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Reliability Calculus: A Theoretical Framework To Analyze Communication Reliability

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Abstract—**Communication reliability is one of the most important concerns and fundamental issues in network systems, such as cyber-physical systems, where network components, sensors, actuators, controllers are interconnected with each other. These systems are prevalent in many safety-critical areas, including aerospace, automotive, civil infrastructure, energy, healthcare, manufacturing, and transportation, etc. In such systems, a single link failure, or communication delay could lead to catastrophic consequences. Hence, there is an urgent demand on efficient methodologies to model and analyze the delay distribution of control messages or feedback signals, especially when networks grow more complex and more heterogenous. In this paper, a calculus based on frequency domain analysis is developed to address this goal, so we can model and analyze the reliability of communication in large-scale compositional networked systems. Several network structures (e.g. serial, parallel, circular and backup) are defined as building blocks to model a wide variety of connections in networked systems. The advantages of the proposed theoretical framework over the traditional time-domain approaches include the capability to capture higher order moments of system characteristics, scalability to analyze the reliability of complex systems, efficiency in calculation and practicability in simulation.**

I. INTRODUCTION

As an emerging area, cyber-physical system (CPS) research is the synergy of physical processes, computing, communication and control into multiple levels of information processing and operation management [1]. Cyber-physical systems are composed of and interconnected by various components (e.g., sensors, monitors, controllers, actuators, and embedded computers) through networks. Such systems have gained increasing attentions in a wide range of application domains, such as nuclear power plants, highway traffic, embedded medical devices, oil refiners, power grids, and healthcare. In these systems, communication reliability is of paramount concern. For example, an undelivered or delayed control message or feedback signal may cause degraded performance, and even disastrous consequences. Designing reliable communication protocols with a specified/desired confidence degree requires a powerful tool to precisely model and analyze the end-to-end communication delay distribution, because the cumulative distribution function (CDF) of the message delays can be used as a metric to infer communication reliability. In this paper, we focus on the study of communication reliability of networked systems, which is defined as the probability that a message in the underlying network is successfully delivered end-to-end within a given time. Note that the communication reliability (end-to-end delay distribution) depends on the delay distribution of individual components and associated links. It requires that we precisely model the end-to-end delay distribution over possible routes and under different protocols to provide reliability or quality of service (QoS) service guarantees. However, it is challenging to model and analyze the communication reliability for large-scale networked systems due to the diversity of application characteristics, and the complicated network topology and communication protocols.

Previous work on end-to-end delay modeling and analysis has been primarily focused on solutions in the time domain. There are several intrinsic drawbacks of such time-domain-based approaches. *First*, many time-domain solutions leverage Markov-Chain-based (queueing) models [2], [3], [4], [5], [6], which simplify real-world systems. However, these methods are usually hard to generalize to capture the higher-order delay (latency) characteristics, which are important for understanding the behavior of real-world systems and designing solutions accordingly. *Second*, computational complexity in time-domain-based approaches is prohibitively high in many cases, since they require calculating convolutions over per-hop delay distributions. Further, they are not scalable and practical for large-scale networks as a change in a single network component may require designers to rebuild and re-solve the whole model from scratch. As system designers, on the other hand, we often desire to evaluate various alternatives of a component in a large-scale network in the design phase. Moreover, delays in each component may be time-varying according to various traffic conditions. The complexity associated with the computation of time-domain-based solutions often renders the end-to-end delay analysis inaccurate or

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infeasible in these scenarios. *Third*, it is sometimes hard for time-domain-based approaches to yield closed-form solutions. The reliability analysis is usually based on ad-hoc design, because the computational complexity can be very high.

To address the aforementioned drawbacks, we develop *Reliability Calculus*, a new framework to model and analyze communication reliability in networked systems. The framework is based on frequency-domain modeling and analysis. It is computationally efficient and can obtain closed-form solution for end-to-end communication reliability. Further, it is more scalable and fits the compositional nature of many networked systems (e.g., CPS). In this framework, first, we define four structures including serial, parallel, circular and backup as building blocks for modeling and analysis. Then, we demonstrate the properties of frequency-domain analysis and the benefits of using the proposed *Reliability Calculus* for communication reliability analysis of compositional networked systems. The *Reliability Calculus* framework provides a general architecture for communication reliability (or end-to-end delay) analysis that is applicable to a wide range of networked applications, and applicable even beyond the horizons of "communication networks", including power grids and transportation networks.

The remainder of this paper is organized as follows. Section II introduces related work for communication reliability analysis. Section III presents the basic building blocks to model communication reliability in frequency domain for networked compositional systems. In Section IV, we discuss the properties of the frequency-domain functions which form the foundations of reliability analysis in our framework. Finally, Section VI summarizes the paper with conclusions and the future work.

II. RELATED WORK

A. Reliability Modeling

In reliability engineering, the probability that communication messages are successfully delivered from a source to a destination is defined as 2-terminal reliability (2TR), which concerns the existence of a good path between a source-destination pair [7]. Fratta and Montanari [8] provided a method to calculate the 2-terminal reliability for a fixed network using Boolean algebra. Feo and Johnson [9] proposed an approximation approach to the 2TR problem based on the partial factoring technique. Ball [10] gave an overview of network connectivity reliability analysis and analyzed the computational complexity. Lida [11] proposed a method for estimating the reliability of links within transportation networks. Then Fishman et. al. [12] proposed a method based on the Monte Carlo (MC) Sampling to analyze all-terminal reliability (ATR) of connectivity networks. ATR is defined as the multi-probability that all terminals (devices or nodes) in the network can communicate with each other with existing paths in a predetermined time. Following these efforts, simulation-based approaches have been extensively studied to approximate the values for the communication connectivity. For example, Marseguerra et al. [13] described a method to integrate uncertainty into reliability calculations through Monte Carlo simulation and Genetic Algorithms. Chen et al. [14] proposed several different route choice models to study network connectivity reliability in transportation systems. Ball and Provan [15] investigated the problem of generating efficient computational methods to get the upper and lower bounds of ATR through cut and path set methods.

