishing) among the $p_i(j)$, n is the length of the code and M is the number of words in the code. To prove this, note that any two words have a possible output word in common, namely the word consisting of the sequence of common output letters when the two input words are compared letter by letter. Each of the two input words has a probability at least $p_{\min}^n$ of producing this common output word. In using the code, the particular input words will each occur $\frac{1}{M}$ of the time and will cause the common output $\frac{1}{M}p_{\min}^n$ of the time. This output can be decoded in only one way. Hence, at least one of these situations leads to an error. This error, $\frac{1}{M}p_{\min}^n$, is assigned to this code word, and from the remaining $M-1$ code words another pair is chosen. A source of error to the amount $\frac{1}{M}p_{\min}^n$ is assigned in similar fashion to one of these, and this is a disjoint event. Counting in this manner, we obtain a total of at least $\frac{M-1}{M}p_{\min}^n$ as probability of error.

If it is not true that the input letters are all adjacent to each other, it is possible to transmit at a positive rate with zero probability of error. The least upper bound of all rates which can be achieved with zero probability of error will be called the zero error capacity of the channel and denoted by C_0. If we let $M_0(n)$ be the largest number of words in a code of length n, no two of which are adjacent, then C_0 is the least upper bound of the numbers $\frac{1}{n}M_0(n)$ when n varies through all possible integers.

One might expect that C_0 would be equal to $\log M_0(1)$, that is, that if we choose the largest possible set of non-adjacent letters and form all sequences of these of length n, then this would be the best error free code of length n. This is not, in general, true, although it holds in many cases, particularly when the number of input letters is small. The first failure occurs with five input letters with the channel in Fig. 2. In this channel, it is possible to choose at most two non-adjacent letters, for example 0 and 2. Using sequences of these, 00, 02, 20, and 22 we obtain four words in a code of length two. However, it is possible to construct a code of length two with five members no two of which are adjacent as follows: 00, 12, 24, 31, 43. It is readily verified that no two of these are adjacent. Thus, C_0 for this channel is at least $\frac{1}{2}\log 5$.

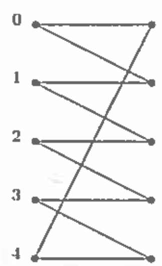

Fig. 2

Figure 1.1 Shannon's Capacity. Reproduced from [22] with kind permission of I.E.E.E. Publications.

it can be shown that those graphs which represent sets of intervals and which have been considered here by G. Hajós, are also Gallai graphs.

A graph is called a *semi-Gallai graph* if every odd elementary cycle of length > 3 has a chord. Hajnal and Surányi[8] have shown that in a Gallai graph G the minimum number of cliques in which the set of nodes X can be partitioned[9] is the internal stability number of G, i.e. the maximum number of nodes that can be chosen such that no two of them are connected by an edge.

If G is a semi-Gallai graph, then $\omega(G) = \gamma(G)$ holds 'almost always'. This can be seen by the following examples.

Example 1: *Gallai graphs.* A Gallai graph is obviously a semi-Gallai graph. One can easily show that for a Gallai graph G, $\omega(G) = \gamma(G)$ holds.

Example 2: *Simple graphs.* G is called a *simple graph* if it consists of two disjoint node sets X_1 and X_2 and some edges connecting X_1 to X_2. Since such a graph has no odd cycles, it is obviously a semi-Gallai graph. On the other hand,

$\omega(G) = 2 = \gamma(G)$, if G has at least one edge;
$\omega(G) = 1 = \gamma(G)$, if G has no edge.

Example 3: *Adjoint graphs of simple graphs.* The adjoint graph G^* of the graph G is, by definition, the graph whose nodes represent the edges of G such that two nodes are joined by an edge if and only if the edges that they represent share a common node in G.

If G is a simple graph, then the adjoint graph G^* is a semi-Gallai graph, otherwise G^* would have a chordless cycle of length $2k+1 > 3$, namely $u_1^*, u_2^*, \ldots, u_{2k+1}^*$. This would correspond to a sequence of edges $u_1, \ldots, u_{2k+1}$ giving rise to an elementary cycle of length $2k+1$, which is impossible, since G is a simple graph.

On the other hand, $\omega(G^*) = \gamma(G^*)$ since $\gamma(G^*)$ is the minimum number of colours allowing a colouring of the edges of G, which is the maximum degree of G, i.e. $\omega(G^*)$, by a well-known theorem of König for simple graphs.[10]

Example 4: The perfect graphs by Shannon:[11] One can show that these graphs, considered in information theory, are semi-Gallai graphs, and satisfy the relation $\omega(G) = \gamma(G)$.

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[8] [10] p. 113.
[9] That is, every node of G belongs to exactly one of the cliques.
[10] We have proved the following result in addition. A necessary and sufficient condition for a graph G to be an adjoint graph of a simple multigraph is that:

1. G is a semi-Gallai graph.
2. For every node x, the set $\{x\} \cup \Gamma x$ is the union of two cliques A_x and B_x, where the disjoint sets $A_x - B_x$ and $B_x - A_x$ are not connected by any edge. (Γx denotes the set of nodes adjacent to node x.)

