Bertrand and Cournot Competition in Two-Sided Markets

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Abstract. We study the price of anarchy of a natural pricing mechanism in which producers specify prices for their goods, and consumers directly specify how much of each good they desire. Specifically, each producer submits a linear pricing function that specifies a per-resource price $p(d)$ as a function of the demand d that it faces.

Assuming producers have convex marginal costs, we show that the price of anarchy is a function $\rho_{\text{dem}}(\rho_{\text{sup}})$ of the supply-side price of anarchy ρ_{sup} that reduces to the demand-side price of anarchy when $\rho_{\text{sup}} = 1$. We then derive an expression for ρ_{sup} as a function of the market's structure; combined with ρ_{dem}, this expression leads to tight price of anarchy bounds in the two-sided market.

Most notably, we find that the price of anarchy varies dramatically across different types of markets. It approaches zero when the consumers utilities yield highly inelastic demand; however, for any fixed demand elasticity, the price of anarchy improves to 2/3 as competitiveness within the market increases. In a monopsony setting, the mechanism achieves its highest price of anarchy of 3/4.

Our results hold in multi-resource settings such as bandwidth markets over parallel-serial networks and, under an additional assumption on user demand, bandwidth markets over arbitrary networks. In the context of parallel-serial graphs, our analysis reveals how network structure affects the economic efficiency of the mechanism.

1 Introduction

Determining how to share scarce resources has been a central question in economics for centuries. Today, the need to efficiently distribute scarce resources often arises in practice within numerous fields of engineering. Power engineers, for example, are concerned with distributing electricity demand among generators in a way that minimizes total cost. Network engineering requires sharing limited bandwidth among competing network users, such as large Internet providers.

The most natural way of sharing resources is through a pricing mechanism. One of our aims in this paper is to better understand how efficiently can pricing mechanisms achieve that goal. Our interest in this theoretical question is motivated by a practical problem within networking.

1.1 Practical motivation

The current Internet is presently composed of numerous autonomous systems owned by competing companies. There is demand for connectivity among these independent networks: for example, networks owned by Internet service providers better serve their clients by peering with each other. In recent years, there has also been a surge of demand for connectivity with a good quality of service, mainly for the purpose of transmitting high-quality multimedia content such as movies.

There is also a willingness among autonomous systems to supply connectivity, that is to carry each other's traffic for a price. Presently, this connectivity is determined by contractual agreements among the different subnetworks. The Internet thus represents a natural market setting where a scarce resource, bandwidth, is produced and consumed by competing economic agents. A natural question is how should this market be designed.

Naturally, a market mechanism for Internet bandwidth should be efficient economically. It should allocate bandwidth to users that value it more, and have it provided by firms with lower costs. Such a mechanism must also be extremely scalable if it is to be used on a network the size of the Internet. In addition, it should also be intuitive for the users, so that they are willing to adopt the system in practice.

Perhaps the most natural market mechanism consists in simply asking providers to price their resources and then have consumers specify the resources quantities they desire. In economics, mechanisms that ask users for direct prices are called *Bertrand*; mechanisms that directly ask for resource amounts are called *Cournot*.

Although there has been effort to implement this natural Bertrand-Cournot mechanism in practice (Valancius et al., 2008), its economic efficiency is not well understood. In particular, most existing results hold only when one side of the market competes for a fixed supply or for a fixed demand. Here, we offer a theoretical analysis of Bertrand and Cournot competition in *two-sided* markets having both consumers and producers. Our results can help explain and predict the real-world performance of similar mechanisms.

1.2 Theoretical motivation

By analyzing Bertrand and Cournot competition, we address our main question, which is theoretical.

At the most basic level, we wish to know how efficiently can a pricing mechanism determine production and allocation of resources among competing consumers and firms. In particular, we want to understand what is the tradeoff between the computational complexity of a pricing mechanism and the social welfare it produces among its users.

To answer this broader question, we first examine the efficiency that can be achieved by the *most scalable* mechanisms one can imagine. Among such scalable mechanisms, we study the *most natural* pricing approach: the Bertrand-Cournot mechanism.

Bertrand and Cournot mechanisms have also been studied in the context of bandwidth markets over networks in order to understand the impact of combinatorial market structure on economic efficiency (Chawla and Roughgarden, 2008, Correa et al., 2010). Networks have been used to model two types of market competition: *horizontal*, in which firms provide goods that are perfect substitutes to each other, and *vertical*, in which the firms' goods can only be consumed in conjunction with each other. Our analysis of the effects of network structure on economic efficiency also illustrates how horizontal and vertical competition influence social welfare.

2 Related Work

2.1 The bandwidth market problem

The main practical problem we address is the design of a pricing mechanism for Internet autonomous systems.

Numerous mechanisms have been studied empirically. Esquivel et al. (year) propose Routebazaar, a system that is very similar to the Bertrand-Cournot mechanism outlined above. Using numerical simulations, the authors demonstrate that Routebazaar's bandwidth allocations produce high social welfare among users. Valancius et al. propose a similar system called MINT which also accepts per-unit prices for bandwidth from providers and desired consumption bundles from consumers. The authors discuss numerous issues concerning the implementation of MINT, and show empirically that MINT generates higher profits for providers than the BGP routing protocol. They also provide numerous references to other bandwidth markets, many of which have been deployed in practice.

From a theoretical perspective, the problem of sharing bandwidth has been receiving attention for more than fifteen years, with most results consisting in adapting the theory of VCG mechanisms. Varian et al. (1995) originally proposed a *smart market* for bandwidth that would charge a price per packet in a way that elicits truthful behavior from the users. The problem has been considered with the framework of algorithmic mechanism design by Nisan and Ronen (2000) and by Feigenbaum et al. (2001) In particular, Feigenbaum et al. propose a distributed mechanism that induces providers to reveal their true costs to the users, thus allowing Internet traffic to take the cheapest route.

The main limitation of VCG-based approaches is to assume that demand is fixed, that it remains constant no matter what the providers do. Real bandwidth markets on the other hand are *two-sided*: they also contain both consumers that compete with each other in response to the providers' prices. VCG techniques may not be particularly appropriate within two-sided markets, as we cannot incentivize both producers and consumers to be truthful without bringing in money from outside the market. If we increase payments to providers so that they act truthfully, there will be no money left to incentivize the consumers.

For this reason, our pricing mechanisms do not follow the classical VCG approach. Instead, we define a scalable mechanism and measure its economic efficiency in terms of price of anarchy. The interest of our particular mechanism comes from its use of Cournot and Bertrand bids.

2.2 The economic efficiency of scalable pricing mechanisms

One of the most studied scalable pricing methods is the proportional allocation mechanism (PAM) of Johari and Tsitsklis (2004). When consumers compete for a fixed supply, this approach yields a price of anarchy of 3/4, and this is the best guarantee that can be achieved within a large class of pricing mechanisms. When providers compete for a fixed demand, the PAM achieves a price of anarchy of 1/2. Kuleshov and Vetta (2010) study a generalization of the PAM to markets with both consumers and providers. They show that it yields a price of anarchy of about 58%, and that this is the best guarantee that can be achieved among a family of similar mechanisms.

Unfortunately, the two-sided PAM requires users to submit bids that do not have a simple interpretation, which may limit its adoption. Additionally, the analysis of the PAM by Kuleshov and Vetta assumes that providers submit pricing information without anticipating its effect on user demand. This may not accurately represent real-world behavior.

Other scalable mechanisms that have received significant attention include demand-side Cournot mechanisms, which accept desired resource quantities from consumers and price the good according to a fixed pricing function (Johari and Tsitsklis, 2005, Harks and Miller, 2009). Their price of anarchy depends entirely on how the link is priced. Linear functions result in a price of anarchy of 2/3; this number degrades to zero as the functions' curvature increases.

A third well-studied scalable pricing approach is the supply-side Bertrand mechanism, in which providers simply announce per-unit prices. Acemoglu and Ozdaglar (2007) examine its price of anarchy in markets over networks consisting of parallel edges in which demand is fixed. They also establish several negative efficiency results in parallel-serial networks. Correa et al. (2010) consider a variation of the Bertrand approach where instead of submitting fixed prices for one unit of resource, providers submit linear *pricing functions* $p(d) = \alpha d$. These functions specify a per-unit price $p(d)$ as a function of the total demand d faced by a provider. When market demand is fixed, Correa et al. (2008, 2010) derive necessary and sufficient conditions for the existence of equilibria among providers.

Recently, Chawla and Roughgarden (2008) have looked at a two-sided Bertrand-Cournot market where providers submit fixed prices as in the Acemoglu and

Ozdaglar model. They derive a series of negative results, showing that the price of anarchy equals zero in most types of markets.

Here, we consider a different version of the Bertrand-Cournot mechanism that combines the Cournot demand side of Johari and Tsitsiklis (2005) and the Bertrand supply side of Correa et al. (2010).

3 Results

We study a scalable and natural approach to market pricing that consists in asking each provider for a linear pricing function that specifies a per-unit resource price $p(d)$ when the provider faces a total demand of d. Consumers directly specify the amount of each good they want.

We find that the performance of this mechanism varies significantly across different types of markets. We first examine markets over simple networks of parallel edges. We show that the price of anarchy is a function $\rho_{\text{dem}}(\rho_{\text{sup}})$ of the supply-side price of anarchy ρ_{sup} that reduces to the demand-side price of anarchy when $\rho_{\text{sup}} = 1$. We derive an expression for ρ_{sup} as a function of the number of edges; this expression leads to several tight price of anarchy bounds in the two-sided market when combined with ρ_{dem}.

When users have linear valuations, we find that price of anarchy equals 2/3, matching an existing result on demand-sided Cournot mechanisms. However, when users' utilities yield demand functions that are very inelastic with respect to resource prices, the price of anarchy approaches zero. Fortunately, for any fixed demand elasticity, the price of anarchy approaches 2/3 when the number of providers in the market increases.

We subsequently extend these results to markets of parallel-serial networks, and we show how the structure of the market affects the price of anarchy. We find that the worst mechanism performance occurs in networks where edges are connected in serial. Although like in the previous case, the price of anarchy can reach zero when demand is highly inelastic, as the number of disjoint paths in the network increases, the price of anarchy improves to 2/3.

Finally, we show that most of our results carry over to markets over arbitrary networks under a technical assumption on user demand.

4 Definition of the Mechanism

We now present our mechanism in the context of our motivating practical problem: bandwidth sharing.

Consider a computer network of V nodes. A set of P paths run through the network. Two nodes might be joined by more than one direct connection; therefore we model the network by a multigraph $G = (V, E)$. We refer to a set of parallel edges between two nodes of the multigraph as a *link* and we denote the set of all links by L. In a general market setting, different links correspond to different goods. A path in the graph (V, L) induced by the links is referred to as a *route*. We denote the set of all routes by T.

Two types of users operate on the graph: Q consumers and R providers. By a slight abuse of notation, we use E, L, V, P, Q, R, and T to simultaneously denote the sets of elements as well as the sizes of these sets.

Each consumer q owns a source and a sink $s_q, t_q \in V$. Each provider $r \in R$ operates on some edge $e \in E$. For simplicity, we assume every edge e is uniquely identified with a provider r.

The consumers' goal is to send flow across the network from their source to their target along various available paths; if consumer q can transmit flow along path p, we denote this by $p \in q$. Consumer q will be able to send flow by buying bandwidth from providers located on edges along available paths. We write $e \in p$ to denote that edge e is on path p.

4.1 The two-sided mechanism

The two-sided mechanism specifies rules by which bandwidth is produced, sold, and consumed. It first accepts from each provider r a linear *pricing function* $p_r(f) = \gamma_r f$ with slope $\gamma_r > 0$. Equivalently, the strategy of a provider in the game induced by the mechanism is a scalar $\gamma_r > 0$. Since edges are in a one-to-one correspondence with providers, we equivalently refer to γ_r as γ_e.

This pricing function specifies the price provider r will charge per unit of capacity if his total demand is f. Thus if consumer q buys d_{qr} from r, he will pay r a total of $p_r(f_r)d_{qr} = (\gamma_r f_r)d_{qr}$, where f_r is the total amount of bandwidth r is asked to provide. Although γ_r specifies a pricing function, for the sake of brevity we will often simply call γ_r a price.

Given this pricing information, consumer q then directly specifies the amount of bandwidth d_{qp} it wishes to send over every path p and pays $\sum_{p\in P}\sum_{e\in p} d_{qp}p_e(f_e)$, where $f_e = \sum_{p\in P; e\in P}\sum_{q\in Q} d_{qp}$ is the total demand faced by the provider at edge e. In economics terms, the mechanism induces a two-stage Stackelberg game (ref., year) in which the providers lead.

4.2 Utility functions

Let $d_q = \sum_{p\in P} d_{qp}$ denote the flow sent by consumer q. The utility q obtains from sending d_q equals

$$U_q(d_q) = V_q(d_q) - \sum_{p\in P} d_{qp} \sum_{e\in p} p_e(f_e)$$

where $V_q(d_q)$ is q's *valuation function*. We make the following assumption on the valuation functions.

Assumption 1 *For all $q \in Q$, the valuation functions $V_q(d_q) : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous, increasing and concave. Over $(0,\infty)$, they are differentiable. At $d_q = 0$ the right derivative exists and is denoted by $V_q'(0)$.*

Consider now a provider r facing a demand of f_r. The utility r obtains from serving f_r units of bandwidth is defined as

$$\begin{aligned} U_r(f_r) &= p_r(f_r)f_r - C_r(f_r) \\ &= \gamma_r f_r^2 - C_r(f_r) \end{aligned}$$

where $C_r(f_r)$ is r's cost function. We make the following assumption about the providers' costs.

Assumption 2 *For all $r \in R$, the cost function $C_r(f) : \mathbb{R}_+ \to \mathbb{R}_+$ is of the form*

$$C_r(f) = \int_0^f c_r(x)dx$$

where $c_r(x) : \mathbb{R}_+ \to \mathbb{R}_+$ is the marginal cost *function of r. It is assumed that $c_r(x)$ is continuous, strictly increasing, convex, and $c_r(0) = 0$.*

4.3 Social welfare and equilibrium concept

We use the standard utilitarian objective function to measure welfare. Let $\boldsymbol{d}_q$ denote the vector of strategies $(d_{q1}, ..., d_{qP})$ of consumer q, and let $\boldsymbol{D}$ denote the matrix of all consumer strategies $(\boldsymbol{d}_1, ..., \boldsymbol{d}_Q)$. The vector of provider strategies $(\gamma_1, ..., \gamma_R)$ will be denoted by $\boldsymbol{\gamma}$.