B. End-to-end Delay Analysis

In a communication network, the timeliness of a message delivery is an important design factor. Characterizing the end-to-end delay becomes a fundamental research issue and has been studied extensively for wired networks [16], [17], [18], [2], [3], [4], [19], [20], [21], [22] and for wireless networks [23], [6], [24], [5], [25], [26], [27], [28], [29], [30] respectively. Due to the stochastic nature of network components and protocols, it is desired to capture the complete characterization or distribution of the end-to-end delay in large-scale networks. Previous work [23], [5], [25], [26], [27] has analyzed the latency performance based on first order statistics, i.e., mean and variance. These approaches, however, are not capable to capture the higher-order delay characteristics. Several efforts have been made to provide probabilistic bounds on end-to-end network delays. Cruz in [31], [32] proposed the concept of Network Calculus which provides a theoretical framework for analyzing worst-case message delay bounds in computer networks based on the min-plus algebra. Later, network calculus and its variations [33], [34], [35], [36] have been extensively studied by researchers for different applications and scenarios. The probabilistic extensions of Network Calculus have been studied in [37], [38], [39] to provide statistic delay bounds. However, decision-making based on worst-case delay bounds is often conservative. Similarly, queueing theory or queueing-network-based analysis [5], [40], [41], [18], [23] is usually based on specific traffic patterns or conditions (such as heavy traffic) when modeling delay behavior, hence this kind of approaches are not general enough.

When modeling and analyzing the communication reliability (e.g. end-to-end delay distribution), one of the key challenges is computational complexity. Existing approaches require calculating the convolution of delays of individual components or links, which incurs very heavy computational overhead and thus makes the analysis infeasible under large-scale settings. Hence, one of the most key design goals of the proposed work is to reduce the computational complexity.

III. BUILDING BLOCKS

Communication reliability (or end-to-end delay) distributions are usually given in the time domain. However, time-domain analysis is not always convenient or efficient. We propose to convert the analysis from time domain to frequency domain by Laplace Transform (i.e.$f(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)dt$, where $\mathcal{L}$ is the operator of Laplace Transform). After we get analysis results in frequency domain, we convert the results from frequency domain back to time domain by *Inverse Laplace Transform*, so that people can understand the results in the time domain. Intuitively, by switching analysis to the frequency domain, we will be able to convert the convolution calculation to multiplication, which simplifies the calculation greatly. According to the preliminary investigation of properties of frequency-domain functions, we believe that frequency-domain analysis will bring additional benefits in modeling and simulating end-to-end delay distribution.

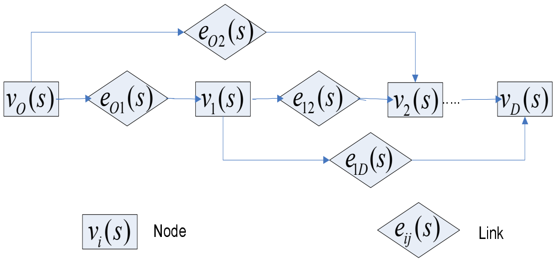


Fig. 1. An example networked system in frequency domain

A networked system can be modeled as a graph with a set of nodes and links. In our frequency domain analysis framework, $V = \{v_i(s)\}$ represents a reliability function of a node (or component or device), and $E = \{e_{ij}(s)\}$ denotes a reliability function of a link in frequency domain. Figure 1 shows an example networked system, where rectangular and rhombic blocks represent states of vertices and links in the frequency domain, respectively. To study the communication reliability, we accumulate the delay on links, and assume that $v_i(s) = 1$ on nodes for the convenience to present our idea.

Most networked systems are composed of serial, parallel, circular and back-up structures [42], [43]. Hence, we can use these structures as the building blocks in frequency-domain analysis, and then generalize the analytical framework for end-to-end delay modeling and reliability analysis in general networked systems. Note that these building blocks can also be used to model protocol behaviors.

Serial Structure: In the serial structure shown in Figure 2, the end-to-end delay is the sum of all delays on individual components or links from v_1 to v_n. In the time domain, the end-to-end delay distribution of such a structure can be estimated as the convolution of delays on individual blocks, i.e., $v_1 * e_{12} * v_2 * e_{23} * \cdots v_n$. Here, we accumulate the delays on links, and assume that $v_i(s) = 1$ on nodes.

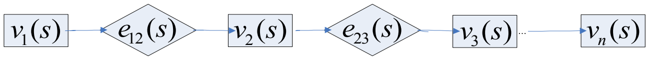


Fig. 2. Serial structure

Let's assume $e_{ij}(s),\ i \neq j,\ \forall i,j = 1,2,\cdots,n$, are independent of each other. According to the *Laplace Transform* [44] and probability theory [45], we express the end-to-end delay of the serial structure $e_{1n}(s)$ as follows:

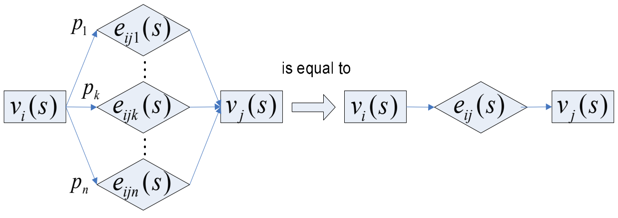
$$e_{1n}(s) = \prod e_{ij}(s) = e_{12}(s) \cdot e_{23}(s) \cdot \cdots \cdot e_{(n-1)n}(s). \quad (1)$$

For example, if the link delay in serial structure follows normal distribution, e.g. $e_{ij}(t) = N(u_{ij}, \sigma_{ij}^2)$ in the time domain. We can use frequency-domain representation to describe the link delay as $e_{ij}(s) = e^{\frac{1}{2}\sigma_{ij}^2 s^2 - \mu_{ij} s}$. By Equation (1), we get the frequency-domain delay distribution of the serial structure. That is $e_{1n}(s) = \prod e_{ij}(s) = e^{\frac{1}{2}s^2 \sum \sigma_{ij}^2 - s \sum \mu_{ij}}$. For the serial structure, the convolution calculation in the time domain is converted into multiplication in the frequency domain. Therefore, computational complexity is reduced greatly.