The first condition alone has the following meaning. Let G^* be a graph adjoint to some graph H. Then G is a semi-Gallai graph if and only if it does not contain any cycle of odd length > 3. The second condition is necessary and sufficient for G to be adjoint to some graph G without any triangles.
[11] Compare C. Berge, *Théorie des Graphes*, Paris, 1959, pp. 37–39.

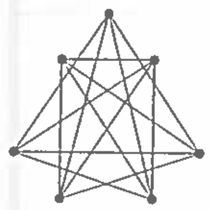

Figure 1.2

In view of such a large number of examples one could conjecture that for all semi-Gallai graphs $\omega(G) = \gamma(G)$ holds, but this is not true, as is shown by the following counterexample given by one of our students, Mr. Ghouila-Houri: G is a graph with nodes a, b, c, d, e, f, g and edges $ac, ad, ae, af, bd, be, bf, bg, ce, cf, cg, df, dg, eg$. One can easily show that G is a semi-Gallai graph with $\omega(G) = 3$ but $\gamma(G) = 4$. (See Figure 1.2).

1.6 Indian Report

Figure 1.3 is a reproduction of what are, as far as we are aware, the first explicit written statements of the strong and weak prefect graph conjectures [4].

VI CONJECTURES

The problem of characterizing α-perfect and γ-perfect graphs seems difficult, but the preceding results enable us to state several conjectures.

For instance:

Conjecture 1. A graph is α-perfect if and only if it is γ-perfect

Conjecture 2. A graph is γ-perfect if and only if it does not contain an elementary odd cycle of one of the following types:

type 1: the cycle is of length greater than 3 and does not possess any chord;

type 2: the cycle is of length greater than 3, and does not possess any triangular chord, but possess all its non-triangular chords (a chord is triangular if it determines a triangle with the edges of the cycle)

Conjecture 3. A graph is α-perfect if and only if it does not contain an elementary odd cycle of type 1 or 2.

It is easy to show that conjecture 2 is equivalent to conjecture 3, and implies conjecture 1. It is also easy to show that if a graph is γ-perfect (or α-perfect), then it does not contain an elementary odd cycle of type 1 or 2.

Figure 1.3 Indian Report. Reproduced from [4] with kind permission of the Indian Statistical Institute (Macmillan, Calcutta).

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2

From Conjecture to Theorem

Bruce A. Reed

CNRS, Equipe Combinatoire, Université Pierre et Marie Curie, Paris 6, France

We recall Berge's definition of a perfect graph and his two seminal conjectures concerning these objects, which were discussed in the previous chapter.

A graph G is *perfect* if for each induced subgraph H of G, the chromatic number of H, denoted $\chi(H)$, is equal to its clique number, $\omega(H)$. A graph is *minimal imperfect* if it is not perfect but all of its proper induced subgraphs are. Berge proposed:

The Weak Perfect Graph Conjecture *A graph is perfect if and only if its complement is perfect.*

The Strong Perfect Graph Conjecture *A graph is minimal imperfect if and only if it is a chordless cycle with* $2k+1$ *vertices for* $k \geq 2$ *or the complement of such a cycle.*

To provide evidence for the weak perfect graph conjecture, Berge noted that for many well-known classes of graphs, both the graphs in the class and their complements are perfect. The two best-known examples are the triangulated graphs and the comparability graphs.

Comparability graphs, which are discussed in more detail in the next chapter, permit a transitive orientation of their edges, that is, an acyclic orientation such that if $\vec{xy}$ is an arc and $\vec{yz}$ is an arc then $\vec{xz}$ is an arc. This implies that every oriented path of the graph induces a clique. Further, acyclicity implies that if $\vec{xy}$ is an arc then any oriented path ending at x can be extended to an oriented path ending at y. These two remarks imply that we can $\omega(G)$-colour a comparability graph G by colouring each vertex with the length of a longest oriented path ending at that vertex. Since every subgraph of a comparability graph is also a comparability graph (we restrict the transitive orientation to the subgraph), these graphs are perfect.

The fact that complements of comparability graphs are perfect is equivalent to a classical theorem of Dilworth [5]. His theorem, which was stated in terms of chains and antichains in partial orders, says that the minimum number of oriented paths

Perfect Graphs. Edited by J. L. Ramírez Alfonsín and B. A. Reed

needed to partition the vertices of a comparability graph is equal to the size of a largest stable set in the graph. Since these oriented paths induce cliques, we obtain that comparability graphs are perfect.

Remark 2.1 *Dilworth's theorem is also equivalent to earlier theorems of König, Menger and others, although these equivalences are not quite as direct (see [13]). One of these theorems states that bipartite graphs are Δ edge-colourable which, as noted by Berge [1], implies that the line-graphs of bipartite graphs are perfect.*

In the next section of this brief chapter, we prove that triangulated graphs and their complements are perfect. We then go on to discuss two proofs of the weak perfect graph conjecture and their polyhedral and algebraic consequences. In particular, we shall see that Lovász's proof of the weak perfect graph conjecture pointed out the fundamental role that perfect graphs play in polyhedral combinatorics.