Definition 1. *The* social welfare *of the mechanism equals*

$$W(\boldsymbol{D}, \boldsymbol{\gamma}) = \sum_{q \in Q} V_q(d_q) - \sum_{r \in R} C_r(f_r)$$

We use the subgame perfect Nash equilibrium as the solution concept for the two-stage game induced by the mechanism. Let $\boldsymbol{D}_{-q}$ represent the matrix of consumer strategies with q's entries deleted. Similarly, let $\boldsymbol{\gamma}_{-r}$ denote the vector of provider strategies with r's strategy deleted.

We will also use $f_r(\boldsymbol{\gamma})$ to denote the demand faced by provider r when all the pricing functions are linear with slopes as in $\boldsymbol{\gamma}$ and when consumers pick their equilibrium flows. Nash equilibria in demand-side Cournot mechanisms with linear pricing functions always exist and are unique (Johari and Tsitsklis, 2005), which implies that $f_r(\boldsymbol{\gamma})$ is well-defined for all r and $\boldsymbol{\gamma}$.

Definition 2. *A* subgame perfect Nash equilibrium *of the mechanism is a vector of strategies $(\boldsymbol{D}, \boldsymbol{\gamma})$ such that*

1. *For every consumer $q \in Q$,*

$$\boldsymbol{d}_q = \arg\max_{\boldsymbol{d}} U_q(\boldsymbol{d}, \boldsymbol{D}_{-q}, \boldsymbol{\gamma}).$$

   That is, $\boldsymbol{d}_q$ maximizes U_q with all other strategies being fixed.

2. For every provider $r \in R$,

$$\gamma_r = \arg\max_{\gamma} \left(\gamma f_r(\gamma, \boldsymbol{\gamma}_{-r}) - C_r(f_r(\gamma, \boldsymbol{\gamma}_{-r}))\right)$$

That is, γ_r is r's best response when all other pricing functions are fixed and when r anticipates the consumers' equilibrium response to his change.

We measure economic efficiency using the standard notion of price of anarchy.

Definition 3. *The* price of anarchy *is defined as the ratio*

$$\frac{W(\boldsymbol{D}^E, \boldsymbol{\gamma}^E)}{\max_{\boldsymbol{D},\boldsymbol{\gamma}} W(\boldsymbol{D}, \gamma)}$$

where $(\boldsymbol{D}^E, \boldsymbol{\gamma}^E)$ is a subgame perfect Nash equilibrium of the mechanism with the lowest social welfare.

4.4 Discussion

Even though in the real AS graph, providers are located at vertices, we choose to associate them with edges to be consistent with the vast majority of existing models of network resource allocation, such as the models of Nisan et al. (2000), Feigenbaum et al. (2002), Johari and Tsitsiklis (2004), Kuleshov and Vetta (2010). The combinatorial structure of many types of markets, such as markets for transportation, is more accurately modeled with edge-based providers.

The choice to model the mechanism by a two-stage game represents the providers' expected price-anticipating behavior. We expect any provider to take a few initial pricing guesses to estimate its demand curve. Once he has estimated the curve well enough, he will choose how to price his resource. This behavior should be closely approximated by assuming the provider has perfect knowledge of the consumers' responses. The standard model in non-cooperative game theory for settings in which some players anticipate other's actions is a two-stage game; the subgame perfect Nash equilibrium is its appropriate solution concept.

There can be many ways to collect pricing information from providers, the simplest of which would be to ask for a single uniform price. However, in optimal allocations, providers price resources at marginal cost. Fixed prices are not flexible enough for the providers to express information about their marginal cost structure.Our subsequent price of anarchy analysis demonstrates that linear pricing functions express that structure well. In addition, Harks and Miller (2009) recently established that linear pricing functions result in the highest social welfare among consumers in one-sided Cournot markets. Our analysis shows that these good efficiency properties carry over to two-sided markets.

Our last comment concerns Assumptions 1 and 2. Both have been previously made in the literature on resource allocation mechanisms, for example in the work of Johari, Tsitsiklis and Mannor (2006), Harks and Miller (2009), Kuleshov and Vetta (2010) or Correa et al. (2010). Although Assumption 1 is relatively modest, the requirement of convex marginal costs in Assumption 2 is much stronger. However, a form of Assumption 2 is necessary to obtain good efficiency guarantees.

5 Technical Preliminaries

In this section, we establish some technical results that will be used in all subsequent sections.

5.1 Elasticity of demand

Provider behavior in the two-stage game is determined by consumers' responses to price. For the purposes of our analysis, this response is entirely summarized by the *elasticity* of demand with respect to price.

Definition 4. *Let $f(x) : \mathbb{R} \to \mathbb{R}$ be a differentiable function. The elasticity of f with respect to x is a function $\epsilon_x f(y) : \mathbb{R} \to \mathbb{R}$ defined as*

$$\epsilon_x\, f(y) = \frac{df(y)}{dx} \frac{y}{f(y)}$$

Informally, elasticity at a point y is the ratio

$$\frac{\Delta\% f(x)}{\Delta\% x}$$

of the percentage change of $f(x)$ given a certain percentage change $\Delta\% x$ of x at y. Point elasticity is the limit of that ratio as $\%\Delta x \to 0$. Elasticity possesses the following useful properties. They can be derived using simple algebra.

Lemma 1. *Let $f(x), g(x) : \mathbb{R} \to \mathbb{R}$ be differentiable functions. Let $a \in \mathbb{R}$. Then the following holds:*

1. $\epsilon_x\, af = \epsilon_x\, f$ *for* $a > 0$.
2. $\epsilon_x\, f^a = a\, \epsilon_x\, f$
3. $\epsilon_x\, f(g(\cdot)) = \epsilon_g\, f\ \epsilon_x\, g$
4. $\epsilon_x\, (f + g) = \frac{f}{f+g} \epsilon_x\, f + \frac{g}{f+g} \epsilon_x\, g$
5. *A function has constant point elasticity if and only if it is a monomial.*

We mainly consider the elasticity $\epsilon_{\gamma_r} f_r$ of the demand a provider faces with respect to the slope of its pricing function. When referring to $\epsilon_{\gamma_r} f_r$, we drop the γ_r subscript and simply write ϵf_r.

5.2 Necessary and sufficient equilibrium conditions

We next identify necessary and sufficient conditions for a vector of bids $(\boldsymbol{D}, \boldsymbol{\gamma})$ to form a Nash equilibrium.

Lemma 2. *A vector of strategies $(\boldsymbol{D}, \boldsymbol{\gamma})$ is a subgame perfect Nash equilibrium of the two-sided mechanism if and only if the following holds for all q, r, p:*

$$V_q'(d_q) \geq \sum_{p'} \sum_{l \in p \cap p'} \gamma_l d_{p'} + \sum_{l \in p} \gamma_l f_l \tag{1}$$

$$V_q'(d_q) \leq \sum_{p'} \sum_{l \in p \cap p'} \gamma_l d_{p'} + \sum_{l \in p} \gamma_l f_l \text{ if } d_p > 0 \tag{2}$$

$$c_r(f_r) \leq \gamma_r f_r \left(2 - \frac{1}{\epsilon f_r}\right) \tag{3}$$

$$c_r(f_r) \geq \gamma_r f_r \left(2 - \frac{1}{\epsilon f_r}\right) \text{ if } f > 0 \tag{4}$$

Proof. Follows by differentiating the users' utilities with respect to d_p and f_r. □

6 Single-Link Graphs

In this section, we apply the two-sided mechanism to the simplest possible setting: that of a single resource. In the networking context, this corresponds to a network with two nodes, s and t. All users have s, t as their source and their target; every provider r offers to carry flow from s to t over the edge e_r.

The end result of this section is a series of theorems that identify the price of anarchy for different types of consumer valuation functions and different numbers of providers.

We establish these theorems in several steps. First, we show that our mechanism is equivalent to one where, from the consumers' perspective, there is only a single edge between s and t. This game is simpler to analyze and is easier to play for consumers. Secondly, we show that given any equilibrium $\mathcal{E}$ of this simplified game, there exist linear valuation and linear marginal cost functions such that the price of anarchy of the mechanism when the users have these valuations and costs is at least as bad as that of the original equilibrium $\mathcal{E}$. This implies that without loss of generality we can analyze our mechanism in the context of linear valuations and quadratic costs. Thirdly, we establish a lemma that expresses the price of anarchy as a function $\rho_{\text{dem}}(\rho_{\text{sup}})$ of a parameter ρ_{sup} that summarizes the extent by which providers selfishly overcharge the consumers. This parameter can be interpreted as the price of anarchy on the supply side of the market. The function ρ_{dem} combines demand-side market inefficiency with the supply-side inefficiency provided by ρ_{sup} to return the price of anarchy in the two-sided market. When $\rho_{\text{sup}} = 1$, it returns the price of anarchy of the one-sided Cournot resource allocation mechanism for consumers. Finally, we compute ρ_{sup} within several illustrative types of markets and obtain the two-sided price of anarchy by applying the previous lemma.

6.1 A simpler game

In this section we show that instead of asking from consumers for a flow on every edge, we can simply ask them for how much they want to send in total, and distribute that flow for them. This simplifies our subsequent price of anarchy analysis; it also makes the mechanism more scalable and easier to use by consumers.

The price that consumers will be charged for sending flow from s to t equals $p(f) = \Gamma f$, where f is the total flow sent by all consumers and

$$\Gamma = \frac{1}{\sum_e 1/\gamma_e}.$$

Thus, we essentially replace e parallel edges by a single edge with a new linear pricing function $p(f)$[3].

Our first lemma describes how the consumers' total flow should be distributed, and justifies our definition of Γ.

Lemma 3. *Suppose that at every link e there is a linear pricing function with slope γ_e. The cheapest cost for sending a flow of f is Γf^2 and the cheapest per-unit price that can be obtained is Γf, where*

$$\Gamma = \frac{1}{\sum_e 1/\gamma_e}.$$

There is also only one way to send the flow at that cost: the flow f_e at edge e equals

$$f_e = \frac{1/\gamma_e}{\sum_{e' \in E} 1/\gamma_{e'}} f$$

Proof. Suppose that f units of flow have to be sent from s to t. Let f_e, denote the flow on edge e when f is sent in the cheapest way. The price of sending an extra bit of flow on each edge must be equal, otherwise we could transfer a small amount of flow from one edge to the other. Thus we must have $\gamma_e f_e = \gamma_{e'} f_{e'}$ for all e, e'. Combining this with the identity $\sum_{e=1}^{n} f_e = f$, we obtain

$$f_e = \frac{1/\gamma_e}{\sum_{e' \in E} 1/\gamma_{e'}} f$$

The total cost of sending f is then

$$\sum_{e \in E}^{n} \gamma_i f_i^2 = \sum_{e \in E} \frac{1/\gamma_e}{(\sum_{e' \in E} 1/\gamma_{e'})^2} f^2 = \frac{1}{\sum_{e' \in E} 1/\gamma_{e'}} f^2$$

and this yields our expression for Γ. □

[3] Interestingly, edge aggregation mechanism follows the same principles as the aggregation of several parallel resistors into one.

When multiple users want to send flow from s to t, they should ideally distribute their flows in an optimal way across the edges. Our next lemma establishes that if we take the entire flow a user wishes to send and distribute that flow for them, the resulting mechanism has the same structure as the original mechanism. Specifically, it has exactly the same equilibria.

Definition 5. *The* aggregate *mechanism at a single link accepts from every consumer q a total flow of d_q and allocates that flow according to Lemma 3. User q receives a utility of*

$$U_q(d_q) = V_q(d_q) - \mathit{\Gamma} f d_q.$$

On the supply side, the aggregate mechanism is identical to the standard one.

Lemma 4. *The Nash equilibria of the standard and aggregate mechanisms are equal. At equilibrium, the utilities of each player are the same.*

Proof. Applied to the single-link game, the necessary and sufficient conditions of Lemma 2 state that:

$$V_q'(d_q) \geq \gamma_e(d_{qe} + f_e) \text{ for all } e \tag{5}$$

$$V_q'(d_q) \leq \gamma_e(d_{qe} + f_e) \text{ for all } e \text{ s.t. } d_{qe} > 0 \tag{6}$$

$$C_r'(f_r) \leq \gamma_r f_r \left(2 - \frac{1}{\epsilon f_r}\right) \tag{7}$$

$$C_r'(f_r) \geq \gamma_r f_r \left(2 - \frac{1}{\epsilon f_r}\right) \text{ if } f > 0 \tag{8}$$

It is not hard to show that the efficiency conditions for the aggregate game are:

$$V_q'(d_q) \geq \mathit{\Gamma}(d_q + f) \text{ for all } q \tag{9}$$

$$V_q'(d_q) \leq \mathit{\Gamma}(d_q + f) \text{ for all } q \text{ s.t. } d_q > 0 \tag{10}$$

$$C_r'(f_r) \leq \gamma_r f_r \left(2 - \frac{1}{\epsilon f_r}\right) \tag{11}$$

$$C_r'(f_r) \geq \gamma_r f_r \left(2 - \frac{1}{\epsilon f_r}\right) \text{ if } f_r > 0 \tag{12}$$

Take any demand-side equilibrium $\boldsymbol{d}$ of the aggregate game. We have to show that conditions (5-6) hold for

$$d_{qe} = \frac{1/\gamma_e}{\sum_{e'} 1/\gamma_{e'}} d_q$$

Condition 6 for any edge e follows by simple algebra:

$$V_q'(d_q) \leq \mathit{\Gamma}(d_q + f) = \frac{\gamma_e/\gamma_e}{\sum_{e'} 1/\gamma_{e'}}(d_q + f) = \gamma_e\left(\frac{1/\gamma_e}{\sum_{e'} 1/\gamma_{e'}} d_q + \frac{1/\gamma_e}{\sum_{e'} 1/\gamma_{e'}} f\right) = \gamma_q(d_{qe} + f_e)$$

Condition 5 follows similarly.