Parallel Structure: In the parallel structure shown in Figure 3(a), several alternative paths exist from a source to a destination. A path e_{ijk} from v_i to v_j is selected with a certain probability p_k. Without loss of generality, we assume $p_1 + p_2 + \cdots + p_n = 1$ under the trustable routing agent.

Suppose that the delay distribution of an individual link is $e_{ijk}(s), k = 1,2,\cdots,n$ in the frequency domain and they are independent of each other. We can describe the end-to-end delay distribution of the parallel structure $e_{ij}(s)$ in Figure 3(b) as follows:

$$\begin{aligned} e_{ij}(s) &= \sum_{k=1}^{n} p_k \cdot e_{ijk}(s) \quad (2) \\ &= p_1 \cdot e_{ij1}(s) + p_2 \cdot e_{ij2}(s) + \cdots + p_n \cdot e_{ijn}(s). \end{aligned}$$



(a) (b)

Fig. 3. Parallel structure

For example, if the delay distribution functions of two parallel links follow different exponential distributions, i.e. $e_{ij1}(s) = \frac{1}{s+\lambda_{ij1}}, e_{ij2}(s) = \frac{1}{s+\lambda_{ij2}}$ and $p_1 = 0.5, p_2 = 0.5$, we obtain the end-to-end delay distribution of parallel structure as follows,

$$e_{ij}(s) = \frac{1}{2} \cdot \frac{1}{s + \lambda_{ij1}} + \frac{1}{2} \cdot \frac{1}{s + \lambda_{ij2}}.$$

Circular Structure: The circular structure shown in Figure 4(a) contains two nodes $v_i(s)$ and $v_j(s)$, and two links $e_{ij}(s)$ and $e_{ii}(s)$, where $e_{ii}(s)$ is a self-circular or a feedback loop. To get the closed form of end-to-end delay probability density function of circular structure represented by $e^*_{ij}(s)$ in Figure 4(c), we use the block shown in Figure 4(b) to approximate the circular structure. For an infinite n, Figure 4(a) and Figure 4(b) are equivalent. Note that we can use the circular structure to model packet loss and retransmission behavior.

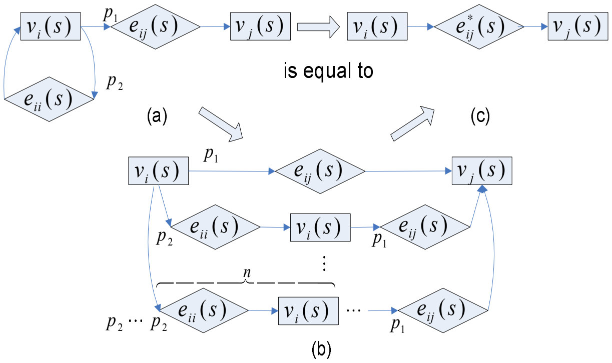


Fig. 4. Circular structure

Suppose that the delay distribution functions of two links $e_{ij}(s), e_{ii}(s)$ are independent of each other. Based on the equivalent Figure 4(b) and above serial parallel structure, we can come up with the end-to-end delay distribution of the circular structure $e^*_{ij}(s)$ as follows:

$$\begin{aligned} e^*_{ij}(s) &= p_1 \cdot e_{ij}(s) + p_2 \cdot e_{ii}(s) \cdot p_1 \cdot e_{ij}(s) + \\ &\quad [p_2 \cdot e_{ii}(s)]^2 \cdot p_1 \cdot e_{ij}(s) + \cdots \\ &= p_1 \cdot e_{ij}(s) \cdot [1 + \sum_{n=1}^{\infty} (p_2 \cdot e_{ii}(s))^n]. \end{aligned}$$

Based on power series $1 + x + x^2 + \cdots = (1-x)^{-1}$, we have

$$e^*_{ij}(s) = \frac{p_1 \cdot e_{ij}(s)}{1 - p_2 \cdot e_{ii}(s)}. \tag{3}$$

If we assume that link delay follows negative exponential distribution, i.e., $e_{ij}(s) = \frac{\lambda_1}{\lambda_1+s}$ and $e_{ii}(s) = \frac{\lambda_2}{\lambda_2+s}$, we obtain the end-to-end delay distribution of the circular structure in the frequency domain as follows,

$$\begin{aligned} e^*_{ij}(s) &= \frac{p_1 \cdot \frac{\lambda_1}{\lambda_1+s}}{1 - p_2 \cdot \frac{\lambda_2}{\lambda_2+s}} \\ &= \frac{p_1 \cdot \lambda_1 \cdot (\lambda_2 + s)}{(\lambda_2 + s - p_2 \cdot \lambda_2)(\lambda_1 + s)} \\ &= \frac{p_1 \cdot \lambda_1 \cdot (\lambda_2 + s)}{(p_1 \cdot \lambda_2 + s)(\lambda_1 + s)}. \end{aligned}$$

Back-up Structure: The back-up structure shown in Figure 5 includes a source $v_i(s)$, a destination $v_j(s)$, and links $e_{ij1}(s), e_{ij2}(s), \cdots, e_{ijn}(s)$. The back-up structure is different from parallel structure, because there are concurrent routes, instead of alternative routes, from a source to a destination in the back-up structure. Hence, the back-up structure can be used to model broadcasting message delivery.