2.1 Gallai's Graphs

We call a graph triangulated or chordal if it contains no induced cycle of length 4 or more. As discussed in Chapter 1, in the late 1950s and early 1960s these were known as the Gallai graphs. Indeed, Berge called the graphs without odd holes semi-Gallai, and would have used this term in the title of his famous 1961 talk, except that Gallai objected.

The proof that Gallai graphs are perfect relies on the following result, first proved by Dirac [6].

Definition 2.2 *For two vertices x and y in a graph G, a set S of vertices of G is an (x, y)-separator if there are no paths between x and y in $G - S$. S is a minimal vertex separator if there is no (x, y)-separator properly contained within S.*

Theorem 2.3 *Every minimal vertex separator in a Gallai graph is a clique.*

Proof. Let S be a minimal (x, y)-separator for some pair of nonadjacent vertices x and y in G. Let U_x be the component of $G - S$ containing x and U_y be the component containing y. Assume that S contains two nonadjacent vertices u and v. The minimality of S implies that every vertex of S sees a vertex of U_x. Thus there is a chordless path P_x between u and v in $u + v + U_x$. Similarily, there is a chordless path P_y between u and v in $u + v + U_y$. The union of these two paths is a chordless cycle of length at least 4 in the Gallai graph G. This is a contradiction. □

Corollary 2.4 *A graph is Gallai precisely if, for each of its induced subgraphs H, every minimal separator of H is a clique.*

Proof. One direction of the implication follows by applying the above theorem to each of the (Gallai) subgraphs H of a Gallai graph. The other direction holds because every induced cycle with four or more vertices contains a minimal separator which is not a clique. Indeed, every minimal separator in such a graph is a pair of nonadjacent vertices. □

Definition 2.5 *A clique cutset is a cutset whose vertices induce a clique.*

Corollary 2.6 *Every Gallai graph which is not a clique has a clique cutset.*

Proof. For every nonadjacent pair of vertices x and y in a graph, $V - x - y$ is an (x, y)-separator and therefore contains a minimal (x, y)-separator. □

It follows easily that no minimal imperfect graph is Gallai:

Theorem 2.7 *No minimal imperfect graph contains a clique cutset.*

Proof. Let C be a clique cutset in a minimal imperfect graph G. Let U be a component of $G - C$. We can obtain an $\omega(G)$-colouring of $G - U$ since it is perfect. We can obtain an $\omega(G)$-colouring of the subgraph of G induced by $C + U$ since it is perfect. As C is a clique, both these colourings use $|C|$ distinct colours on C. By renaming the colours we can make the two colourings agree on C. We thereby obtain an $\omega(G)$-colouring of G. This contradiction yields the desired result. □

Corollary 2.8 [2] *Every Gallai graph is perfect.*

Proof. Any imperfect Gallai graph contains a minimal imperfect Gallai graph. This graph cannot be a clique, so by Corollary 2.6 it has a clique cutset. But this contradicts Theorem 2.7. This is impossible. □

This theorem is a useful tool in a much more general setting (it also has even more useful generalizations; see Chapters 6 and 8). Gallai [8] proved that the class of *i-triangulated* graphs, i.e. those in which every cycle of odd length has at least two noncrossing chords, is a class of perfect graphs. He did so by proving that any such graph either has a clique cutset or is complete multipartite (i.e. can be partitioned into k stable sets such that any two vertices in distinct stable sets of the partition are adjacent) and hence clearly satisfies $\chi = \omega$, and then applying the theorem above.

Gallai did not prove that the complements of the i-triangulated graphs are perfect. To do so, he would only have needed to show that the complement of a minimal imperfect graph does not have a clique cutset, but it is not immediately apparent how to do this.

There is, however, a strengthening of Corollary 2.6, due to Dirac [6], which does allow us to prove simply that the complements of Gallai graphs are perfect.

Definition 2.9 *The neighbourhood of a vertex v, denoted $N(v)$, is the set of vertices to which it is adjacent.*

Definition 2.10 *A vertex in a graph G is simplicial if its neighbourhood induces a clique.*

Theorem 2.11 *Every Gallai graph G which is not a clique contains two non-adjacent simplicial vertices.*

Proof. We proceed by induction, the base case when G is a vertex being trivial. By Corollary 2.6, G has a clique cutset C. By the inductive hypothesis, for each component U of $G - C$, the graph H_U induced by $V(U) \cup C$ either is a clique or contains two nonadjacent simplicial vertices. In the first case, every vertex of U is simplicial in H_U. In the second case, at least one of any pair of nonadjacent simplicial vertices of H_U is in U. Hence, in either case, there is a simplicial vertex of H_U in U, which is clearly also simplicial in G. By considering two different components of $G - C$, the result follows. □

Lemma 2.12 *If G is minimal imperfect then $\overline{G}$ does not contain a simplicial vertex.*

Proof. For any vertex x of G, $\omega(N(x)) \leq \omega(G)-1$, and hence $N(x)$ has an $(\omega(G)-1)$-colouring. If x is simplicial in $\overline{G}$ then $G-N(x)$ is a stable set S. So S, along with our $(\omega(G)-1)$-colouring of $N(x)$, yields an $\omega(G)$-colouring of G, a contradiction. □