Recall that demand-side Cournot games have unique equilibria. Thus both games have the same equilibria, and for any $\boldsymbol{\gamma}$, the demand at each edge remains the same in the aggregate game. Thus the function $f_r(\boldsymbol{\gamma})$ is unchanged, and the equilibrium prices $\boldsymbol{\gamma}$ of the aggregate game also form a supply-side equilibrium in the original game. □

It is interesting that if the providers' costs C_r are quadratic — that is, of the form $C_r(f) = \frac{\beta_r}{2} f^2$ — Lemma 3 also describes the socially optimal way to distribute a flow of f across the edges.

Lemma 5. *Suppose that at every link e the cost function is $C_e(f) = \frac{\beta_e}{2} f^2$. The cheapest cost of sending a flow of f is Bf^2 where*

$$B = \frac{1}{\sum_e 1/\beta_e}.$$

There is also only one way to send the flow at that cost.

Proof. When f units of flow are sent in a socially optimal way across the link, the marginal costs $\beta_e f_e$ at each edge e must be equal. Thus

$$f_e = \frac{1/\beta_e}{\sum_{e' \in l} 1/\beta_{e'}} f$$

and the total cost to society is

$$\sum_{e \in l} \frac{\beta_e}{2} f_e^2 = \frac{1}{2} \sum_{e \in l} \frac{1/\beta_e}{(\sum_{\epsilon' \in l} 1/\beta_{e'})^2} f^2 = \frac{1}{2 \sum_{e' \in l} 1/\beta_{e'}} f^2.$$

□

6.2 Worst-case utility functions

In this section, we show that selfish users experience the greatest welfare loss when their valuation and marginal cost functions are linear.

In the next section, we compute the price of anarchy by finding the exact valuations and costs for which users experience the greatest welfare loss. The following lemmas in a sense tell us that it is enough to search in the subset of linear valuations and quadratic costs.

Our first two lemmas deal with valuation functions.

Lemma 6. *Let $(\boldsymbol{d}, \boldsymbol{\gamma})$ be any vector of strategies, and let $\boldsymbol{d}^*, \boldsymbol{\gamma}^*$ be welfare-maximizing strategies. Then*

$$\frac{\sum_{q \in Q} V_q(d_q) - \sum_{r \in R} C_r(f_r)}{\sum_{q \in Q} V_q(d_q^*) - \sum_{r \in R} C_r(f_r^*)} \geq \frac{\sum_{q \in Q} V_q'(d_q) d_q - \sum_{r \in R} C_r(f_r)}{\max_{\bar{\boldsymbol{f}} \geq 0} \left((\max_{q \in Q} V_q'(d_q)) \sum_r \bar{f}_r - \sum_{r \in R} C_r(\bar{f}_r) \right)}$$

Proof. By concavity, for any q we have $V_q(d_q^*) \leq V_q(d_q) + V_q'(d_q)(d_q^* - d_q)$. Since $V_q(0) \geq 0$ by assumption, $V_q(d_q) \geq V_q'(d_q)d_q$. Using these two inequalities we obtain

$$\begin{aligned}
\frac{\sum_{q\in Q} V_q(d_q) - \sum_{r\in R} C_r(f_r)}{\sum_{q\in Q} V_q(d_q^*) - \sum_{r\in R} C_r(f_r^*)} &\geq \frac{\sum_{q\in Q}(V_q(d_q) - V_q'(d_q)d_q) + \sum_{q\in Q} V_q'(d_q)d_q - \sum_{r\in R} C_r(f_r)}{\sum_{q\in Q}(V_q(d_q) - V_q'(d_q)d_q) + \sum_{q\in Q} V_q'(d_q)d_q^* - \sum_{r\in R} C_r(f_r^*)} \\
&\geq \frac{\sum_{q\in Q} V_q'(d_q)d_q - \sum_{r\in R} C_r(f_r)}{\sum_{q\in Q} V_q'(d_q)d_q^* - \sum_{r\in R} C_r(f_r^*)} \\
&\geq \frac{\sum_{q\in Q} V_q'(d_q)d_q - \sum_{r\in R} C_r(f_r)}{\max_{\bar{\boldsymbol{f}}\geq 0}\left(\left(\max_{q\in Q} V_q'(d_q)\right)\sum_r \bar{f}_r - \sum_{r\in R} C_r(\bar{f}_r)\right)}.
\end{aligned}$$

In the last line, observe that the consumer with the highest marginal utility receives all the supply. □

We use Lemma 6 to establish the following result.

Lemma 7. *Let $(\boldsymbol{d}, \boldsymbol{\gamma})$ be a vector of strategies such that*

1. *The vector $\boldsymbol{d}$ forms a demand-side equilibrium given pricing functions $\boldsymbol{\gamma}$.*
2. *The vector $\boldsymbol{\gamma}$ forms a supply-side equilibrium given elasticity functions $\boldsymbol{\epsilon f}$.*

Let $(d_q^*)_{q\in Q}$ be a welfare-maximizing flow allocation. There exist $\alpha_q > 0$ for all $q \in Q$ such that

$$\frac{\sum_{q\in Q} V_q(d_q) - \sum_{r\in R} C_r(f_r)}{\sum_{q\in Q} V_q(d_q^*) - \sum_{r\in R} C_r(f_r^*)} \geq \frac{\sum_{q\in Q} \alpha_q d_q - \sum_{r\in R} C_r(f_r)}{\max_{\bar{f}_r}\left(\left(\max_{q\in Q} \alpha_q\right)\sum_r \bar{f}_r - \sum_{r\in R} C_r(\bar{f}_r)\right)} \tag{13}$$

The vector $\boldsymbol{d}$ is still a demand-side equilibrium given the pricing functions $\boldsymbol{\gamma}$ and the new valuation functions.

Proof. The inequality follows by applying Lemma 6 at $(\boldsymbol{d}, \boldsymbol{\gamma})$ and choosing $\alpha_q = V_q'(d_q)$ for all q.

Observe that when valuations are linear the derivatives at the equilibrium $\boldsymbol{d}$ are the same as with the original valuations. Thus by the first-order conditions of Lemma 2, $\boldsymbol{d}$ remains an equilibrium. □

Assuming that valuations are linear, we now derive a similar bound for linear marginal cost functions. Our first lemma is again a technical result that is used in the proof of the second lemma.

Lemma 8. *Suppose consumers have linear valuations, and let $(\boldsymbol{d}, \boldsymbol{\gamma})$ be a vector of strategies such that*

1. *The vector $\boldsymbol{d}$ forms a demand-side equilibrium given pricing functions $\boldsymbol{\gamma}$.*
2. *The vector $\boldsymbol{\gamma}$ forms a supply-side equilibrium given elasticity functions ϵf_e that satisfy $|\epsilon f_e| \leq 1$.*

Let $(d_q^*)_{q\in Q}$ be a welfare-maximizing flow allocation and let $f_e = \sum_{q\in Q} d_{qe}$, $f_e^* = \sum_{q\in Q} d_{qe}^*$. Then $f_e \leq f_e^*$ for all e.

Proof. First observe that for any $q \in Q$, $e \in E$,

$$V'_q(d_q) \underbrace{\geq}_{\text{by (1)}} \gamma_e f_e (1 + \frac{d_{qe}}{f_e}) \underbrace{\geq}_{f, d_q \geq 0} \gamma_e f_e \underbrace{\geq}_{\text{by } |\epsilon f_r| \leq 1} c_e(f_e) \tag{14}$$

Recall that c_e denotes the marginal cost of the provider at edge e.

Now suppose for a contradiction that $f_e > f_e^*$ for some e. Then there exists a q such that $d_{qe} > d_{qe}^*$, and $d_q > 0$. Since we assumed the V_q were linear, $V'_q(d_q^*) \geq V'_q(d_q)$, and by the strict monotonicity of c_e, $c_e(f) > c_e(f^*)$. Combining this with (14), we obtain $V'_q(d_q^*) > c_e(f^*)$. But at optimum, we must have $V'_q(d_q^*) = c_e(f^*)$ for all e and all q such that $d'_q > 0$, and thus we arrive at a contradiction. □

Using Lemma 8, we obtain the following result, very similar to a result of Johari (2004).

Lemma 9. *Let* $(\boldsymbol{d}, \boldsymbol{\gamma})$ *be such that*

1. *The demands* $\boldsymbol{d}$ *form a demand-side equilibrium given* $\boldsymbol{\gamma}$.
2. *The prices* $\boldsymbol{\gamma}$ *form a supply-side equilibrium given some vector of elasticity functions* $\boldsymbol{\epsilon f}$.

Suppose that consumers have linear valuations with slopes $(\alpha_q)_{q \in Q}$. *Then there exist* $\beta_r > 0$ *for all* $r \in R$ *such that*

$$\frac{\sum_{q\in Q} \alpha_q d_q - \sum_{r\in R} C_r(f_r)}{\sum_{q\in Q} \alpha_q d_q^* - \sum_{r\in R} C_r(f_r^*)} \geq \frac{\sum_{q\in Q} \alpha_q d_q - \sum_{r\in R} \beta_r (f_r)^2}{\sum_{q\in Q} \alpha_q d_q^{**} - \sum_{r\in R} \beta_r (f_r^{**})^2} \tag{15}$$

where d_q^{**} *and* f_r^{**} *are welfare-maximizing allocations when the cost function of every provider* r *is* $C_r(f_r) = \frac{\beta_r}{2} f_r^2$.

Moreover, $\boldsymbol{\gamma}$ *is still a supply-side equilibrium given* $\boldsymbol{\epsilon f_e}$ *and the new cost functions.*

Proof. Given a provider r, let f_r be the equilibrium flow at their edge, and define $\beta_r = c_r(f_r)/f_r$. Recall that c_r denotes the marginal cost of provider r. We will establish the lemma for these β_r in two steps. First, we will define an intermediary cost function $\hat{C}_r(f)$ and show that replacing the C_r by the $\hat{C}_r$ can only reduce the left-hand side in (15). Then, we will show that going from $\hat{C}_r$ to $\frac{\beta_r}{2} f^2$ yields the lower bound in the right-hand side of (15), thus establishing the claim.

Let $\hat{C}_r(f)$ be the cost function uniquely determined by the marginal cost function

$$\hat{c}_r(f) = \begin{cases} c_r(f) & \text{if } f \leq f_r \\ \beta f & \text{if } f_r \leq f \end{cases}$$

Observe that $\boldsymbol{d}$ and $\boldsymbol{\gamma}$ are still respectively demand-side and supply-side equilibria once we replace the C_r by the $\hat{C}_r$. That is because $c_r(f_r) = \hat{c}_r(f_r)$ at the equilibrium flow f_r, and the necessary and sufficient conditions (2)-(3) still hold at that point.

Moreover, $C(f_r) = \hat{C}(f_r)$, and the social welfare at $(\boldsymbol{d}, \boldsymbol{\gamma})$ is unchanged. The optimal social welfare, on the other hand, can only improve, as

1. For $f_r \leq f$ $\hat{c}_r(f) = \beta f \leq c_r(f)$ by convexity of the marginal cost function
2. For $f \leq f_r$, $\hat{c}_r(f) = c_r(f)$

These two observations imply that $\hat{C}_r(f) \leq C_r(f)$ for all f, r. Thus replacing the C_r by the $\hat{C}_r$ can only reduce the left-hand side in (15). Formally, we have shown that

$$\frac{\sum_{q \in Q} \alpha_q d_q - \sum_{r \in R} C_r(f_r)}{\sum_{q \in Q} \alpha_q d_q^* - \sum_{r \in R} C_r(f_r^*)} \geq \frac{\sum_{q \in Q} \alpha_q d_q - \sum_{r \in R} \beta_r (f_r)^2}{\sum_{q \in Q} \alpha_q \hat{d}_q^* - \sum_{r \in R} \hat{C}_r(\hat{f}_r^*)^2}$$

where $\hat{d}_q^*$ and $\hat{f}_r^*$ denote optimal allocation when the cost functions are $\hat{C}_r$.

Now consider what happens when we pass from $\hat{C}_r$ to $\frac{\beta_r}{2} f^2$. Observe that the marginal cost at equilibrium is βf_r for both functions, and therefore $\boldsymbol{d}$ and $\boldsymbol{\gamma}$ remain equilibrium points. Let $\hat{f}^*$ be the flow at optimum when the cost functions are $\hat{C}_r$. Since valuations are linear, by Lemma 8, $f_r \leq \hat{f}^*$. Observe also that

1. When $f_r \leq f$, $\beta f = \hat{c}(f)$.
2. When $f \leq f_r$, $\beta f \geq \hat{c}(f)$ by convexity of the marginal cost function.

Thus $\frac{\beta}{2} f^2 \geq \hat{C}(f)$ for all f.

Let f_r^{**} be the flow at optimum at edge r when the cost functions are $\frac{\beta_r}{2} f^2$. Then $\frac{\beta_r}{2} (f_r^{**})^2 - \hat{C}(\hat{f}_r^*) = \frac{\beta}{2} (f_r)^2 - \hat{C}(f_r) > 0$. This value represents the area on the graph between the function $\hat{c}_r(f)$ and $\beta_r f$. Now, the price of anarchy can be bounded as follows:

$$\begin{aligned}
\frac{\sum_{q \in Q} \alpha_q d_q - \sum_{r \in R} \hat{C}_r(f_r)}{\sum_{q \in Q} \alpha_q \hat{d}_q^* - \sum_{r \in R} \hat{C}_r(\hat{f}_r^*)} &\geq \frac{\sum_{q \in Q} \alpha_q d_q - \sum_{r \in R} \hat{C}_r(f_r) - \sum_{r \in R} (\frac{\beta_r}{2} (f_r)^2 - \hat{C}_r(f_r))}{\sum_{q \in Q} \alpha_q \hat{d}_q^* - \sum_{r \in R} \hat{C}(\hat{f}_r^*) - \sum_{r \in R} (\frac{\beta_r}{2} (f_r^{**})^2 - \hat{C}_r(\hat{f}^*))} \\
&= \frac{\sum_{q \in Q} \alpha_q d_q - \sum_{r \in R} \frac{\beta_r}{2} (f_r)^2}{\sum_{q \in Q} \alpha_q \hat{d}_q^* - \sum_{r \in R} \frac{\beta_r}{2} (f^{**})^2} \\
&\geq \frac{\sum_{q \in Q} \alpha_q d_q - \sum_{r \in R} \frac{\beta_r}{2} (f_r)^2}{\sum_{q \in Q} \alpha_q d_q^{**} - \sum_{r \in R} \frac{\beta_r}{2} (f^{**})^2}
\end{aligned}$$

The latter ratio is the right-hand side of (15). This completes the proof. □

Later in this section, we bound the price of anarchy by finding the type of users (specifically, their valuations and costs) that experience the greatest welfare loss. The above lemma in a sense tells us that it is enough to search in the space of users that have quadratic costs.