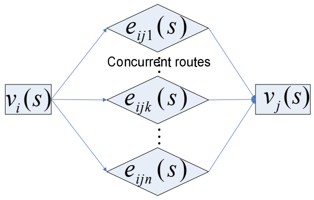


Fig. 5. Back-up structure

Suppose $e_{ijk}(s), k = 1, 2, \cdots, n$ are independent of each other. We can describe the end-to-end delay distribution function of the back-up structure $e_{ij}(s)$ as follows:

$$e_{ij}(s) = \min\{e_{ij1}(s), e_{ij2}(s), \cdots, e_{ijn}(s)\}. \tag{4}$$

where we obtain the minimum $e_{ij}(s)$ by comparing these expected values $-e'_{ijk}(s)\Big|_{s=0}, k = 1, 2, \cdots, n.$

When the delay distribution of all links follow negative exponential distributions in the frequency domain, i.e., $e_{ijk}(s) = \frac{\lambda_k}{s+\lambda_k}, k = 1, 2, \cdots, n$. It means that the delay distribution function $e_{ijk}(t) = \lambda_k e^{-\lambda_k t}, t \geq 0$ in time-domain. To compare $-e'_{ijk}(s)\Big|_{s=0}$, we only need to check $\lambda_k, k = 1, 2, \cdots, n$, because $e_{ijk}(s)$ is the minimum when the value λ_k is the maximum among $k = 1, 2, \cdots, n$

$$-e'_{ijk}(s)\big|_{s=0} = -(\frac{\lambda_k}{s+\lambda_k})'\Big|_{s=0} = -\frac{-\lambda_k}{(s+\lambda_k)^2}\Big|_{s=0} = \frac{1}{\lambda_k}.$$

IV. PROPERTIES OF FREQUENCY-DOMAIN FUNCTIONS

The following properties of frequency-domain functions illustrate the advantages of frequency-domain analysis in reliability modeling and simulation.

Property 1: The n-th order derivative for s of any frequency-domain functions, where $s = 0$, is equal to the product of n-th order origin moment of random variable T and the n-th power of -1 in the time domain, i.e.,

$$f^n_T(s)|_{s=0} = (-1)^n E(T^n), \tag{5}$$

where T is a random variable in the time domain, and $E(T^n)$ represents the n-th order origin moment of random variable T.

Proof:

$$\begin{aligned} \frac{\partial f_T(s)}{\partial s}|_{s=0} &= [\frac{\partial}{\partial s} \int_{-\infty}^{+\infty} e^{-st} f(t) dt]_{s=0} \\ &= [\int_{-\infty}^{+\infty} (-t) \cdot e^{-st} f(t) dt]_{s=0} \\ &= \int_{-\infty}^{+\infty} (-t) \cdot f(t) dt = -E[T]. \end{aligned}$$

Similarly,

$$\begin{aligned} f_T^2(s)|_{s=0} &= [\int_{-\infty}^{+\infty}(-t)^2\cdot e^{-st}f(t)dt]_{s=0} \\ &= (-1)^2E(T^2)=E(T^2). \\ f_T^3(s)|_{s=0} &= [\int_{-\infty}^{+\infty}(-t)^3\cdot e^{-st}f(t)dt]_{s=0} \\ &= (-1)^3E(T^3)=-E(T^3). \\ &\cdots\cdots \\ f_T^n(s)|_{s=0} &= (-1)^nE(T^n). \end{aligned}$$

With this property, we can get higher order derivatives of delay distribution functions, when $s=0$. Therefore, we can not only get the expectation, but also the higher order characteristics of end-to-end delay distribution conveniently.

Property 2: For an arbitrary probability distribution, if any order origin moment of its random variable exists, the frequency-domain function can be described as the series formatted as the following equation:

$$f_T(s)=\sum_{n=0}^{\infty}(-1)^n(\frac{s^n}{n!})E(T^n). \tag{6}$$

where T is a random variable in the time domain, and $E(T^n)$ represents n-th order origin moment of random variable T.

Proof: Since $f_T(s)=E(e^{-sT})$, and $e^{-x}=1-\frac{x}{1!}+\frac{x^2}{2!}-\cdots+(-1)^n\frac{x^n}{n!}+\cdots=\sum\limits_{n=0}^{\infty}(-1)^n(\frac{x^n}{n!})$, we have:

$$f_T(s)=E[\sum_{n=0}^{\infty}(-1)^n\frac{(sT)^n}{n!}]=\sum_{n=0}^{\infty}(-1)^n\frac{s^n}{n!}E[T^n].$$

This property implies that for any probability density function in the time domain, its frequency-domain function can be expressed as a series format, which is convenient for us to carry out qualitative analysis, quantitative calculation, and computer simulations. Also, the computational complexity scales linearly with n.

For example, if a delay distribution of a link T follows negative exponential distribution $T\sim f(t)=\lambda e^{-\lambda t}, t\geq 0$ in the time domain, we have $T\sim f(s)=\frac{\lambda}{\lambda+s}$ in the frequency domain. According to *Property 1*, we get

$$\begin{aligned} E(T) &= -\frac{\partial f(s)}{\partial s}|_{s=0}=-\frac{\partial}{\partial s}(\frac{\lambda}{\lambda+s})|_{s=0} \\ &= -\frac{-\lambda}{(\lambda+s)^2}|_{s=0}=\frac{1}{\lambda}. \end{aligned}$$

$$\begin{aligned} E(T^2) &= \frac{\partial^2 f(s)}{\partial s^2}|_{s=0}=\frac{\partial^2}{\partial s^2}(\frac{\lambda}{\lambda+s})|_{s=0} \\ &= \frac{2\lambda}{(\lambda+s)^3}|_{s=0}=\frac{2}{\lambda^2}. \end{aligned}$$