Corollary 2.13 *The complement of any Gallai graph is perfect.*

Proof. Since stable sets are perfect, this result follows immediately from Lemma 2.12 and Theorem 2.11. □

Remark 2.14 *This corollary was originally proven by Hajnal and Surányi in 1958 (see [10]). They were motivated by a result of Gallai which showed that the complements of interval graphs, a special type of Gallai graph, satisfied $\chi(G)=\omega(G)$. Hajnal and Surányi used the fact that for any vertex x in a Gallai graph, any pair of vertices in $N(x)$ joined by a path in $G-x-N(x)$ are adjacent. This fact also holds in the infinite case, in which they were interested (in contrast, infinite Gallai graphs need not contain simplicial vertices; consider an infinite tree).*

Gallai's contribution to the study of perfect graphs was not limited to the study of triangulated and i-triangulated graphs. He also wrote a seminal article characterizing comparability graphs, a translation of which can be found in the next chapter.

In that paper he studied the notion of the *geschlossene Menge* (literally: closed set) which nowadays we would call a module or a homogeneous set. This notion has turned out to be a very useful tool in algorithmic graph theory (see Chapter 5, where its importance in the study of P_4-structures is highlighted). It is also the key to the proof of the perfect graph theorem.

Remark 2.15 *Lovász introduced this idea independently in the context of normal hypergraphs.*

2.2 The Perfect Graph Theorem

Lovász [11] proved the weak perfect graph conjecture in 1972. For this reason, it is now also called the perfect graph theorem. The key to Lovász's proof is the replication lemma.

Definition 2.16 *We replicate a vertex x in a graph G by adding a vertex x' adjacent to $x+N(x)$.*

Definition 2.17 *Substituting a graph H for a vertex x in a graph G yields a new graph with vertex set $V(G)-x+V(H)$ such that the vertices of $G-x$ induce $G-x$, the vertices of H induce H and each vertex y of H is adjacent to precisely those vertices of $G-x$ which are in $N(x)$; see Figure 2.1.*

Definition 2.18 *A homogenous set in a graph G is a set S of vertices with $2 \leq |S| \leq |V(G)-1|$ such that for every vertex v not in S, either v sees all of S or v sees none of S.*

Remark 2.19 *Replicating a vertex x is the same as substituting an edge for x. A graph has a homogeneous set if and only if it arises via substitution.*

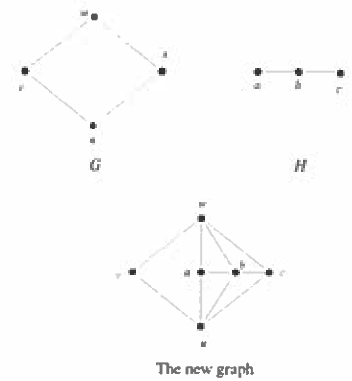

Figure 2.1 A substitution.

Lemma 2.20 (The replication lemma) *If G is obtained from a perfect graph by replicating a vertex then G is perfect.*

Proof. Consider a graph G' obtained from a perfect graph G by replicating a vertex x. Every induced subgraph H' of G' is either a subgraph of G or obtained from a subgraph H of G by replicating x. So, it is enough to show that for any graph H' obtained by replicating a vertex x in a perfect graph H, we have: $\chi(H')=\omega(H')$.

To do so, we note that if x is in a maximum clique of H then $\omega(H')=\omega(H)+1$, so we can colour H with $\omega(H)$ colours and use a new colour on x'. This yields the desired $\omega(H')$-colouring of H'.

If x is not in a maximum clique of H then we consider an $\omega(H)$-colouring of H and let S be the colour class containing x. Clearly, S meets every maximum clique of H, and hence so does $S-x$. Thus since $H-(S-x)$ is perfect, it has an $\omega(H)-1$ colouring. Combining this with the stable set $S-x+x'$ yields the desired $\omega(H')$ colouring of H'. □

Corollary 2.21 *If G is obtained by substituting a clique for a vertex in a perfect graph then G is perfect.*

Proof. We can perform this substitution via a series of replications. □

Theorem 2.22 [11] *If G is perfect then so is $\overline{G}$.*

Proof. It is enough to show that if G is perfect then $\chi(\overline{G})=\omega(\overline{G})$, i.e. that the vertices of G can be partitioned into $\alpha(G)$ cliques. Let G be a minimal counterexample to this statement.

Let t be the total number of maximum stable sets of G and, for each vertex x, let t_x be the number of maximum stable sets containing x. Create a new graph G^* by substituting a clique C_x of size t_x for each vertex x of G (whatever order these substitutions are performed in, we obtain the same G^*). We label each vertex of C_x by a distinct maximum stable set containing x. These labels point out a t-colouring of G^*.