6.3 Demand-side analysis

It turns out that the price of anarchy of the two-sided mechanism can be very cleanly separated into two components: one arising from demand-side inefficiency and the other from supply-side inefficiency.

Demand-side inefficiency can be bounded using techniques developed by Johari and Tsitsklis (2006). In fact, the following lemmas have proofs that are identical to those given by Johari and Tsitsklis. The second lemma establishes the price of anarchy in the two-sided market as function of only a single parameter $0 \leq \rho \leq 1$ that entirely captures the inefficiency on the supply-side of the market.

Throughout the paper, in order to prove inefficiency bounds for the two-sided mechanism, we will always first measure supply-side inefficiency by bounding ρ, and then apply the main lemma of this section to combine demand and supply-side inefficiencies into a single bound for the two-sided market.

Lemma 10. *The price of anarchy of the mechanism is lower-bounded by the solution to the following minimization problem:*

$$
\begin{aligned}
\min \quad & \frac{d_1 + \sum_{q=2}^{Q} \alpha_q d_q - B/2}{1/2B} \\
s.t. \quad & \alpha_q \leq \Gamma + \Gamma d_q \;\; \textit{if } d_q > 0,\ \forall q \in Q \\
& \alpha_q \geq \Gamma + \Gamma d_q,\ \forall q \in Q \\
& \beta_r \geq \gamma_r \left(2 - \frac{1}{\epsilon f_r}\right) \;\; \textit{if } f_r > 0,\ \forall r \in R \quad \beta_r \leq \gamma_r \left(2 - \frac{1}{\epsilon f_r}\right),\ \forall r \in R \\
& \sum_{q \in Q} d_q = 1 \\
& 0 < \alpha_q \leq 1,\ \forall q \in Q \\
& 0 \leq d_q, \Gamma, B
\end{aligned}
$$

When valuation functions are linear, this bound is tight.

Proof. Let V_q, $q \in Q$ denote valuation functions satisfying Assumption 1 and let C_r, $r \in R$ denote cost functions satisfying Assumption 2. Let $(\boldsymbol{d}, \boldsymbol{\gamma})$ be an equilibrium of the mechanism. Let $\boldsymbol{\epsilon f}$ be the vector of elasticity functions at every edge when valuation functions are V_q. The price of anarchy is the ratio

$$\frac{\sum_{q \in Q} V_q(d_q) - \sum_{r \in R} C_r(f_r)}{\sum_{q \in Q} V_q(d_q^*) - \sum_{r \in R} C_r(f_r^*)}$$

where the d_q form a demand-side equilibrium given pricing functions $\boldsymbol{\gamma}$, and the f_r are defined by the d_q and by the rule for distributing bandwidth that we defined in Lemma 3.

By Lemma 7, there exist a set of $\alpha_q > 0$ for $q \in Q$ such that the following bound holds

$$\frac{\sum_{q\in Q} V_q(d_q) - \sum_{r\in R} C_r(f_r)}{\sum_{q\in Q} V_q(d_q^*) - \sum_{r\in R} C_r(f_r^*)} \geq \frac{\sum_{q\in Q} \alpha_q d_q - \sum_{r\in R} C_r(f_r)}{\max_{\bar{f}_r} \left((\max_{q\in Q} \alpha_q) \sum_r \bar{f}_r - \sum_{r\in R} C_r(\bar{f}_r)\right)} \tag{16}$$

and $\boldsymbol{d}$ is still a demand-side equilibrium given $\boldsymbol{\gamma}$. The vector $\boldsymbol{\gamma}$ is still a supply-side equilibrium given the original elasticities $\boldsymbol{\epsilon f}$. Observe that these are not the elasticities that arise from linear valuations. Thus, with linearized valuations, $(\boldsymbol{d}, \boldsymbol{\gamma})$ is no longer an equilibrium of the two-sided mechanism.

Now by Lemma 9, there exist $\beta_r > 0$ for all $r \in R$ such that

$$\frac{\sum_{q\in Q} \alpha_q d_q - \sum_{r\in R} C_r(f_r)}{\max_{\bar{f}_r} \left((\max_{q\in Q} \alpha_q) \sum_r \bar{f}_r - \sum_{r\in R} C_r(\bar{f}_r)\right)} \geq \frac{\sum_{q\in Q} \alpha_q d_q - \sum_{r\in R} \beta_r (f_r)^2}{\max_{\bar{f}_r} \left((\max_{q\in Q} \alpha_q) \sum_r \bar{f}_r - \sum_{r\in R} \frac{\beta_r}{2} (\bar{f}_r)^2\right)}$$

and $\boldsymbol{\gamma}$ is still a supply-side equilibrium given $\boldsymbol{\epsilon f}$ when providers' cost functions are replaced by $C_r(f) = \frac{\beta_r}{2} f^2$.

Using Lemma 5, we can compute the optimal flow when users have linear valuations and linear marginal costs. By Lemma 5, it costs the providers $\frac{B}{2} f^2$ to send a flow of f. On the other hand, the highest-value consumer derives a utility of $\max_q \alpha_q f$. Thus, the optimal welfare is the maximum of

$$\max_q \alpha_q f - \frac{B}{2} f^2,$$

which is

$$\frac{\max_q \alpha_q^2}{2B}.$$

Therefore, by Lemma 7 and Lemma 9, to compute the price of anarchy, it is enough to find a set of linear valuations $V_q(d) = \alpha_q d$, quadratic costs $C_r(f) = \frac{\beta_r}{2} f^2$, and elasticities $\boldsymbol{\epsilon f}$ that result in the worst welfare loss at a point $(\boldsymbol{d}, \boldsymbol{\gamma})$ that satisfies:

1. The vector $\boldsymbol{d}$ is a demand-side equilibrium given $\boldsymbol{\gamma}$.
2. The vector $\boldsymbol{\gamma}$ is a supply-side equilibrium given $\boldsymbol{\epsilon f}$.

Formally, the price of anarchy is lower-bounded by the optimum of the following optimization problem:

$$\begin{aligned}
\min \quad & \frac{d_1 + \sum_{q=2}^{Q} \alpha_q d_q - B/2}{1/2B} \\
\text{s.t.} \quad & \alpha_q \le \Gamma + \Gamma d_q \text{ if } d_q > 0,\ \forall q \in Q \\
& \alpha_q \ge \Gamma + \Gamma d_q,\ \forall q \in Q \\
& \beta_r \ge \gamma_r \left(2 - \frac{1}{\epsilon f_r}\right) \text{ if } f_r > 0,\ \forall r \in R \quad \beta_r \le \gamma_r \left(2 - \frac{1}{\epsilon f_r}\right),\ \forall r \in R \\
& \sum_{q \in Q} d_q = 1 \\
& 0 < \alpha_q \le 1,\ \forall q \in Q \\
& 0 \le d_q, \Gamma, B
\end{aligned}$$

The objective function is precisely the bound of Lemma 9; it is minimized over all possible linear valuations with slopes α_q, all possible quadratic costs with parameters β_r, and all possible strategies $\boldsymbol{d}$, $\boldsymbol{\gamma}$ under the constraints we informally defined above.

Observe also that in the above program, we are assuming that the total flow f in the graph equals 1 and that

$$1 = \alpha_1 \ge \alpha_2 \ge \ldots \ge \alpha_Q > 0$$

This poses no loss of generality because we can always normalize the numerator and the denominator of the objective function so that these assumptions hold.
□

Lemma 11. *The price of anarchy of the two-sided mechanism for a single link is*

$$\frac{2\rho(2-\rho)}{4-\rho}$$

where $0 \le \rho \le 1$ *is an overcharging parameter that equals*

$$\frac{B}{\Gamma}.$$

When valuations are linear, this bound is tight.

Proof. By Lemma 10, the price of anarchy is lower-bounded by the solution to the following minimization problem.

$$\min \quad \frac{d_1 + \sum_{q=2}^{Q} \alpha_q d_q - B/2}{1/2B} \tag{17}$$

$$\text{s.t.} \quad \alpha_q \leq \Gamma + \Gamma d_q \text{ if } d_q > 0, \forall q \in Q \tag{18}$$

$$\alpha_q \geq \Gamma + \Gamma d_q, \forall q \in Q \tag{19}$$

$$\beta_r \geq \gamma_r \left(2 - \frac{1}{\epsilon f_r}\right) \text{ if } f_r > 0, \forall r \in R \tag{20}$$

$$\beta_r \leq \gamma_r \left(2 - \frac{1}{\epsilon f_r}\right), \forall r \in R \tag{21}$$

$$\sum_{q \in Q} d_q = 1 \tag{22}$$

$$0 < \alpha_q \leq 1, \forall q \in Q \tag{23}$$

$$0 \leq d_q, \Gamma, B \tag{24}$$

Without loss of generality, we can assume that (19)-(22) hold with equality. Indeed, if $d_q = 0$ for some q, we can reduce α_q until equality holds without affecting the price of anarchy. If $\Gamma = 0$, then $\alpha_q = 0$ for all q, and the theorem holds trivially. Thus, we can consider the following program:

$$\min \quad \frac{d_1 + \sum_{q=2}^{Q} \alpha_q d_q - B/2}{1/2B} \tag{25}$$

$$\text{s.t.} \quad 1 = \Gamma + \Gamma d_1, \tag{26}$$

$$\alpha_q = \Gamma + \Gamma d_q, \text{ for} q = 2, ..., Q \tag{27}$$

$$\beta_r = \gamma_r \left(2 - \frac{1}{\epsilon f_r}\right), \forall r \in R \tag{28}$$

$$\sum_{q \in Q} d_q = 1 \tag{29}$$

$$0 < \alpha_q \leq 1, \forall q \in Q \tag{30}$$

$$0 \leq d_q, \Gamma, B \tag{31}$$

Plugging in (27), (28) and (29) into (26) we obtain

$$\min \quad \frac{\frac{1-\Gamma}{\Gamma} + \sum_{q=2}^{Q}(\Gamma + \Gamma d_q)d_q - B/2}{1/2B} \tag{32}$$

$$\text{s.t.} \quad \sum_{q=2}^{Q} d_q = 1 - \frac{1-\Gamma}{\Gamma} \tag{33}$$

$$\Gamma + \Gamma d_q \leq 1, \forall q \in Q \tag{34}$$

$$\beta_r = \gamma_r \left(2 - \frac{1}{\epsilon f_r}\right), \forall r \in R \tag{35}$$

$$0 < \Gamma \leq 1 \tag{36}$$

$$0 \leq d_q \tag{37}$$

Observe that (33) is symmetric and convex in the variables d_q. Since a convex function admits a unique minimum, all d_q must be equal at that point. Thus, we must have $d_q = \frac{1-(1-\Gamma)/\Gamma}{Q-1}$. Observe that that solution is feasible if and only if

$$\frac{1}{Q} \leq \frac{1-\Gamma}{\Gamma} \leq 1$$

The optimization problem becomes:

$$\begin{aligned}
\min \quad & \frac{\frac{1-\Gamma}{\Gamma} + (\Gamma + \Gamma\frac{1-(1-\Gamma)/\Gamma}{Q-1})(1-(1-\Gamma)/\Gamma) - B/2}{1/2B} \\
\text{s.t.} \quad & \frac{1}{Q} \leq \frac{1-\Gamma}{\Gamma} \leq 1 \\
& \beta_r = \gamma_r \left(2 - \frac{1}{\epsilon f_r}\right), \forall r \in R \\
& 0 < \Gamma \leq 1
\end{aligned}$$

Observe that the function is decreasing in Q. Thus the minimum occurs when we let $Q \to \infty$. This yields the program

$$\min \quad \frac{\frac{1-\Gamma}{\Gamma} + \Gamma(1-(1-\Gamma)/\Gamma) - B/2}{1/2B} \tag{38}$$

$$\text{s.t.} \quad 0 \leq \frac{1-\Gamma}{\Gamma} \leq 1 \tag{39}$$

$$\beta_r = \gamma_r \left(2 - \frac{1}{\epsilon f_r}\right), \forall r \in R \tag{40}$$

To get rid of B, we introduce the ratio $\rho = B/\Gamma$.

$$\min \quad \frac{\frac{1-\Gamma}{\Gamma} + \Gamma(1-(1-\Gamma)/\Gamma) - \rho\Gamma/2}{1/2\rho\Gamma} \tag{41}$$

$$\text{s.t.} \quad 0 \leq \frac{1-\Gamma}{\Gamma} \leq 1 \tag{42}$$

$$\rho = B/\Gamma \tag{43}$$

$$\beta_r = \gamma_r \left(2 - \frac{1}{\epsilon f_r}\right), \forall r \in R \tag{44}$$

This program finally evaluates to

$$\min \quad \frac{2\rho(2-\rho)}{4-\rho} \tag{45}$$

$$\text{s.t.} \quad \rho = B/\Gamma \tag{46}$$

$$\beta_r = \gamma_r \left(2 - \frac{1}{\epsilon f_r}\right), \forall r \in R \tag{47}$$

□

6.4 Reducing the number of users

To obtain a bound on ρ we need to analyze the elasticity of demand faced by each provider. Our first step will be to show that we can assume there is only one user. We will establish our result for monomial valuation functions.