According to *Property 2*, we get

$$\begin{aligned} f(s) &= \frac{\lambda}{\lambda+s}=1-\frac{s}{\lambda}+\frac{s^2}{\lambda^2}-\cdots+(-1)^n\frac{s^n}{\lambda^n}+\cdots \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{s^n}{n!}\cdot\frac{n!}{\lambda^n}=\sum_{n=0}^{\infty}(-1)^n\frac{s^n}{n!}E(T^n). \end{aligned}$$

Hence, $E(T^n)=\frac{n!}{\lambda^n}$, and $E(T)=\frac{1}{\lambda}$, $E(T^2)=\frac{2}{\lambda^2}$, $\cdots$, $E(T^n)=\frac{n}{\lambda^n}, \cdots\cdots$. With these properties, the computation becomes simplified in the frequency domain, because the complex expression in the time domain has been changed to simple polynomials in the frequency domain. We will further investigate the properties of frequency-domain functions to benefit the delay modeling and communication reliability analysis.

Property 3: If the frequency-domain function of an arbitrary probability distribution exists, the function is unique.

For instance, the communication probability density function of Γ distribution is

$$f(t)=\frac{\lambda(\lambda t)^{n-1}e^{-\lambda t}}{(n-1)!}, t\geq 0, n\in \text{positive integer}.$$

We get its frequency-domain function:

$$\begin{aligned} f_T(s) &= \frac{\lambda^n}{(\lambda+s)^n}\int_0^{\infty}\frac{(\lambda+s)[(\lambda+s)t]^{n-1}e^{-(\lambda+s)t}}{(n-1)!}dt \\ &= (\frac{\lambda}{\lambda+s})^n \end{aligned}$$

and

$$\begin{aligned} E(T) &= -f_T'(s)|_{s=0}=-\frac{\partial}{\partial s}(\frac{\lambda}{\lambda+s})^n|_{s=0} \\ &= -\frac{-n\lambda^n}{(\lambda+s)^{n+1}}|_{s=0}=\frac{n}{\lambda} \end{aligned}$$

$$\begin{aligned} E(T^2) &= f_T''(s)|_{s=0}=\frac{\partial^2}{\partial s^2}(\frac{\lambda}{\lambda+s})^n|_{s=0} \\ &= \frac{-n\cdot\lambda^n\cdot-(n+1)}{(\lambda+s)^{n+2}}|_{s=0}=\frac{n^2+n}{\lambda^2} \end{aligned}$$

$$D(T)=E(T^2)-E^2(T)=\frac{n^2+n}{\lambda^2}-\frac{n^2}{\lambda^2}=\frac{n}{\lambda^2}.$$

The computation result in frequency domain indicates that it is the same as in time domain.

V. General Framework for Analysis

In this section, we present the general framework for communication reliability analysis based on the building blocks discussed in section III. We first define the *reliability adjacency matrix*, and develop matrix operations according to the network topology and the nature of the protocols. Our objective is to obtain a closed-form solution through the matrix operations, and infer communication reliability (i.e., end-to-end message delay). We also conduct sensitivity analysis to provide guidelines on how to improve the communication reliability of networked systems.

A. Reliability Adjacency Matrix

Definition 1: For a networked system with a number of components $v_1, v_2, \cdots, v_n$ and links $e_{ij}, i, j = 1, 2, \cdots, n$, the *reliability adjacency matrix* can be expressed as follows:

$$[A_{ij}]_{n\times n}, i, j = 1, 2, \cdots, n, \tag{7}$$

where A_{ij} represents the on-hop (or direct) delay relation from v_i to v_j. We have,

$$A_{ij} = \begin{cases} 0, & \text{no direct links from } v_i \text{ to } v_i; \\ p_{ij}e_{ij}, & \text{direct connection;} \\ p_{ik}e_{ik} \cdot p_{kj}e_{kj}, & \text{serial structure;} \\ p_1 e_{ij1} + p_2 e_{ij2}, & \text{parallel structure;} \\ \frac{p_1 \cdot e_{ij}(s)}{1 - p_2 \cdot e_{ii}(s)}, & \text{circular structure;} \\ \min\{e_{ij1}(s), e_{ij2}(s)\}, & \text{backup structure.} \end{cases} \tag{8}$$

We use the example system in Figure 6, and assume parameters and delay distributions of links are known and given in the figure. In such a system, v_1 is the source and v_6 is the destination, we assume $p_1 + p_2 = 1$, $p_3 + p_4 = 1$, $p_5 + p_6 + p_7 = 1$, $v_k(s) = 1, k = 1, 2, \cdots, 6$, and $e_{ij}(s)$ follows various delay distributions, which can be determined through experimental or experiential statistics. We can we obtain the frequency-domain *reliability adjacency matrix* of the system as follows.

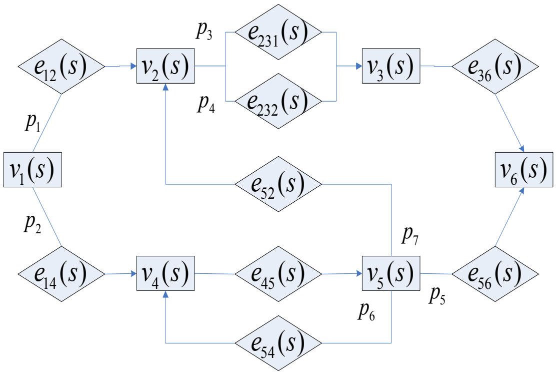
$$A = \begin{bmatrix} 0 & p_1 e_{12} & 0 & p_2 e_{14} & 0 & 0 \\ 0 & 0 & p_3 e_{231} + p_4 e_{232} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e_{36} \\ 0 & 0 & 0 & 0 & e_{45} & 0 \\ 0 & p_7 e_{52} & 0 & p_6 e_{54} & 0 & p_5 e_{56} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix A_{ij} describes all direct connections between any two components in the given system.