Clearly, G^* has $t\alpha(G)$ vertices and $\alpha(G^*)=\alpha(G)$. Thus $\chi(G^*) \geq \frac{|V(G^*)|}{\alpha(G^*)}=t$. So, $\chi(G^*)=t$ and since, by Corollary 2.21, G^* is perfect, we know that G^* contains a clique C of size t. No two vertices of C can be labelled by the same stable set and so C contains a vertex labelled by each stable set. Thus, if we let C' be the set of those vertices x in G such that $C_x \cap C \neq \emptyset$ then C' intersects every maximum stable set of G, so $\alpha(G - C') \leq \alpha(G) - 1$. Furthermore, C' is clearly a clique because C is.

Now, by our inductive hypothesis, $G - C'$ can be covered with $\alpha(G) - 1$ cliques. These, along with C', yield a covering of G using $\alpha(G)$ cliques, a contradiction. □

2.3 Some Polyhedral Consequences

A linear programming problem (an LP) in n real-valued variables $x_1, \ldots, x_n$, involves maximizing a linear function of these variables subject to a set of linear inequalities. That is, for an integer m, and reals $c_1, \ldots, c_n, b_1, \ldots, b_m, a_{1,1}, \ldots, a_{1,n}, a_{2,1}, \ldots, a_{2,n}, \ldots, a_{m,1}, \ldots, a_{m,n}$, we wish to solve:

$$\begin{aligned} &\text{maximize} && \sum_{j=1}^{n} c_j x_j \\ &\text{subject to:} && \\ & && \sum_{j=1}^{n} a_{i,j} x_j \leq b_i \text{ for } i = 1, \ldots, m \\ & && x_j \geq 0 \text{ for } j = 1, \ldots, n. \end{aligned}$$

Or, in matrix terminology, for some $m \times n$ matrix A, row vector c of length n and column vector b of length m, we want to solve:

$$\begin{aligned} &\text{maximize} && c \cdot x \\ &\text{subject to:} && \\ & && Ax \leq b, \\ & && x \geq 0. \end{aligned}$$

In polyhedral terms we want to find a point of the polytope $P_{A,b}$ given by $Ax \leq b, x \geq 0$, which is furthest in the direction pointed to by c.

In an integer programmming problem (an IP), we wish to solve:

$$\begin{aligned} &\text{maximize} && c \cdot x \\ &\text{subject to:} && \\ & && Ax \leq b, \\ & && x \geq 0, \\ & && x_i \text{ integer for } i = 1, \ldots, n. \end{aligned}$$

Thus, we want to find the point in $P_{A,b}$ with integer-valued coordinates which is furthest in the direction pointed to by c.

As discussed in Chapter 11, there is a polynomial-time algorithm to solve LPs. In contrast, determining if the solution to an IP is above a certain threshold is NP-complete.

Given an IP in the form above, its *fractional relaxation* is the LP obtained by dropping the integrality condition; see Figure 2.2. Often an optimal, or near optimal, solution can be obtained for an IP from an optimal solution to its fractional relaxation. In particular, if there is an optimal solution which has integer-valued coordinates then it is clearly also an optimal solution to the IP. It is of great interest to determine when this will be the case.

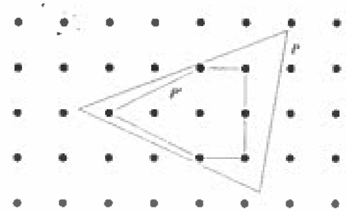

Figure 2.2 Feasible regions for an LP and corresponding IP.

In this context, it is important to note that an IP problem is really just a special type of LP presented in a compact form. To see this, we consider the polytope $P'_{A,b}$ which is obtained by taking the convex hull of the points in $P_{A,b}$ with integer-valued coordinates (i.e. a vector x is in $P'_{A,b}$ if and only if it is a convex combination of points in $P_{A,b}$ with integer-valued coordinates). The vertices of this polyhedron clearly have integer-valued coordinates. Now, for any objective function c the maximum of $c \cdot x$ over a polyhedron (if it exists) is obtained at a vertex. In particular, then, the maximum of $c \cdot x$ over $P'_{A,b}$ is obtained at a point y of $P_{A,b}$ with integer-valued coordinates. So, since every point z of $P_{A,b}$ with integer-valued coordinates is in $P'_{A,b}$, and hence satisfies $c \cdot z \leq c \cdot y$, we see that this maximum is indeed the solution to the corresponding IP.

The main reason why IPs are harder to solve than LPs is that the constraints needed to define $P'_{A,b}$ may be difficult to obtain from those used to define $P_{A,b}$. For one thing, there may be many (e.g. exponentially) more of them. However, if the IP is well structured we may be able to efficiently construct a description of $P'_{A,b}$ from the constraints for $P_{A,b}$. In particular, if every vertex of $P_{A,b}$ has integer-valued coordinates then the two polyhedra are the same.

It turns out that, as Chvátal [4] (see also [7]) pointed out, the perfect graph theorem implies that perfect graphs can be used to characterize those $(0, 1)$ matrices A such that $P_{A,1}$ has vertices with integer-valued coordinates (the 1 here and below denotes a vector all of whose coordinates are 1). This characterization depends heavily on the perfect graph theorem. To state it precisely, we need some definitions.