Formally, suppose that the Q consumers have valuation functions $V_q(d_q) = \alpha_q d_q^x$, where $0 < x \leq 1$ and $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_Q > 0$. Let $f = \sum_{q \in Q} d_q$. Let Γ be the aggregate price for the link. If there was only one consumer, the flow would be the maximizer of that conumer's utility function:

$$U(f) = \alpha f^x - \Gamma f^2.$$

We can check that

$$f = \left(\frac{x\alpha}{2\Gamma}\right)^{1/(2-x)}$$

maximizes $U()$. Consequently, the elasticity of demand with respect to Γ equals

$$\epsilon_\Gamma f = -\frac{1}{2-x}.$$

The goal of this section is to show that $\epsilon_\Gamma f = -1/(2-x)$ no matter how many consumers there are. To establish that, we will use the following technical lemma:

Lemma 12 (Matrix determinant). *Suppose A is an invertible $n \times n$ matrix, and U, V are $n \times m$ matrices. Then*

$$\det(A + UV^T) = \det(I + V^T A^{-1} U) \det(A)$$

Lemma 13. *Suppose there are Q consumers with valuation functions $V_q(d_q) = \alpha_q d_q^x$, where $0 < x \leq 1$ and $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_Q > 0$. Let $f = \sum_{q \in Q} d_q$. Let Γ be the aggregate price for the graph. Then*

$$\epsilon_\Gamma f = -\frac{1}{2-x}$$

Proof. At equilibrium, for any q,

$$\alpha_q x d_q^{x-1} = \Gamma(f + d_q)$$

Taking the elasticity on both sides (and using the identities in Lemma 1), we obtain

$$(x-1)\epsilon_\Gamma d_q = 1 + \sum_{r \neq q} \frac{d_r}{\sum_{r \neq q} d_r + 2d_q} \epsilon_\Gamma d_r + \frac{2d_q}{\sum_{r \neq q} d_r + 2d_q} \epsilon_\Gamma d_q$$

The function $\epsilon_\Gamma d_q$ must satisfy the above equation. It is easy to check that $\epsilon_\Gamma d_q = -1/(2-x)$ for all q (the elasticity in the single-consumer case) is a solution to this linear system of equations. It remains to verify that there are no other solutions to the system.

Observe that we can rewrite this system using matrices as

$$-1 = ((1-x)I + ABC)\epsilon$$

where ϵ is a vector of variables, $A = diag(1/(\sum_{r \neq 1} d_r + 2d_1), ..., 1/(\sum_{r \neq Q} d_r + 2d_Q))$, $C = diag(d_1, ..., d_Q)$, and B is a $Q \times Q$ matrix having the following structure

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Observe that we can rewrite the determinant of $(1-x)I + ABC$ as $(1-x)^Q \det(I + \frac{1}{1-x}ABC)$. To evaluate this determinant, we will use the matrix determinant lemma.

Applying the lemma to our determinant, we obtain

$$(1-x)^Q \det(I + \frac{1}{1-x}ABC) = (1-x)^Q \det(B^{-1} + \frac{1}{1-x}AC)\det(B^{-1})$$

It can be verified that the eigenvalues of B are $n+1, 1$, where n is the dimension of B. Thus B and B^{-1} are positive definite. Clearly, AC is also positive definite. Thus $B^{-1} + \frac{1}{1-x}AC$ is positive definite, and hence invertible. It follows that the determinant of $(1-x)I + ABC$ is non-zero and that the solution is unique.

From Lemma 1, we obtain

$$\epsilon_\Gamma f = \epsilon_\Gamma \sum_q d_q = \sum_q \frac{d_q}{\sum_q d_q} \epsilon_\Gamma d_q$$

and since we have just established that $\epsilon_\Gamma d_q = -1/(2-x)$, it follows that $\epsilon_\Gamma f = -1/(2-x)$. □

Thus the elasticity of the flow in the multi-consumer case is identical to the single-consumer case.

6.5 Computing ρ

By Lemma 13, we can assume there is only one consumer.

Suppose first there is one edge e. In that case, $\Gamma = \gamma_e$ and

$$\epsilon_{\gamma_e} f = \epsilon_\Gamma f \epsilon_{\gamma_e} \Gamma = -\frac{1}{2-x}.$$

By substituting into Lemma 11, we immediately obtain the following results.

Theorem 1. *Suppose there is a single link e. Suppose users have linear valuation functions. The price of anarchy is $2/3$. This bound is tight.*

Theorem 2. *Suppose there is a single link e. Suppose users have monomial valuation functions. The price of anarchy is bounded by.*

$$\frac{2/(2-x)(2-1/(2-x))}{4-1/(2-x)} = \frac{6-4x}{4(2-x)^2-2+x}.$$

Theorem 3. *As $x \to 0$, the price of anarchy goes to zero. That is, the lower bound of the previous theorem is also tight in the limit.*

Proof (Sketch). Suppose there is only one user with a valuation function $V(f) = f^x$ and one provider with a cost of f^2. As $x \to 0$, the price γ goes to infinity, and the equilibrium welfare goes to zero. □

Therefore, the two-sided mechanism can have arbitrarily bad efficiency if the users' valuation functions have sufficient curvature. This efficiency loss is similar to the one observed in the demand-side Cournot market. The notable difference is that in the demand market, efficiency degraded with the curvature of the *cost* function, while in a two-sided market, in degrades with the *valuation* function. However, for any well-behaved valuation function of bounded elasticity, the price of anarchy is constant.

This behavior resembles what we observe in real-world markets. The markup a provider can afford to set on their product depends on the elasticity of demand. If demand is highly inelastic, such as demand for medicine, consumers absolutely desire the good, and are willing to pay any amount for it. On the other hand, if the goods in questions are luxuries — like diamonds, for example — demand will be elastic and providers will be forced to price close to their marginal costs.

At first, these results may appear rather disappointing. However, they are somewhat mitigated by the following theorems regarding competition among providers.

Theorem 4. *Suppose users have valuation functions $\alpha_q d_q^x$. Let E be the number of edges. Assume that $\beta_e/\beta_{e'} \leq \Delta$ for some $\Delta > 0$ and for all e, e'. Then as $E \to \infty$, the price of anarchy goes to $2/3$. This bound is tight.*

Proof. Suppose that there are E parallel edges on the graph. Let f_e denote the flow across edge e. There is again only one user with utility

$$U(\boldsymbol{f}) = (\sum_e f_e)^x - \sum_e \gamma_e f_e^2$$

At equilibrium, the first order conditions on $U()$ must hold, as well as $\Gamma f_e = \Gamma f_{e'}$ for all e, e'. From that, we can derive an expression for the flow on edge e:

$$f_e = \left(\frac{x(1 + \sum_{e' \neq e} \gamma_e/\gamma_{e'})^{x-1}}{2\gamma_e} \right)^{1/(2-x)}$$

Using Lemma 1 and some algebra, we can show that

$$\begin{aligned}
\epsilon_{\gamma_i} f_i &= \frac{1}{2-x} \epsilon_{\gamma_e} \frac{(1 + \sum_{e \neq e'} \gamma_e/\gamma_{e'})^{(x-1)}}{2\gamma_e} \\
&= \frac{1}{2-x} \left(-(x-1)\epsilon_{\gamma_e} \frac{1/\gamma_e}{\sum_{e'} 1/\gamma_{e'}} - 1 \right) \\
&= \frac{1}{2-x} \left((x-1) \frac{\gamma_e}{\sum_{e'} \gamma_{e'}} - 1 \right) \\
&= -\frac{1}{2-x} \left(\frac{\gamma_e}{\sum_{e'} \gamma_{e'}} + (2-x) \frac{\sum_{e' \neq e} \gamma_e}{\sum_{e'} \Gamma_{e'}} \right)
\end{aligned} \tag{48}$$

Thus the elasticity is always higher (in absolute value) when there are more parallel routes. Thus we can expect the single route result to lower-bound the price of anarchy.

Observe now that from the first order condition and from formula (72), it follows that

$$2\gamma_e \geq \beta_e \geq x\gamma_e$$

So for all e, we have

$$\frac{\gamma_e}{\sum_{e'} \gamma_{e'}} \leq \frac{\beta_e/2}{\sum_{e'} \beta_{e'}/x} \to 0$$

as $E \to \infty$ because of our assumption that $\beta_e/\beta_{e'} \leq \Delta$ for all e, e'.

So for all e, $\gamma_e \to \beta_e$ as $E \to \infty$, and $\rho \to 1$. By Lemma 11, it follows that in the limit as $E \to \infty$, the price of anarchy tends to 2/3. □

We also obtain the following corollary.

Corollary 5 *In a monopsony, that is when there is only one user and an infinite number of providers, the price of anarchy equals 3/4.*

Proof. By Lemma 7, the worst price of anarchy occurs when the user's valuation is linear. As $E \to \infty$, $\Gamma \to B$. In a monopsony, there is no demand-side competition, and the socially optimal allocation is the maximizer of

$$f - \frac{B}{2} f^2,$$

which yields an optimum welfare of $f = 1/2B = 1/2\Gamma$. The equilibrium welfare, on the other hand, will equal the maximizer of

$$f - \Gamma f^2,$$

which yields a welfare of $f = 3/8\Gamma$.

The ratio of these two welfares is 3/4. □

This guarantee is slightly better than that of the two-sided proportional allocation mechanism studied by Kuleshov and Vetta (2010). The best possible guarantee that could be achieved in that setting was 0.71 (in a monopsony as well). On the other hand, the worst price of anarchy guarantee of that mechanism was about 0.58. Their analysis is not entirely comparable with ours however. They do not model the mechanism as a two-stage game; providers, therefore, do not anticipate how their pricing changes will affect this price and do not overcharge when demand is highly inelastic. We believe our approach better models reality.

It is also interesting to note that to obtain these guarantees, we need to assume a minimum level of competitiveness for every provider. This assumption is formalized through the requirement that $\beta_e/\beta_{e'} \leq \Delta$ for some $\Delta > 0$ and for all e, e'. This requirement is very natural: if a provider's cost is much higher than those of other provders, he cannot effectively compete. The high-cost providers' impact on the market is very small, as they cannot afford to price their product competitively.

7 Demand-Side Analysis of General Graphs

We now extend our results for networks of parallel edges to arbitrary networks. We have formally defined how the mechanism operates in this setting in Section 4.

We analyze the mechanism's performance in the same way as we did for a single link. We first analyze the demand-side of the market and prove a lemma that establishes a bound on the two-sided price of anarchy as a function of an over-charging parameter ρ that summarizes supply-side inefficiency. Specifically, we show that the price of anarchy in a graph is lower-bounded by the price of anarchy at the worst link in that graph.

We then analyze the supply-side of the market; specifically, we compute ρ as a function of the structure of the graph.

We pay particular attention to *parallel-serial* graphs, in which all users share the same source and sink, which are joined by edges connected either in series or in parallel. One reason for considering such graphs is that they are used to model combinatorial market structure, particularly horizontal and vertical competition among firms (see Correa et al., 2010). Another reason is that our supply-side analysis of general graphs is based on the analysis of parallel-serial graphs.

7.1 A simpler game

Like in the single-link setting, we first show that the game in which users choose a flow on every path is equivalent to a game in which they choose a flow on every edge. Although this does not simplify the game computationally, the game with bids on every edge is easier to analyze mathematically.

Formally, recall that in the game where users submit rates on paths, we defined the strategy of user q as a vector $\boldsymbol{d}_q = (d_{qp}, p \in P_q)$. The utility of user q was

$$U_q(\boldsymbol{d}_q, \boldsymbol{D}_{-q}) = V_q(\sum_{p \in P_q} d_{qp}) - \sum_{p \in P_q} d_{qp} \sum_{e \in p} \gamma_e f_e$$

In the game where users submit rates on edges, we define the strategy of user q as a vector $\boldsymbol{x}_q = (x_{qe}, e \in E)$. After having bought a vector of edge capacities $\boldsymbol{x}_q$, user q sends a max (s_q, t_q)-flow of size $d_q(\boldsymbol{x}_q)$ in the $\boldsymbol{x}_q$-capacitated graph G. The resulting flow on path p is denoted $d_{qp}(\boldsymbol{x_q})$. The utility of user q in the new game is therefore

$$\bar{U}_q(\boldsymbol{x}_q, \boldsymbol{x}_{-q}) = V_q(d_q(\boldsymbol{x}_q)) - \sum_{e \in E} x_{qe} \gamma_e \sum_{q' \in Q} x_{q'e}$$

Lemma 14 (Johari, 2004). *At equilibrium, the pricing functions and edge flows in the two above games are identical and the utilities of every player are the same.*

Proof. Let the pricing functions at every edge be fixed.

By Theorem 3.23 in Johari (2004), if $\boldsymbol{D}$ is a demand-side equilibrium in the game where users submit rates on paths, then the vector of edge rates $x_{qe} = \sum_{p \in P_q} \sum_{e \in p} d_{qp}$ forms a demand-side equilibrium in the game where users submit rates on edges.

Conversely, if $\boldsymbol{x}$ is a demand-side equilibrium of the bid-per-edge game, then the $d_{qp}(x_r)$ form a demand-side equilibrium of the bid-per-path game.

Since we fixed pricing functions at every edge at arbitrary values, the above holds for any provider strategy, and in particular the demand function at every edge remains the same. Thus the demand elasticity faced by any provider remains the same, and the supply-side equilibria of both games are identical.

Therefore, the two-sided equilibria are equivalent, and the utilities of every player are the same because of how we defined them. □

Given that result, we can focus on bounding demand-side efficiency loss in the game with bids at every edge.

7.2 Bounding the graph price of anarchy by the price of anarchy at an edge

The next lemma, originally established in Johari (2004), states that we can bound the demand-side price of anarchy by that of a single-edge game.

Lemma 15. *Let $(\boldsymbol{d}, \boldsymbol{\gamma})$ be a vector of strategies such that*

1. *The vector $\boldsymbol{d}$ forms a demand-side equilibrium given pricing functions $\boldsymbol{\gamma}$ in the game with bids at every edge.*
2. *The vector $\boldsymbol{\gamma}$ forms a supply-side equilibrium given elasticity functions $\boldsymbol{\epsilon f}$.*

Let $\boldsymbol{d}^*$ and $\boldsymbol{f}^*$ denote welfare-maximizing allocations. Then there exist $\alpha_{qe} > 0$ for all $q \in Q$, $e \in E$ such that

$$\frac{\sum_{q\in Q} V_q(d_q) - \sum_{r\in R} C_r(f_r)}{\sum_{q\in Q} V_q(d_q^*) - \sum_{r\in R} C_r(f_r^*)} \geq \min_{e\in E} \frac{\sum_{q\in Q} \alpha_{qe} d_{qe} - C_e(f_e)}{\max_{\bar{f}}(\max_{q\in Q} \alpha_{qe})\bar{f} - C_e(\bar{f})}. \tag{49}$$

At every edge e, the $(d_{qe})_{q\in Q}$ form a demand-side Nash equilibrium of the single-edge Cournot game in which the pricing function is $\gamma_e f_e$ and in which consumers have linear valuations with slopes $(\alpha_{qe})_{q\in Q}$.