Definition 2: *Reliability adjacency matrix* in r hops is defined as A^r and $A^r = AA^{r-1}, r = 1, 2, \cdots$, where A^r_{ij} denotes the relationship from v_i to v_j within r hops along serial links.

According to *Definition 2*, *reliability adjacency matrix* A^r describes the end-to-end communication reliability along all links between any two components in the given system within r hops. Hence, we use $A^r_{16}(r > 1)$ to describe the end-to-end communication reliability distribution from node v_1 to v_6 in r hops. Let us denote $A^2 = AA = [A^2_{ij}]_{n\times n} = \sum_{k=1}^{n} A_{ik}A_{kj}$. Similarly,

$$A^r = AA^{r-1} = [A^r_{ij}]_{n\times n}, r = 1, 2, \cdots. \tag{9}$$



Links	Delay Distribution
$e_{12}(s)$	$\frac{1}{1+s}, p_1 = 0.6$
$e_{14}(s)$	$\frac{1}{s+2}, p_2 = 0.4$
$e_{231}(s)$	$\frac{1}{s+2}, p_3 = 0.5$
$e_{232}(s)$	$\frac{1}{s+3}, p_4 = 0.5$
$e_{36}(s)$	$\frac{3}{3+s}$
$e_{45}(s)$	$\frac{1}{1+s}$
$e_{54}(s)$	$\frac{2}{2+s}, p_6 = 0.3$
$e_{52}(s)$	$\frac{1}{1+s}, p_7 = 0.3$
$e_{56}(s)$	$\frac{2}{2+s}, p_5 = 0.4$

Fig. 6. An example networked system and parameters

Therefore, we have

$$\begin{aligned} A^2_{i6} &= A \cdot \begin{bmatrix} 0 \\ 0 \\ e_{36} \\ 0 \\ p_5 e_{56} \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ (p_3 e_{231} + p_4 e_{232}) e_{36} \\ 0 \\ e_{45} p_5 e_{56} \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

$$\begin{aligned} A^3_{i6} &= A \cdot \begin{bmatrix} 0 \\ (p_3 e_{231} + p_4 e_{232}) e_{36} \\ 0 \\ e_{45} p_5 e_{56} \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} p_1 e_{12} (p_3 e_{231} + p_4 e_{232}) e_{36} + p_2 e_{14} e_{45} p_5 e_{56} \\ 0 \\ 0 \\ 0 \\ p_7 e_{52} (p_3 e_{231} + p_4 e_{232}) e_{36} + p_6 e_{54} e_{45} p_5 e_{56} \\ 0 \end{bmatrix}. \end{aligned}$$

$$A_{i6}^4 = A \cdot \begin{bmatrix} p_1e_{12}(p_3e_{231}+p_4e_{232})e_{36} \\ +p_2e_{14}e_{45}p_5e_{56} \\ 0 \\ 0 \\ 0 \\ p_7e_{52}(p_3e_{231}+p_4e_{232})e_{36} \\ +p_6e_{54}e_{45}p_5e_{56} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ e_{45}p_7e_{52}(p_3e_{231}+p_4e_{232})e_{36} \\ +e_{45}p_6e_{54}e_{45}p_5e_{56} \\ 0 \\ 0 \end{bmatrix}.$$

$$A_{i6}^5 = A \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ e_{45}p_7e_{52}(p_3e_{231}+p_4e_{232}) \\ e_{36}+e_{45}p_6e_{54}e_{45}p_5e_{56} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} p_2e_{14}e_{45}p_7e_{52}(p_3e_{231}+p_4e_{232}) \\ e_{36}+p_2e_{14}e_{45}p_6e_{54}e_{45}p_5e_{56} \\ 0 \\ 0 \\ p_6e_{54}e_{45}p_7e_{52}(p_3e_{231}+p_4e_{232}) \\ e_{36}+p_6e_{54}e_{45}p_6e_{54}e_{45}p_5e_{56} \\ 0 \\ 0 \end{bmatrix}.$$

When we look at A_{16}^r, $r = 1, 2, ...$, which is the first element in A_{i6}^r, we find three connected paths from $v_1(s)$ to $v_6(s)$ within 5 hops, they are:

(1) $v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_6 : p_1e_{12}(p_3e_{231}+p_4e_{232})e_{36}$, we have

$$e_{1236}(s) = p_1e_{12}(s)[p_3e_{231}(s)+p_4e_{232}(s)]e_{36}(s);$$

(2) $v_1 \rightarrow v_4 \leftrightarrow v_5 \rightarrow v_6$: $p_2e_{14}e_{45}p_5e_{56}$, $p_2e_{14}e_{45}p_6e_{54}e_{45}p_5e_{56}, \cdots$,

$$e_{1456}(s) = \frac{p_2e_{14}(s)e_{45}(s)p_5e_{56}(s)}{1-p_6e_{54}(s)e_{45}(s)};$$

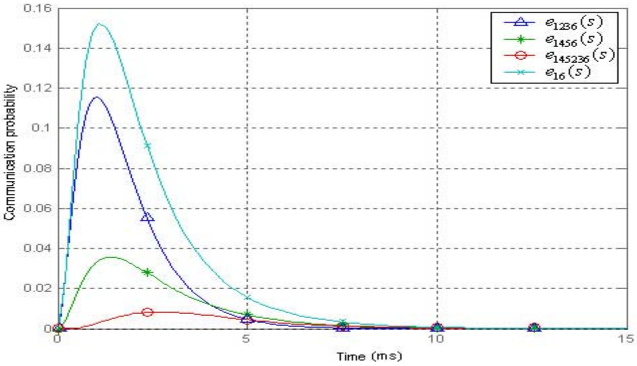
(3) $v_1 \rightarrow v_4 \leftrightarrow v_5 \rightarrow v_2 \rightarrow v_3 \rightarrow v_6$: $p_2e_{14}e_{45}p_7e_{52}(p_3e_{231}+p_4e_{232})e_{36}$, $p_2e_{14}e_{45}p_6e_{54}e_{45}p_7e_{52}(p_3e_{231}+p_4e_{232})e_{36}, \cdots$

$$e_{145236}(s) = \frac{p_2e_{14}(s)e_{45}(s)p_7e_{52}(s)[p_3e_{231}(s)+p_4e_{232}(s)]e_{36}(s)}{1-p_6e_{54}(s)e_{45}(s)}.$$