Definition 2.23 *A vector x dominates a vector y if each of its coordinates is at least as large as the corresponding coordinate of y.*

Definition 2.24 *A matrix is reduced if none of its rows is dominated by another.*

Observation 2.25 *If a row of a matrix A is dominated by another row then, letting A' be the matrix obtained from A by deleting the dominated row, we have $P_{A,1} = P_{A',1}$.*

Proof. For any nonnegative x, the constraint given by the dominated row will hold if the constraint given by the row dominating it does. □

It follows form our observation that in characterizing those matrices for which $P_{A,1} = P'_{A,1}$, we need only consider reduced matrices.

Definition 2.26 *A matrix A is a clique-node incidence matrix of a graph G if its rows are the characteristic vectors of the (inclusionwise) maximal cliques of G. That is, for a graph G with $V(G) = v_1, \ldots, v_n$ and some enumeration of the maximal cliques of G as $C_1, \ldots, C_m$, we have that the entry in the intersection of the ith column of A with the jth row of A is 1 if v_i is in C_j and 0 otherwise.*

Note that we can move between any two clique-node incidence matrices of G by reordering rows, and thus the polytope $P_{A,1}$ is the same for any clique-node incidence matrix A for G. So, we sometimes speak of *the* clique-node incidence matrix A_G for G.

The characterization we are interested in is the following:

Theorem 2.27 *For a (0,1) reduced matrix A, we have that $P_{A,1} = P'_{A,1}$ if and only if A is the clique-node incidence matrix of a perfect graph.*

Remark 2.28 *As we shall see in Chapter 11, this characterization allows us to optimize over $P_{A_G,1}$ for perfect G, and thereby solve the colouring and clique number problems on such graphs.*

We prove this theorem in a moment.

First, however, we note that a vector x satisfies $A_G x \leq 1$ precisely if it satisfies $\sum_{v_i \in C} x_i \leq 1$ for every clique of G. Thus, a vector x with integer-valued coordinates is in $P_{A_G,1}$ if and only if it is the characteristic vector of a stable set (that is, for some stable set S, we have $x_i = 1$ if $v_i \in S$ and $x_i = 0$ otherwise).

We define the *stable set polytope for G*, denoted $STAB(G)$, to be $P'_{A_G,1}$, that is, the convex hull of the characteristic vectors of the stable sets for G. We define the *fractional stable set polytope for G*, denoted $QSTAB(G)$ (the Q here stands for the rationals) to be $P_{A_G,1}$. We see that $STAB(G) \subseteq QSTAB(G)$, and our theorem tells us that these two polytopes are the same precisely if G is perfect.

Proof. We first show that if $P_{A,1} = P'_{A,1}$ then A must be the clique-node incidence matrix of some graph. We then show that this graph cannot be imperfect. Finally, we show that for any perfect graph G, $P_{A(G),1} = P'_{A(G),1}$ thereby completing the proof.

We turn now to the first step. Consider a matrix A, and the graph G_A whose vertices are the columns of A and in which two vertices are adjacent precisely if there is a row with a 1 in both of the corresponding columns.

Every row of A corresponds to a clique of G_A. So, if A is not a clique-node incidence matrix for G_A then either there is a maximal clique of G_A which does not correspond to a row of A or one of the rows of A corresponds to a nonmaximal clique of G_A. Since A is reduced, in either case, there is a clique of G_A such that no row of A has a 1 in all of the columns of this clique. We let K be a smallest such clique.

Since each edge of G_A comes from a row of A, $|K| \geq 3$. Further, by the minimality of K, we have:

> for every C in K, there is a row of A which has a 1 in all of the columns of $K - C$. (*)

We let c be the cost vector such that c_i is 1 if the ith column is in K and 0 otherwise. We claim that the maximum of $c \cdot x$ over x in $P_{A,1}$ exceeds max $c \cdot y$ over the y in $P_{A,1}$ with integer-valued coordinates, which implies $P_{A,1} \neq P'_{A,1}$. To prove our claim, we note first that by (*), any vector y in $P_{A,1}$ with integer-valued coordinates has a 1 in at most one of the columns of K, and hence $c \cdot y \leq 1$. On the other hand, consider the vector x for which $x_i = 0$ if the ith column is not in K and $x_i = \frac{1}{|K|-1}$ if the ith column is in K. Since no row of A has a 1 in all the columns of K, this vector is clearly in $P_{A,1}$. Further, $c \cdot x = \frac{|K|}{|K|-1} > 1$. This proves the claim and hence shows that $P_{A,1} \neq P'_{A,1}$.

We now show that for an imperfect graph G, $P_{A_G,1} \neq P'_{A_G,1}$, i.e. $STAB(G) \neq QSTAB(G)$. To this end consider a minimal imperfect subgraph H of G. Let c be the vector such that $c_i = 0$ if $v_i \notin H$ and c_i is the number of maximum cliques of H containing v_i for $v_i \in H$. Let y be the vector such that y_i is 0 if v_i is not in H and $y_i = \frac{1}{\omega(H)}$ for v_i in H.