Proof (Sketch). Let the suppliers strategies γ_e be fixed. For simplicity, we will denote the pricing functions by $p_e(\boldsymbol{x}_q, \boldsymbol{x}_{-q})$.

Let $\boldsymbol{x}$ be a Nash equilibrium of the per-link bid game. By definition, for all q, $\boldsymbol{x}_q$ maximizes $\bar{U}_q$:

$$\boldsymbol{x}_q \in \arg\max_{\bar{\boldsymbol{x}}} \left(V_q(d_q(\bar{\boldsymbol{x}})) - \sum_{e\in E} \bar{x}_e p_e(\bar{x}_e, \boldsymbol{x}_{-q}) \right)$$

As usual, $d_q(\bar{\boldsymbol{x}})$ is the size of the maximum (s_q, t_q)-flow in G when edge capacities equal $\bar{\boldsymbol{x}}$.

The function $-\bar{U}_q$ is proper and convex; therefore the subdifferential $\partial(-\bar{U}_q)$ is non-empty at $\boldsymbol{x}_q$. In particular, since $\boldsymbol{x}_q$ maximizes $\bar{U}_q$, $\mathbf{0}$ is a subgradient of $-\bar{U}_q$.

It can be established using a theorem in convex analysis (specifically, Theorem 23.8 in Rockafellar, 1970) that

$$\partial(-\bar{U}_q(\bar{x})) = \partial(-(V_q(d_q(\bar{x})) + \partial(\sum_{e\in E} \bar{x}_e p_e(\bar{x}_e, \boldsymbol{x}_{-q}))$$

where + denotes summation of sets: $A + B = \{a + b | a \in A, b \in B\}$.

Thus there exist $\boldsymbol{\alpha}_q \in -\partial(-(V_q(d_q(\bar{x})))$ and $\boldsymbol{\beta}_q \in -\partial(\sum_{e\in E} \bar{x}_e p_e(\bar{x}_e, \boldsymbol{x}_{-q}))$ such that $\boldsymbol{\alpha}_q = -\boldsymbol{\beta}_q$. Since V_q is non-decreasing in $\bar{\boldsymbol{x}}$, $\boldsymbol{\alpha}_q \geq 0$.

But then by the same theorem, $\mathbf{0}$ is also going to be a subgradient of $\boldsymbol{\alpha}_q^\top \bar{\boldsymbol{x}} - \sum_{e\in E} \bar{x}_e p_e(\bar{x}_e, \boldsymbol{x}_{-q})$ at the Nash equilibrium $\boldsymbol{x}_q$. In particular, we will have:

$$\boldsymbol{x}_q \in \arg\max_{\bar{\boldsymbol{x}}} \left(\boldsymbol{\alpha}_q^\top \bar{\boldsymbol{x}} - \sum_{e\in E} \bar{x}_e p_e(\bar{x}_e, \boldsymbol{x}_{-q}) \right)$$

which implies that for all e:

$$\boldsymbol{x}_{qe} \in \arg\max_{\bar{x}} (\alpha_e \bar{x} - \bar{x}_e p_e(\bar{x}_e, \boldsymbol{x}_{-q}))$$

Thus the bids received at edge e form a demand-side Nash equilibrium of a single-edge game at e in which consumers have linear valuations with slopes α_{qe}.

To prove the theorem, we need to lower-bound the price of anarchy of the network game by that of the worst edge. Let $\boldsymbol{x}^*$ be the set of per-link bids that maximizes social welfare in the network game. Observe that by the definition of a subderivative, we have

$$V_q(\boldsymbol{x}_q^*) \leq V_q(\boldsymbol{x}_q) + \boldsymbol{\alpha}_q^\top(\boldsymbol{x}_q^* - \boldsymbol{x}_q).$$

Also, it is clear that

$$\sum_{q\in Q} \boldsymbol{\alpha}_q^\top \boldsymbol{x}_q^* - \sum_{e\in E} C_e(f_e^*) = \sum_{e\in E}\left(\sum_{q\in Q} \alpha_{qe} x_{qe}^* - C_e(f_e^*)\right) \leq \sum_{e\in E} \max_{\bar{f}} \left(\max_q \alpha_{qe}\bar{f} - C_e(\bar{f})\right).$$

Finally, by the definition of the subderivative and the fact that $V_q(d_q(\mathbf{0}) \geq 0$, it follows that

$$V_q(\boldsymbol{x}_q) - \boldsymbol{\alpha}_q^\top \boldsymbol{x}_q \geq 0.$$

Applying the first, the second, and then the third inequality, we obtain

$$\begin{aligned}
\frac{\sum_{q\in Q} V_q(\boldsymbol{x}_q) - \sum_{e\in E} C_e(f_e)}{\sum_{q\in Q} V_q(\boldsymbol{x}_q^*) - \sum_{e\in E} C_e(f_e^*)} &\geq \frac{\sum_{q\in Q}\left(V_q(\boldsymbol{x}_q) + \boldsymbol{\alpha}_q^\top \boldsymbol{x}_q - \boldsymbol{\alpha}_q^\top \boldsymbol{x}_q\right) - \sum_{e\in E} C_e(f_e)}{\sum_{q\in Q}\left(V_q(\boldsymbol{x}_q) + \boldsymbol{\alpha}_q^\top(\boldsymbol{x}_q^* - \boldsymbol{x}_q)\right) - \sum_{e\in E} C_e(f_e^*)} \\
&= \frac{\sum_{q\in Q}\left(V_q(\boldsymbol{x}_q) - \boldsymbol{\alpha}_q^\top \boldsymbol{x}_q\right) + \sum_{q\in Q} \boldsymbol{\alpha}_q^\top \boldsymbol{x}_q - \sum_{e\in E} C_e(f_e)}{\sum_{q\in Q}\left(V_q(\boldsymbol{x}_q) - \boldsymbol{\alpha}_q^\top \boldsymbol{x}_q\right) + \sum_{q\in Q} \boldsymbol{\alpha}_q^\top \boldsymbol{x}_q^* - \sum_{e\in E} C_e(f_e^*)} \\
&\geq \frac{\sum_{q\in Q}\left(V_q(\boldsymbol{x}_q) - \boldsymbol{\alpha}_q^\top \boldsymbol{x}_q\right) + \sum_{e\in E}\left(\sum_{q\in Q} \alpha_{qe} x_{qe} - C_e(f_e)\right)}{\sum_{q\in Q}\left(V_q(\boldsymbol{x}_q) - \boldsymbol{\alpha}_q^\top \boldsymbol{x}_q\right) + \sum_{e\in E} \max_{\bar{f}}\left(\max_q \alpha_{qe}\bar{f} - C_e(\bar{f})\right)} \\
&\geq \frac{\sum_{e\in E}\left(\sum_{q\in Q} \alpha_{qe} x_{qe} - C_e(f_e)\right)}{\sum_{e\in E} \max_{\bar{f}}\left(\max_q \alpha_{qe}\bar{f} - C_e(\bar{f})\right)} \\
&\geq \min_{e\in E} \frac{\sum_{q\in Q} \alpha_{qe} x_{qe} - C_e(f_e)}{\max_{\bar{f}} \max_q \alpha_{qe}\bar{f} - C_e(\bar{f})}
\end{aligned}$$

which is exactly equation (49).

Finally, recall that at the beginning we fixed the providers' strategies. Thus for any set of pricing functions, the demand at a Nash equilibrium will be the same as in the original mechanism. □

7.3 Worst-case utilties

By Lemma 15, we can assume that like in the single-link setting, valuations are linear in the worst case. We now show that the worst efficiency also arises when marginal costs are linear.

Lemma 16. *Let* $(\boldsymbol{d}, \boldsymbol{\gamma})$ *be such that*

1. *The bids* $\boldsymbol{d}$ *form a demand-side equilibrium given* $\boldsymbol{\gamma}$ *in the game where bids are sent to every edge.*
2. *The prices* $\boldsymbol{\gamma}$ *form a supply-side equilibrium given some vector of elasticity functions* $\boldsymbol{\epsilon f}$.

Suppose that consumers have linear valuations with slopes $(\alpha_q)_{q\in Q}$. *Then there exist* $\beta_e > 0$ *for all* $e \in E$ *such that*

$$\min_{e\in E} \frac{\sum_{q\in Q} \alpha_{qe} d_{qe} - C_e(f_e)}{\max_{\bar{f}}(\max_{q\in Q} \alpha_{qe})\bar{f} - C_e(\bar{f})} \geq \min_{e\in E} \frac{\sum_{q\in Q} \alpha_{qe} d_{qe} - \frac{\beta_e}{2}(f_e)^2}{\max_{\bar{f}}(\max_{q\in Q} \alpha_{qe})\bar{f} - \frac{\beta_e}{2}(\bar{f})^2}.$$

Moreover, $\boldsymbol{\gamma}$ *is still a supply-side equilibrium given* $\boldsymbol{\epsilon f_e}$ *and the new cost functions.*

Proof. Follows by applying Lemma 8 and Lemma 9 at every edge $e \in E$. □

7.4 Demand-side analysis

Finally, we prove a lemma analogous to Lemma 11 in the single-link setting. It expresses the price of anarchy of the two-sided mechanism as a function of an overcharging-parameter ρ that captures the supply-side inefficiency of the market.

Lemma 17. *The price of anarchy of the two-sided mechanism for a single link is*

$$\frac{2\rho(2-\rho)}{4-\rho}$$

where

$$\rho = \min_{e\in E} \frac{\beta_e}{\gamma_e}.$$

Proof. Follows by applying Lemma 11 and Lemma 10 at every edge $e \in E$ with $\Gamma = \gamma_e$ and $B = \beta_e$. □

Interestingly, this implies that the structure of the graph does not affect demand-side efficiency. However, as we will see in later sections, it affects supply-side efficiency.

8 Demand-Side Analysis of Parallel-Serial Graphs

We now consider a special class of graphs called *parallel-serial* graphs. Although the demand-side analysis of the previous section applies to these graphs as well, they exhibit an interesting structure that deserves its own separate analysis.

In particular, we show in this section that on the demand-side, parallel-serial graphs may be presented to consumers as a single edge with an aggregate pricing function, just like in the single-link case (Section 6). We then show that in the context of parallel-serial graphs, the overcharging coefficient ρ has an alternative formulation that is more intuitive.

We assume in this section that valuations and marginal costs are linear. Recall that in the previous section we established that these are the worst-case utilities.

In the literature, parallel-serial graphs were considered by Correa et al. (2007, 2010) in their equilibria analysis of the supply-side market, as well as by Acemoglu and Ozdaglar. They model horizontal and vertical competition among providers and cleanly represent combinatorial market structure.

8.1 Reducing a parallel-serial graph to a single link

Recall that in Section 6, a consumer that wanted to send a flow of f had a unique optimal way of distributing that flow across the edges. Thus we could define a single aggregate price Γ for sending flow across the graph.

In a parallel-serial graph, the same observation can be made. Essentially, we can define aggregate prices for each link, then add them up to obtain prices for routes, then aggregate any parallel routes as if they were edges, and so on until we get a single aggregate price for the graph.

Consumers can be presented with that single price. They only have to choose the size of the flow they wish to transmit, and the mechanism decides for them how to distribute that flow. The equilibria of the aggregate and the original mechanism turn out to be identical and achieve the same utility. Thus in our analysis, we will consider the simpler aggregate mechanism.

We will formally establish this result in the case for a graph consisting of T parallel routes. The full claim can be formally established using a straightforward but somewhat lengthy induction proof.

Let G be the parallel-serial graph consisting of a source and a target s, t connected by a set of parallel disjoint routes denoted T (by a slight abuse of notation, we use the same letter as for the number of routes). On any route $t \in T$, any two vertices may be connected by a number of parallel edges, with one provider on each edge.

We have seen in the previous section, that there is a unique way of distributing flow across a link, and that at every link, we can define a single aggregate price. Let Γ_{lt} denote the aggregate price of link l on route t. Since flow traversing a route is charged by every provider, the aggregate price for that route is $\Gamma_t = \sum_{l \in t} \Gamma_{lt}$.

Observe now that the parallel routes are structured like edges at a single-link. Thus the aggregate price for G equals

$$\Gamma = \frac{1}{\sum_{t \in T} 1/\Gamma_t}.$$

The same procedure can be repeated to parallel-serial graphs with a more complex structure to obtain aggregate prices for those graphs.

We formally establish this through the following lemma.

Lemma 18. *Let G be parallel-serial graph consisting of a source and a target s, t connected by a set of parallel disjoint routes T. Let Γ be the aggregate price of G. The Nash equilibria in the aggregate game with price Γ are the same as those of the original game. The equilibrium utilities of both games are also identical.*

Proof. The proof is very similar to that of Lemma 4. Let P denote the set of all paths in G. The necessary and sufficient conditions of Lemma 2 applied to G state that for all q and r:

$$V_q'(d_q) \geq \sum_{p'} \sum_{e \in p \cap p'} \gamma_e d_{p'} + \sum_{e \in p} \gamma_e f_e \text{ for all } p \in P \tag{50}$$

$$V_q'(d_q) \leq \sum_{p'} \sum_{e \in p \cap p'} \gamma_e d_{p'} + \sum_{e \in p} \gamma_e f_e \text{ for all } p \in P \text{ such that } d_p > 0 \tag{51}$$

$$C_r'(f_r) \leq \gamma_r f_r \left(2 - \frac{1}{\epsilon f_r}\right) \tag{52}$$

$$C_r'(f_r) \geq \gamma_r f_r \left(2 - \frac{1}{\epsilon f_r}\right) \text{ if } f > 0 \tag{53}$$

The efficiency conditions for the aggregate game are can be expressed as:

$$V_q'(d_q) \geq \Gamma(d_q + f) \text{ for all } q \tag{54}$$

$$V_q'(d_q) \leq \Gamma(d_q + f) \text{ for all } q \text{ s.t. } d_q > 0 \tag{55}$$

$$C_r'(f_r) \leq \gamma_r f_r \left(2 - \frac{1}{\epsilon f_r}\right) \tag{56}$$

$$C_r'(f_r) \geq \gamma_r f_r \left(2 - \frac{1}{\epsilon f_r}\right) \text{ if } f_r > 0 \tag{57}$$

Take any demand-side equilibrium $\boldsymbol{d}$ of the aggregate game. We have to show that conditions (51-50) hold for the path flows d_{qp} that the aggregate mechanism chooses for consumers. Observe that the d_{qp} are chosen so that the flow at an edge e on link l within path t equals

$$\sum_{p: e \in p} d_{qp} = \frac{1/\Gamma_t}{\sum_{t'} 1/\Gamma_{t'}} \frac{1/\gamma_{tle}}{\sum_{e'} 1/\gamma_{tle'}} d_q.$$

Condition (51) for any path $p \in P$ follows by simple algebra. Path p must coincide with some route $t \in T$ and at every link $l \in t$ take some edge $e_p \in l$.