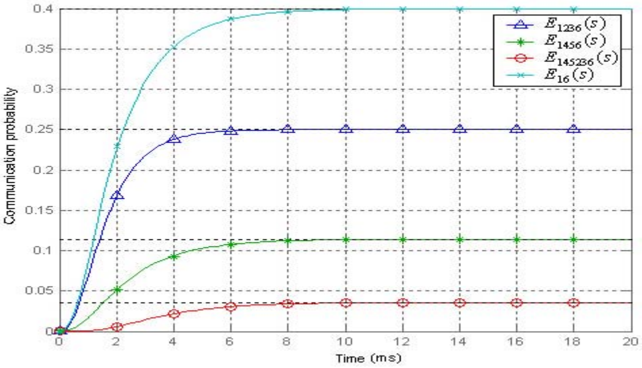
In path (2) and (3), "$\leftrightarrow$" stands for a circular structure. As a result, the communication reliability (or end-to-end delay) distribution on the path from v_1 to v_6 within 5 hops is shown below:

$$\begin{aligned} e_{16}(s) &= e_{1236}(s)+e_{1456}(s)+e_{145236}(s) \\ &= \tfrac{2.12s^3+12.14s^2+21.18s+10.08}{s^6+12s^5+57.4s^4+138.6s^3+175.6s^2+108.6s+25.2}. \end{aligned} \quad (10)$$

Figure 7 illustrates the end-to-end delay distribution $e_{16}(s)$ from node v_1 to node v_6 in the example system shown in Figure 6, as well as the delays along different paths. When we compare delay among different paths, we know that $e_{16}(s) \succ e_{1236}(s) \succ e_{1456}(s) \succ e_{145236}(s)$, where '$\succ$' means smaller delay. If we only consider the paths within 5 hops (e.g. set the *TTL (time-to-live)* be 5), the cumulated probability functions of delays $e_{16}(s), e_{1236}(s), e_{1456}(s), e_{145236}(s)$ are shown in Figure 7(b). Note that the cumulative probability function $E_{16}(s)$ is less than 1. This is because we only consider paths from source v_1 to destination v_6 within 5 hops. Through the above description, we can see that our frequency domain analysis approach has a great potential to reduce computational overhead.



(a) Probability density function of e2e delay (communication reliability)



(b) Cumulative probability function of e2e delay (communication reliability)

Fig. 7. Communication reliability distribution

B. Frequency-domain Analysis

Definition 3: The *general form* of frequency-domain function $f_T(s)$ is denoted as:

$$f_T(s) = \frac{b_1 s^m + b_2 s^{m-1} + \cdots + b_{m+1}}{a_1 s^n + a_2 s^{n-1} + \cdots + a_{n+1}}, \tag{11}$$

where $a_1 \neq 0$, and several a_i, a_j may be zero. Equation (11) can be written as *partial fraction expression* as follows,

$$f_T(s) = k(s) + \frac{r(1)}{s-p(1)} + \frac{r(2)}{s-p(2)} + \cdots + \frac{r(n)}{s-p(n)}. \tag{12}$$

With the above definitions, we will develop the theoretical framework to model and analyze end-to-end delays in general networked systems through the following steps:

Step 1: Obtain the symbolic expression of communication reliability (end-to-end delay) distribution between any pair of nodes in the system through operations over the *reliability adjacency matrix*. Then, substitute the symbols in the expression by delay distribution of individual components or links. So we can get the frequency-domain expression of the end-to-end communication reliability function.

Step 2: Write the frequency-domain expression into the *general form* (see *Definition 3*) in the frequency domain. Then, we get the frequency-domain characteristic by turning the general form into the partial fraction expression [44].

Step 3: Get the end-to-end communication reliability distribution function in time domain either by the *Inverse Laplace Transform* or through the frequency-domain *impulse response*, the response of a system to an impulse, which differs from zero for an infinitesimal time, but whose integral over time is unity [46].

Equation (11) and (12) show two typical forms of *transfer functions*, which are commonly used in signal processing and control theory. Hence, the proposed frequency-domain analytical framework provides good opportunities for us to leverage on control theories for communication reliability analysis.

C. Sensitivity Analysis

In order to illustrate how to enhance the communication reliability of a networked system, sensitivity analysis is conducted. By improving the reliability of certain key component, or blocks, we may achieve much better reliability of the whole system.

In sensitivity analysis, we replace different parameters of links and compare their communication reliability. Systems with different parameters are shown in Table I, *sys1* represents the original system shown in Figure 6; *sys2* represents a system when the link $e_{12}(s)$ in original system is 10 times faster or more reliable (i.e., $e_{12}(s)$ is changed from $\frac{1}{s+1}$ to $\frac{1}{s/10+1}$); similarly, *sys3* represents a system when link $e_{52}(s)$ improves its reliability ten times; and *sys4* represents a system when link $e_{45}(s)$ improves its reliable tenfold. The sensitivity analysis of communication reliability is shown in Figure 8, which indicates the communication reliability (or end-to-end delay distribution) of the system after we improve the performance of different links. The communication reliability of the original system is enhanced when the communication reliability of different links is improved.