Clearly, $y \in QSTAB(G)$. Further, letting $\mathcal{C}(H)$ be the set of maximum cliques of H, we have:

$$c \cdot y = \sum_{v_i \in H} \frac{1}{\omega(H)} \{\text{\# of cliques of } \mathcal{C}(H) \text{ containing } v_i\}$$
$$= \frac{1}{\omega(H)} \sum_{C \in \mathcal{C}(H)} \{\text{\# of vertices in } C\} = |\mathcal{C}(H)|.$$

On the other hand, for the characteristic vector z of a stable set S of G, $c \cdot z$ is the number of maximum cliques of H which have a nonempty intersection with S. Now, S cannot intersect all of $\mathcal{C}(H)$ for then $\omega(H - S) = \omega(H) - 1$ and so an $(\omega(H) - 1)$-colouring of $H - S$ along with $H \cap S$ yields an $\omega(H)$-colouring of H, contradicting the fact that it is minimal imperfect. So, for any vertex z of $STAB(G)$, we have $c \cdot z < c \cdot y$. Thus, for any vector z' in $STAB(G)$ we have $c \cdot z' < c \cdot y$; hence, $STAB(G) \neq QSTAB(G)$.

Finally, to complete the proof, we consider a clique-node incidence matrix A for a perfect graph G. We show that for any nonnegative integer-valued objective function c, the maximum of $c \cdot x$ over x in $QSTAB(G)$ is obtained on a stable set of G. It follows, using basic calculus, that the same statement holds for arbitrary c. Hence $STAB(G) = QSTAB(G)$, i.e. $P_{A,1} = P'_{A,1}$, as required.

Note first that if c is the vector 1, then the fact that G can be covered by $\alpha(G)$ cliques yields that $c \cdot x \leq \alpha(G)$ for any x in $QSTAB(G)$. Thus the characteristic vector of a maximum stable set maximizes $c \cdot x$ over $QSTAB(G)$. For arbitrary c, we will use the replication lemma to obtain an auxiliary graph G_c, such that $\alpha(G_c)$ is the maximum of $c \cdot x$ over all characteristic vectors of stable sets, and then use a clique covering of G_c to show that it is also the maximum of $c \cdot x$ over x in $QSTAB(G)$.

Specifically, let G_c be the graph obtained by substituting a stable set S_i of size c_i for vertex v_i (this involves deleting v_i if $c_i = 0$). Since $\overline{G_c}$ is obtained by substituting cliques into $\overline{G}$, the replication lemma and the perfect graph theorem imply that G_c is perfect.

Now, for a stable set S in G_c, if $S \cap S_i$ is not empty then $S \cup S_i$ is a stable set. So, for any maximum stable set T of G_c, for each v_i, either $T \cap S_i = \emptyset$ or $T \cap S_i = S_i$. Thus, there is a stable set T' of G with characteristic vector z such that $|T| = c \cdot z$. Conversely, given any stable set T' of G, we see that $T = \cup_{v_i \in T'} S_i$ is a stable set of G_c satisfying $|T| = c \cdot z$, where z is the characteristic vector of T'. So, the maximum of $c \cdot z$ over the characteristic vectors of stable sets of G is $\alpha(G_c)$.

On the other hand, since G_c is perfect it has a covering by $\alpha = \alpha(G_c)$ cliques: $K_1, \ldots, K_\alpha$. Letting K'_i be the clique $\{v_j | S_j \cap K_i \neq \emptyset\}$ in G, we see that $K'_1, \ldots, K'_\alpha$ is a family of cliques covering each v_i exactly c_i times. Now, for any vector x in $QSTAB(G)$ and each K'_i, we have: $\sum_{v_i \in K'_i} x_i \leq 1$. Summing over all the α cliques, we see that $c \cdot x \leq \alpha$, as required. □

2.4 A Stronger Theorem

Shortly after proving the perfect graph theorem, Lovász [12] proved the stronger result that a graph is perfect if and only if every induced subgraph H satisfies $\alpha(H)\omega(H) \geq |V(H)|$. His proof was phrased in terms of normal hypergraphs. We give a proof below due to Gasparian [9] which has a more graph-theoretic flavour. If H is perfect then an $\omega(H)$-colouring of H shows $|V(H)| \leq \alpha(H)\omega(H)$. So, to prove Lovász's result, we need only show:

Theorem 2.29 *If G is minimal imperfect then $|V(G)| = \alpha(G)\omega(G) + 1$.*

Proof. For any vertex v of G, the fact that $G - v$ is perfect implies that $|V(G)| = |V(G-v)| + 1 \leq \alpha(G-v)\omega(G-v) + 1 \leq \alpha(G)\omega(G) + 1$. Hence, we need only prove that $|V(G)| \geq \alpha(G)\omega(G) + 1$.

To this end, consider a maximum stable set $S_0 = \{v_1, \ldots, v_{\alpha(G)}\}$ in G. Since G is minimal imperfect, for each $v \in S_0$, $G - v$ has an $\omega(G)$ colouring. We let $S_{(i-1)\alpha(G)+1}, \ldots, S_{i\alpha(G)}$ be the stable sets in the colouring of $G - v_i$. For each i between 0 and $\alpha(G)\omega(G)$, if $\omega(G - S_i) = \omega(G) - 1$ then it is perfect, and so has an $(\omega(G)-1)$-colouring. But this colouring, along with S_i, yields an $\omega(G)$-colouring of G, a contradiction. So, for each such S_i we can choose a maximum clique of G, K_i disjoint from S_i.