We then derive condition (51) as follows.

$$\begin{aligned}
V_q'(d_q) \le \Gamma(d_q+f) &= \Gamma_t\left(\frac{1/\Gamma_t}{\sum_{t'} 1/\Gamma_{t'}} d_q + \frac{1/\Gamma_t}{\sum_{t'} 1/\Gamma_{t'}} f\right) = \Gamma_t(d_{qt}+f_t) \\
&= \sum_{l\in t} \Gamma_{lt}(d_{qt}+f_t) \\
&= \sum_{l\in t} \gamma_{e_p} \frac{1/\gamma_{e_p}}{\sum_{e'\in l} 1/\gamma_{tle'}}(d_{qt}+f_t) \\
&= \sum_{e\in p} \gamma_e \sum_{p'\in P: e\in p'} d_{p'} + \sum_{e\in p} \gamma_e f_e \\
&= \sum_{p'\in P} \sum_{e\in p'\cap p} \gamma_e d_{p'} + \sum_{e\in p} \gamma_e f_e
\end{aligned}$$

Condition (50) follows similarly.

This shows that for any $\boldsymbol{\gamma}$, the demand at each edge remains the same in the aggregate game. Thus the function $f_r(\boldsymbol{\gamma})$ is unchanged, and the equilibrium prices $\boldsymbol{\gamma}$ of the aggregate game also form a supply-side equilibrium in the original game. □

Applied inductively, the above proof can be used to establish the following general lemma.

Lemma 19. *Let G be a parallel-serial graph. There exists an aggregate price Γ for G such that when users choose their flows based on G, the Nash equilibria of the resulting game are identical to those of the original game. The equilibrium utilities of both games are also identical.*

8.2 An alternative formulation for the overcharging coefficient ρ

Because of Lemma 19, most of our technical lemmas for single-link graphs carry over to parallel-serial graphs.

In particular, Lemma 11 holds. Thus the two-sided price of anarchy equals

$$\frac{2\rho(2-\rho)}{4-\rho}$$

where

$$\rho = \frac{B}{\Gamma}.$$

Thus we can view ρ as the ratio of the true price of the graph over the price that the users are charged.

Lemma 13 also carries over directly, as the only thing it assumes is the existence of a single aggregate price for the entire graph. By that lemma, we can analyze the elasticity assuming that the providers face only a single user.

9 Supply-Side Analysis of Parallel-Serial Graphs

Although the demand-side price of anarchy in parallel-serial graphs is the same as for a single link, the overcharging coefficient ρ behaves quite differently. In particular, ρ varies dramatically with the structure of the graph.

In this section, we examine how exactly it varies. We show that serial competition (when providers are located on consecutive edges) encourages overcharging, while parallel competition reduces it. The theorems in this section describe ρ for graphs exhibiting different levels of each type of competition.

9.1 Bounding ρ when the graph is a single route

First, we will assume that G consists of a route composed of L links connected in series and we will refer to this G as a *route graph*. A consumer wishes to send flow from one end of the route to the other. Between any two vertices there can be any number of parallel edges, with one provider per edge.

Suppose first that consumers have linear valuation functions. The effects of the valuations' curvature will be investigated later in this section. The following lemma provides a closed-form expression for the elasticity of f_e with respect to γ_e.

Lemma 20. *Condition $\beta_r f_r = \gamma_r f_r(2 - 1/|\epsilon_r f_r|)$ can be written as*

$$\beta_{le} = \gamma_{le} - \frac{1}{1/\sum_{l' \neq l} \mathit{\Gamma}_{l'} + \sum_{e' \neq e} 1/\gamma_{le'}} \tag{58}$$

Proof. First note that

$$f_{le} = \frac{1/\gamma_{le}}{\sum_{e'} 1/\gamma_{le'}} \frac{1}{\sum_{l'} \mathit{\Gamma}_{l'}} = \frac{\prod_{e' \neq e} \gamma_{le'}}{\sum_{e'} \prod_{k \neq e'} \gamma_{lk}} \frac{1}{\sum_{l'} \mathit{\Gamma}_{l'}}. \tag{59}$$

The denominator in (59) can be expanded as

$$(\sum_{e'} \prod_{k \neq e'} \gamma_{lk})(\sum_{l' \neq l} \mathit{\Gamma}_{l'}) + \prod_{e'} \gamma_{le'}. \tag{60}$$

Using that information, we compute the elasticity.

$$
\begin{aligned}
\epsilon_{le} f_{le} &= \frac{\gamma_{le}}{f_{le}} \frac{\partial f_{le}}{\partial \gamma_{le}} \\
&= \left(\frac{\gamma_{le}}{\frac{\prod_{e' \neq e} \gamma_{le'}}{(\sum_{e'} \prod_{k \neq e'} \gamma_{lk})(\sum_{l' \neq l} \Gamma_{l'}) + \prod_{e'} \gamma_{le'}}} \right) \times \\
&\quad \left(- \frac{\prod_{e' \neq e} \gamma_{le'}}{((\sum_{e'} \prod_{k \neq e'} \gamma_{lk})(\sum_{l' \neq l} \Gamma_{l'}) + \prod_{e'} \gamma_{le'})^2} \right) \times \\
&\quad \left(\frac{\partial}{\partial \gamma_{le}} ((\sum_{e'} \prod_{k \neq e'} \gamma_{lk})(\sum_{l' \neq l} \Gamma_{l'}) + \prod_{e'} \gamma_{le'}) \right) \\
&= -\gamma_{le} \frac{\frac{\partial}{\partial \gamma_{le}} ((\sum_{e'} \prod_{k \neq e'} \gamma_{lk})(\sum_{l' \neq l} \Gamma_{l'}) + \prod_{e'} \gamma_{le'})}{(\sum_{e'} \prod_{k \neq e'} \gamma_{lk})(\sum_{l' \neq l} \Gamma_{l'}) + \prod_{e'} \gamma_{le'}} \\
&= -\gamma_{le} \frac{(\sum_{e' \neq e} \prod_{k \neq e', e} \gamma_{lk})(\sum_{l' \neq l} \Gamma_{l'}) + \prod_{e' \neq e} \gamma_{le'}}{(\sum_{e'} \prod_{k \neq e'} \gamma_{lk})(\sum_{l' \neq l} \Gamma_{l'}) + \prod_{e'} \gamma_{le'}} \\
&= - \frac{(\sum_{e' \neq e} \prod_{k \neq e'} \gamma_{lk})(\sum_{l' \neq l} \Gamma_{l'}) + \prod_{e'} \gamma_{le'}}{(\sum_{e'} \prod_{k \neq e'} \gamma_{lk})(\sum_{l' \neq l} \Gamma_{l'}) + \prod_{e'} \gamma_{le'}}
\end{aligned} \tag{61}
$$

It then follows that

$$
\begin{aligned}
\frac{1}{|\epsilon_{le} f_{le}|} &= \frac{(\sum_{e'} \prod_{k \neq e'} \gamma_{lk})(\sum_{l' \neq l} \Gamma_{l'}) + \prod_{e'} \gamma_{le'}}{(\sum_{e' \neq e} \prod_{k \neq e'} \gamma_{lk})(\sum_{l' \neq l} \Gamma_{l'}) + \prod_{e'} \gamma_{le'}} \\
&= \frac{(\sum_{e' \neq e} \prod_{k \neq e'} \gamma_{lk})(\sum_{l' \neq l} \Gamma_{l'}) + \prod_{e'} \gamma_{le'} + (\prod_{k \neq e} \gamma_{lk})(\sum_{l' \neq l} \Gamma_{l'})}{(\sum_{e' \neq e} \prod_{k \neq e'} \gamma_{lk})(\sum_{l' \neq l} \Gamma_{l'}) + \prod_{e'} \gamma_{le'}} \\
&= 1 + \frac{(\prod_{k \neq e} \gamma_{lk})(\sum_{l' \neq l} \Gamma_{l'})}{(\sum_{e' \neq e} \prod_{k \neq e'} \gamma_{lk})(\sum_{l' \neq l} \Gamma_{l'}) + \prod_{e'} \gamma_{le'}} \\
&= 1 + \frac{1}{\gamma_{le}} \frac{(\prod_{k} \gamma_{lk})(\sum_{l' \neq l} \Gamma_{l'})}{(\sum_{e' \neq e} \prod_{k \neq e'} \gamma_{lk})(\sum_{l' \neq l} \Gamma_{l'}) + \prod_{e'} \gamma_{le'}} \\
&= 1 + \frac{1}{\gamma_{le}} \frac{1}{1/\sum_{l' \neq l} \Gamma_{l'} + \sum_{e' \neq e} 1/\gamma_{le'}}
\end{aligned} \tag{62}
$$

Inserting (62) into $\beta_r f_r = \gamma_r f_r (2 - 1/|\epsilon_r f_r|)$, we obtain the desired result. □

Lemma 20 gives us a simpler form for the supply-side equilibrium constraints. We can now determine the smallest value ρ can attain under these constraints. We will do that using the approach we have already used to evaluate demand-side inefficiency. We will formulate a minimization problem in which ρ is the objective function, and minimize ρ over all possible values of γ_r and β_r that satisfy the necessary and sufficient equilibrium conditions (58).

Before doing that, first observe that there cannot exist a Nash equilibrium when at least two links have only one provider. Indeed, for any two such links l, l' we must have both

$$\beta_l = \gamma_l - \gamma_{l'} - \sum_{k \neq l,l'} \Gamma_k$$
$$\beta_{l'} = \gamma_{l'} - \gamma_l - \sum_{k \neq l,l'} \Gamma_k$$

and clearly the two equations cannot simultaneously hold for positive values of β.

We expect that economic efficiency will improve as the number of providers in the graph increases. Therefore we begin by considering the worst equilibrium situation that can arise: a route with two providers per link.

Theorem 6. *Let G consist of a single route of L links with two edges at each link. Suppose that $1/\Delta \leq \beta_{l1}/\beta_{l2} \leq \Delta$ for some $\Delta > 0$ and for all $l \in L$. For any fixed L the price of anarchy is constant and ρ can be easily computed.*

Proof. When there are two providers per link, ρ has the following form.

$$\rho = \frac{B}{\Gamma} = \frac{\displaystyle\sum_{l=1}^{L} \frac{1}{1/\beta_{l1} + 1\beta_{l2}}}{\displaystyle\sum_{l=1}^{L} \frac{1}{1/\gamma_{l1} + 1/\gamma_{l2}}} \tag{63}$$

Inserting (58) we into (63), we obtain

$$\rho = \frac{\sum_{l=1}^{L} \dfrac{1}{\dfrac{1}{\dfrac{1}{g_{l1}} - \dfrac{1}{g_{l2} + 1/\sum_{l' \neq l} \Gamma_{l'}}} + \dfrac{1}{\dfrac{1}{g_{l2}} - \dfrac{1}{g_{l1} + 1/\sum_{l' \neq l} \Gamma_{l'}}}}}{\sum_{l=1}^{L} \dfrac{1}{g_{l1} + g_{l2}}} \tag{64}$$

where $g_{le} = 1/\gamma_{le}$.

The worst possible efficiency ratio ρ corresponds to the minimum of (64) over all positive values of the g_{le}. Without loss of generality, we can restrict our domain to the set of all g_{le} such that $g_{l1} + g_{l2} \geq 1$. That is because we can take any point and normalize it to obtain a point inside the restricted domain with the original point's objective function value.

Observe now that at any $\boldsymbol{\gamma}$,

$$\frac{\sum_l B_l(\boldsymbol{\gamma})}{\sum_l \Gamma_l(\boldsymbol{\gamma})} \geq \min_l \frac{B_l(\boldsymbol{\gamma})}{\Gamma_l(\boldsymbol{\gamma})}.$$

Since the functions B_l and Γ_l are symmetric in the g_{le}, we can, by relabeling the variables if necessary, assume that $B_1/\Gamma_1 = \min_l B_l/\Gamma_l$ at the minimizer of ρ.

Thus it is enough to minimize the following.

$$\rho = \frac{\dfrac{1}{\dfrac{1}{\dfrac{1}{g_{11}} - \dfrac{1}{g_{12} + 1/\sum_{l' \neq l} \Gamma_{l'}}} + \dfrac{1}{\dfrac{1}{g_{12}} - \dfrac{1}{g_{11} + 1/\sum_{l' \neq l} \Gamma_{l'}}}}}{\dfrac{1}{g_{11} + g_{12}}} \tag{65}$$

Observe that (65) is increasing in $1/\sum_{l' \neq l} \Gamma_{l'}$. Thus at the minimum, we must have $g_{l1} + g_{l2} = 1$ for $l \neq 1$:

$$\rho = \frac{\dfrac{1}{\dfrac{1}{\dfrac{1}{g_{11}} - \dfrac{1}{g_{12} + 1/(L-1)}} + \dfrac{1}{\dfrac{1}{g_{12}} - \dfrac{1}{g_{11} + 1/(L-1)}}}}{\dfrac{1}{g_{11} + g_{12}}} \tag{66}$$

Observe that by the same argument involving symmetry and sums in the numerator and the denominator, (66) is lower-bounded by

$$g_{11} \left(\frac{1}{g_{11}} - \frac{1}{g_{12} + 1/(L-1)} \right)$$

which is the ratio β_e/γ_e at what can be assumed to be the most inelastic edge. Thus we have rederived Lemma 17 (our demand-side efficiency lemma for arbitrary graphs) for the special case of a route graph.