TABLE I
SENSITIVITY ANALYSIS ON COMMUNICATION RELIABILITY

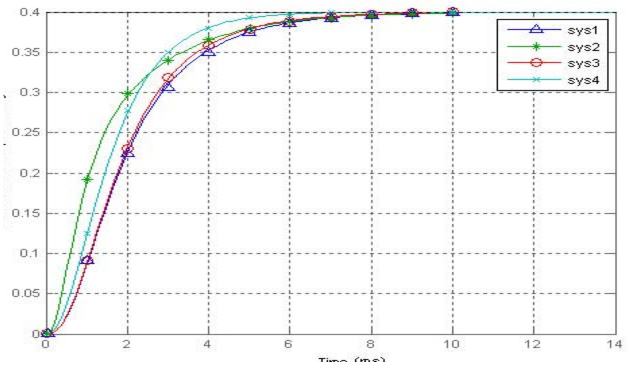
System label	Probability Density Distribution of Individual Components in Frequency Domain
sys1	$e_{12}(s), e_{14}(s), \cdots, e_{56}(s)$ are the same set of parameters as in Figure 6.
sys2	In sys1, $e_{12}(s)$ is improved to $e_{12}(s) = \frac{10}{10+s}$, and other settings are the same as sys1.
sys3	In sys1, $e_{52}(s)$ is improved to$e_{52}(s) = \frac{10}{10+s}$, and other settings are the same as sys1.
sys4	In sys1, $e_{45}(s)$ is improved to$e_{45}(s) = \frac{10}{10+s}$, and other settings are the same as sys1.

The sensitivity analysis clarifies the importance degree of different links. According to Figure 8, the influence of $e_{12}(s)$ and $e_{45}(s)$ for communication reliability is larger than that of $e_{52}(s)$. Figure 8 also implies that if end-to-end communication deadline is tight (less than 2.5 ms), improving reliability of $e_{12}(s)$ is more worthwhile than improving the reliability of $e_{45}(s)$, but if the communication deadline is loose (¿2.5 ms), then it is better to improve the reliability of $e_{45}(s)$ to achieve better robustness. With the sensitivity analysis, when we try to enhance entire system communication reliability with limited budget, we only need to improve the reliability of the most important components.

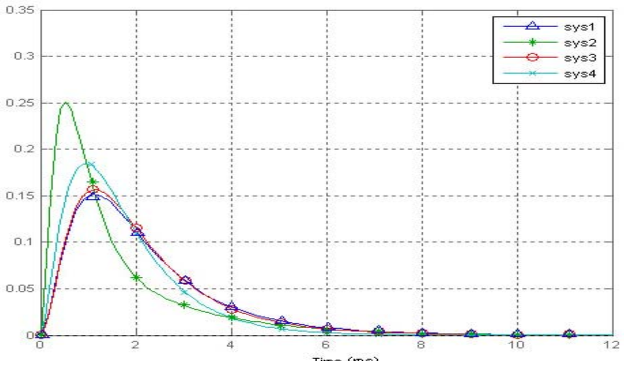
D. Communication Reliability of Larger-scale Systems

For large-scale networked systems, traditional way to evaluate communication reliability (or end-to-end delay distribution) in time domain suffers from the curse of network scale. In order to reduce the computational complexity, we propose a hierarchical approach for the larger-scale networked systems according to the *Reliability Calculus* presented in this paper.

We can divide a large-scaled network into k sub-networks or components based on the size and the network structure as shown in Figure 9. These sub networks can be further divided. As long as we can estimate the reliability functions of individual components (or sub networks), the communication reliability of any source and destination pair (S-D pair) is available through the frequency-domain analysis. The advantage of frequency-domain analysis is that the complexity of the reliability analysis for the compositional system is the *sum* of complexity of the analysis for the individual components (or sub networks), rather than the *product* of the complexity of individual components.



(a) Probability distribution function of communication reliability (or e2e delay)



(b) Probability density function of communication reliability (or e2e delay)

Fig. 8. Sensitivity analysis of different links

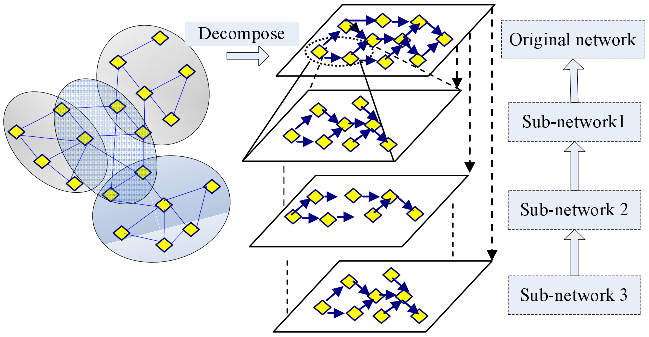


Fig. 9. The hierarchical structure of a larger-scale networked system

VI. CONCLUSIONS

In this paper, we propose a theoretical framework to model and analyze the communication reliability (end-to-end delay distribution) in networked compositional systems. The contribution of this paper can be summarized in the following aspects. First, we are the first to propose reliability analysis in frequency domain, so that the computational complexity can be reduced greatly. According to the properties in frequency domain calculation, we show that the reliability analysis in frequency domain is powerful, flexible and effective. Second, we build the *Reliability Calculus* on several building blocks (serial, parallel, circular and back-up structure) in networked systems, so that the reliability analysis can easily generated to model wide variety of networked structures. We can also apply the proposed *Reliability Calculus* to conduct sensitivity analysis to determine the bottleneck links in a network, and evaluate performance of different communication protocols. The reliability distribution of individual components or links can be obtained based on empirical or online measurements. In the future, we will develop reliability analysis software based on the *Reliability Calculus*. We will design interface to access online modeling tool to get reliability functions of individual network components, and make these individual reliability functions as input the software. In the future, we will also address the situation where communication quality or delays are not independent. We hope this work will bring new ideas and a novel tool, when investigating the communication reliability (end-to-end delay distribution) in large-scale networked systems.

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