The following observation is crucial to the proof:

Observation 2.30 *Each $\omega(G)$ clique is disjoint from exactly one S_i.*

Proof. If an $\omega(G)$-clique K is disjoint from S_0 then it is $\omega(G)$-coloured in the $\omega(G)$-colouring of $G - v_i$ for each v_i in S_0, and hence intersects S_i for all i between 1 and $\alpha(G)\omega(G)$.

If K intersects S_0 in v_i then, for the same reason, it intersects every colour class in our $\omega(G)$-colouring of $G - v_j$ for $j \neq i$. Also, $K - v_i$ intersects exactly $\omega(G) - 1$ colour classes in our colouring of $G - v_i$. The result follows. □

This observation implies that $S_0, \ldots, S_{\alpha(G)\omega(G)}$ are distinct, as are $K_0, \ldots, K_{\alpha(G)\omega(G)}$.

Now, for each S_i, we let x^i be its characteristic vector (recall that this means that we have $x^i = (x_1^i, \ldots, x_n^i)$, where x_j^i is 1 if $v_j \in S_i$ and is 0 otherwise). We claim that the x^i are linearly independent, i.e. for any reals $w_1, \ldots, w_n$ such that $\sum_{i=0}^{\alpha(G)\omega(G)} w_i x^i$ is the zero vector, we have, for all i, $w_i = 0$.

To prove our claim we fix an arbitrary i, and note that since $|S_j \cap K_i| = 1$, for all $j \neq i$, we have $\sum_{v_l \in K_i} w_j x_l^j = w_j$. Thus, $\sum_{j \neq i} \sum_{v_l \in K_i} w_j x_l^j = \sum_{j \neq i} w_j$. Since S_i does not intersect K_i, we have that $\sum_{v_l \in K_i} w_i x_l^i = 0$. Combining these two equations, we obtain $\sum_j \sum_{v_l \in K_i} w_j x_l^j = \sum_{j \neq i} w_j$. If $\sum_{k=0}^{\alpha(G)\omega(G)} w_k x^k$ is the zero vector then $\sum_{v_l \in K_i} \sum_j w_j x_l^j = 0$. Combining the last two equations we obtain $\sum_{j \neq i} w_j = 0$, and hence $w_i = \sum_{j=0}^{\alpha(G)\omega(G)} w_j$. So the w_i must all be equal. But now we see they must all be zero, as claimed.

It is a basic result of linear algebra that we cannot have more than n linearly independent vectors of length n, so $|V(G)| \geq \alpha(G)\omega(G) + 1$, as desired □

Actually, as pointed out by Padberg [15] and Bland, Huang and Trotter [3], linear algebra tells us much more about the sets of maximum cliques and maximum stable sets in a minimally imperfect graph. For example, we have:

Theorem 2.31 *If G is a minimal imperfect graph then:*

(i) G has $|V(G)|$ maximum stable sets,
(ii) G has $|V(G)|$ maximum cliques,
(iii) each vertex is in $\alpha(G)$ maximum stable sets,
(iv) each vertex is in $\omega(G)$ maximum cliques.

Proof. We need only prove that G has at most $|V(G)|$ maximum cliques. Then, by considering $\overline{G}$ we see that G has at most $|V(G)|$ maximum stable sets and hence the set $S_0, \ldots, S_{\alpha(G)\omega(G)}$ of the previous proof is exactly the set of maximum stable sets of G. Thus, (i) and (iii) hold and so, by considering $\overline{G}$, so do (ii) and (iv).

To see that G has at most $|V(G)|$ maximum cliques we consider the $|V(G)|$ by $|V(G)|$ matrix A such that the ith row of A is the characteristic vector x^i of the stable set S_i from the previous proof. For any maximum clique K of G, we let y^K be the column vector which is the characteristic vector of K. Then, Ay^K is a column vector $(z_0^K, \ldots, z_n^K)$, where z_i^K is $|S_i \cap K|$. So, the crucial fact from the previous proof implies that exactly one of the z_i^K is 0 and the rest are 1. That is, $Ay^K = Ay^{K_i}$ for some i. This implies that $A(y^K - y^{K_i}) = 0$. Since the rows of A are independent, we obtain $y^K = y^{K_i}$ and hence $K = K_i$. Thus, $K_0, \ldots, K_{\alpha(G)\omega(G)}$ is the set of maximum cliques of G. □

Further results on the maximum cliques and stable sets of a minimal imperfect graph, proved in the same manner, are discussed in Chapter 9.

We note that the key property of a minimal imperfect graph G used in the proofs of Theorems 2.29 and 2.31 was:

> For every vertex v of G, $G - v$ can be partitioned into α cliques of size ω and ω stable sets of size α.

Graphs which satisfy this property for some α and ω are partitionable. It is easy to mimic the proofs above to show that the characteristic vectors of the maximum cliques in a partitionable graph are independent and that Theorem 2.31 stil holds if we replace *minimal imperfect* by *partitionable*.

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