Now observe that as $\beta_{12} \to \infty$, (66) goes to zero. Therefore in the worst case our bound $\beta_{11}/\beta_{12} \leq \Delta$ must hold with equality:

$$\rho = \frac{\dfrac{1}{\dfrac{1 + \Delta}{\dfrac{1}{g_{11}} - \dfrac{1}{g_{12} + 1/(L-1)}}}}{\dfrac{1}{g_{11} + g_{12}}} \tag{67}$$

Equation (67) is an explicit construction of the worst case ρ we can find. The minimum can be evaluated numerically for specific values of Δ and L under the constraint that the γ and the β are positive. In every case it is strictly positive. □

Just like in the single-link case, we need to assume there is a minimum level of competition at each link. Indeed, if a provider's costs are too high, that provider cannot offer connectivity at a reasonable price and its competitor effectively has a monopoly on the service.

Unfortunately, Theorem 6 also yields several types of markets in which the mechanism achieves very low social welfare.

Corollary 7 *As the competitiveness Δ goes to infinity, the price of anarchy goes to zero.*

Proof. Observe that (67) is attained at any point for which $\Gamma_l = 1$ for all l. Then for any value of g_{l1}, g_{l2}, the ratio goes to zero as $\Delta \to \infty$. □

This behavior, however, is to be expected. Providers must have comparable costs in order to compete with each other. What is more disappointing is that the price of anarchy approaches zero arbitrarily close as the length of the route increases.

Corollary 8 *As $L \to \infty$, the price of anarchy goes to zero.*

Proof. Again consider any solution that attains (67). When $L \to \infty$, the bound becomes

$$\rho = \frac{\dfrac{1}{\dfrac{1}{\dfrac{1}{g_{11}} - \dfrac{1}{g_{12}}} + \dfrac{1}{\dfrac{1}{g_{12}} - \dfrac{1}{g_{11}}}}}{\dfrac{1}{g_{11} + g_{12}}}$$

and since $\beta_{1e} > 0$, the numerator must equal zero. □

Therefore, in the worst case, the two-sided mechanism can have an arbitrarily large inefficiency. Luckily, to avoid this large efficiency loss, we only need to slightly increase provider competition at every link.

Theorem 9. *Let Γ consist of L links of three edges each. The price of anarchy of G is bounded, even in the limit as $L \to \infty$.*

Proof. For three suppliers per link, the expression for ρ becomes

$$\frac{\sum_{l=1}^{L} \dfrac{1}{\dfrac{1}{\dfrac{1}{g_{l1}} - \dfrac{1}{g_{l2} + g_{l3} + 1/\sum_{k \neq l} \Gamma_k}} + \dfrac{1}{\dfrac{1}{g_{l2}} - \dfrac{1}{g_{l1} + g_{l3} + 1/\sum_{k \neq l} \Gamma_k}} + \dfrac{1}{\dfrac{1}{g_{l3}} - \dfrac{1}{g_{l1} + g_{l3} + 1/\sum_{k \neq l} \Gamma_k}}}}{\sum_{l=1}^{L} \dfrac{1}{g_{l1} + g_{l2} + g_{l3}}} \tag{68}$$

We can normalize the g_{le} as in the two provider case so that $g_{l1}+g_{l2}+g_{l3} \geq 1$ for all l. The by the same argument as earlier we can lower bound (68) for any

L by

$$\frac{1}{\dfrac{1}{\dfrac{1}{g_{l1}}-\dfrac{1}{g_{l2}+g_{l3}+1/\sum_{k\neq l}\Gamma_k}}+\dfrac{1}{\dfrac{1}{g_{l2}}-\dfrac{1}{g_{l1}+g_{l3}+1/\sum_{k\neq l}\Gamma_k}}+\dfrac{1}{\dfrac{1}{g_{l3}}-\dfrac{1}{g_{l1}+g_{l3}+1/\sum_{k\neq l}\Gamma_k}}} \Bigg/ \frac{1}{g_{l1}+g_{l2}+g_{l3}}$$

Again the above ratio is minimized when $g_{k1}+g_{k2}+g_{k3}$ goes to one for $k \neq l$. Thus we can further lower bound that ratio by

$$\frac{1}{\dfrac{1}{\dfrac{1}{g_{l1}}-\dfrac{1}{g_{l2}+g_{l3}+1/(L-1)}}+\dfrac{1}{\dfrac{1}{g_{l2}}-\dfrac{1}{g_{l1}+g_{l3}+1/(L-1)}}+\dfrac{1}{\dfrac{1}{g_{l3}}-\dfrac{1}{g_{l1}+g_{l3}+1/(L-1)}}} \Bigg/ \frac{1}{g_{l1}+g_{l2}+g_{l3}}$$

We take the limit as $n \to \infty$ and obtain

$$\frac{1}{\dfrac{1}{\dfrac{1}{g_{l1}}-\dfrac{1}{g_{l2}+g_{l3}}}+\dfrac{1}{\dfrac{1}{g_{l2}}-\dfrac{1}{g_{l1}+g_{l3}}}+\dfrac{1}{\dfrac{1}{g_{l3}}-\dfrac{1}{g_{l1}+g_{l3}}}} \Bigg/ \frac{1}{g_{l1}+g_{l2}+g_{l3}} \tag{69}$$

This ratio can be evaluated numerically for different values of Δ. For $\Delta = 1/2$, the minimum is about 0.45.

Observe also that if there are precisely three providers per link, the lower bound (69) is achieved asymptotically by setting all the g_{le}'s to the worst case values we computed. □

Thus we only need three providers per link to obtain a positive price of anarchy within a route on any size. In the two-sided proportional allocation mechanism, there had to be two providers per link for an equilibrium to exist, so the two guarantees are somewhat comparable.

However, one of the shortcomings that we observed in the previous setting also holds in this case.

Corollary 10 *As the competitiveness Δ goes to infinity, the price of anarchy goes to zero.*

Another source of inefficiency that we have not yet treated stems from the elasticity of demand.

Theorem 11. *As the degree x of $V(f) = f^x$ goes to zero, the price of anarchy goes to zero.*

Proof. (Sketch) Let e be an edge and let S_e denote the normalized fraction of the flow that passes through e. In other words, $f_e = S_e f$. Observe that

$$\epsilon_e f_e = \epsilon_e S_e + \epsilon_e f = \epsilon_e S_e + \epsilon_{\Gamma_i} f \epsilon_e \Gamma_i = \epsilon_e S_e + \frac{1}{2-x}\epsilon_{\Gamma_l} f_{\text{linear}} \epsilon_e \Gamma_l$$

where f_{linear} is the total flow when valuations are linear. In the single link setting on the other hand, we have

$$\epsilon_e f_e = \epsilon_e S_e + \epsilon_{\Gamma_i} f_{\text{linear}} \epsilon_e \Gamma_i$$

which is never worse than the first case.

Observe also that $\epsilon_e S_e$ is negative for a route graph. As x decreases, the elasticity, and hence the ratio β_e/γ_e will tend to zero for every edge. Consequently, ρ will also tend to zero. □

Together, these results indicate that bad behavior is possible within a route, if the route gets too large. Luckily, having enough providers in the market can completely eliminate any supply-side inefficiencies.

Theorem 12. *Suppose there are m providers on each connection. As m goes to infinity, ρ goes to one.*

Proof. Let m denote the number of providers. By an argument we used at the beginning of this section, we can normalize the g_{ij} so that $g_{ij} \geq 1$ for all i, j.

As in the proof of the previous claim, we can use the fractions trick to lower bound the price of anarchy by the ratio corresponding to link 1, which is

$$\frac{\dfrac{1}{\sum_{e=1}^{m} \dfrac{1}{\dfrac{1}{g_{1e}} - \dfrac{1}{\sum_{k\neq e} g_{1k} + 1/\sum_{k\neq 1}\Gamma_k}}}}{\dfrac{1}{\sum_{e=1}^{m} g_{1e}}} \tag{70}$$

For any m, we can relabel the variables so that β_{11} takes the smallest value. Thus we can further lower-bound (70) by:

$$\frac{\dfrac{1}{\dfrac{m\Delta}{\dfrac{1}{g_{11}} - \dfrac{1}{\sum_{k\neq e} g_{1k} + 1/\sum_{k\neq 1}\Gamma_k}}}}{\dfrac{1}{\sum_{e=1}^{m} g_{1e}}} \tag{71}$$

where $\Delta \geq 1$

Using the fraction trick again, ratio (71) can be bounded by

$$\frac{g_{11}}{\dfrac{\Delta}{\dfrac{1}{g_{11}} - \dfrac{1}{\sum_{k\neq 1} g_{1k} + 1/\sum_{k\neq 1}\Gamma_k}}}$$

As $m \to \infty$, the term

$$\frac{1}{\sum_{k \neq 1} g_{1k} + 1/\sum_{k \neq 1} \Gamma_k}$$

goes to zero (since $g_{1k} \geq 1$ by assumption), and the price of anarchy goes to one. □

9.2 Bounding ρ in general parallel-serial graphs

In the previous section we have seen that concave valuation functions can reduce the price of anarchy to zero. However, just like in the single-link case, adding more parallel routes in the network eliminates the effect of the valuations' curvature.

Theorem 13. *Let G be a parallel-serial graph and suppose that $\beta_e/\beta_{e'} \leq \Delta$ for some $\Delta > 0$ and all $e, e' \in E$. When the number of parallel routes of G goes to infinity, the elasticity at each edge tends to that obtained from linear valuations.*

Proof. The beginning of the proof follows closely that of Theorem 4. As before, by Lemma 13 we can assume that there is only one user. Let f_t denote flow sent across parallel route t in T. The user's utility function is

$$U() = (\sum_t f_t)^x - \sum_t \Gamma_t f_t^2$$

At equilibrium, the first order conditions on $U()$ must hold, as well as $\Gamma f_t = \Gamma f_{t'}$ for all t, t'. From that, we can derive an expression for the flow on route t:

$$f_t = \left(\frac{x(1 + \sum_{t' \neq t} \Gamma_t/\Gamma_{t'})^{x-1}}{2\Gamma_t} \right)^{1/(2-x)}$$

Using Lemma 1 and some algebra, we can show that

$$\begin{aligned}
\epsilon_{\Gamma_t} f_t &= \frac{1}{2-x} \epsilon_{\Gamma_t} \frac{(1 + \sum_{t' \neq t} \Gamma_t/\Gamma_{t'})^{(x-1)}}{2\Gamma_t} \\
&= \frac{1}{2-x} \left(-(x-1)\epsilon_{\Gamma_t} \frac{1/\Gamma_t}{\sum_{t'} 1/\Gamma_{t'}} - 1 \right) \\
&= \frac{1}{2-x} \left((x-1) \frac{\Gamma_t}{\sum_{t'} \Gamma'_t} - 1 \right) \\
&= -\frac{1}{2-x} \left(\frac{\Gamma_t}{\sum_{t'} \Gamma_{t'}} + (2-x) \frac{\sum_{t' \neq t} \Gamma_{t'}}{\sum_{t'} \Gamma_{t'}} \right) \qquad (72)
\end{aligned}$$

Thus the elasticity is always higher (in absolute value) when there are more parallel routes. Thus we can expect the single route result to lower-bound the price of anarchy.

Formally, we can apply the same argument as in Theorem 4 to show that as the number of parallel routes goes to infinity, $\epsilon_{\Gamma_t} f_t \to -1$.

Now let e denote an edge on route t. We can write $f_e = S f_t$ where S is some product of fractions that represents the fraction of f_t that passes through e.

Observe that

$$\epsilon_e f_e = \epsilon_e S + \epsilon_e f_t = \epsilon_e S + \epsilon_{\Gamma_t} f_t \epsilon_e \Gamma_t.$$

By the argument we used in the analysis of the single-link graph, $\epsilon_{\Gamma_t} f_t = 1/(2 - x)$. Observe that that is the only elasticity term that depends on the valuation function. All the other ones depend only on graph structure. Now if the number of routes increases, $\epsilon_{\Gamma_t} f_t \to 1$, which is the elasticity we would have derived from linear valuation functions. □

General parallel-serial graphs give the users more alternatives for how to send their flow across the graph. Thus adding more routes can only improve the price of anarchy, which is what the following theorem formalizes.

Theorem 14. *Let G be a parallel-serial graph. The price of anarchy of G is lower-bounded by that of a route graph.*

Proof. We will call a *top-level route* a portion of a parallel-serial graph that is connected in parallel to the remainder of a graph. We will denote such routes by the letter i.

The efficiency ratio ρ can be bounded by that of a top-level route:

$$\frac{B}{\Gamma} = \frac{\dfrac{1}{\sum_i 1/B_i}}{\dfrac{1}{\sum_i 1/G_i}} \geq \min_i \frac{1/\Gamma_i}{1/B_i}.$$

Then this argument can be applied inductively to the serial elements of every top-level route i that also contain sub-routes. Eventually, we will get that

$$\frac{B}{\Gamma} = \frac{\dfrac{1}{\sum_i 1/B_i}}{\dfrac{1}{\sum_i 1/G_i}} \geq \frac{1/\Gamma_t}{1/B_t}$$

for some route t.

By the proof of the previous theorem, demand at every link of a parallel-serial graph is is more elastic when there are several parallel paths. Thus, passing to route elasticities cannot improve the ratio

$$\frac{1/\Gamma_t}{1/B_t}.$$

Thus the price of anarchy for a single route is a lower bound for any parallel-serial graph. □

10 Supply-Side Analysis of General Graphs

Consider the model we defined at the beginning of the paper. Let G be an arbitrary graph. We will show that given a natural assumption on the elasticity of the providers, the bounds for a "route" obtained in the previous section lower-bound the efficiency of an arbitrary graph.

10.1 Supply-side analysis

The ratio β_e/γ_e is a function of the elasticity at edge e. The elasticity, in turn, depends on the structure of the graph, and on the degree of vertical and horizontal competition among providers. The main claim of this section is that the elasticity at an edge in an arbitrary graph is not worse than if that graph was a route. Thus the results from the previous section bound the price of anarchy in an arbitrary graph.

Luckily, we can assume by a previous lemma that marginal costs are linear. That makes the analysis somewhat tractable. However, it will be more convenient to analyze a slightly different and simpler "aggregate" game, like we did in previous sections. Let G be an arbitrary multi-graph. We can construct an aggregate price Γ_l at each link l as in Section 6. In the new game, users select a rate on each route of the aggregated graph (V, L) with a price of Γ_l at every link.

Lemma 21. *The Nash equilibria of the game in which users choose flows on routes are the same as when they choose flows on paths. The equilibrium utilities are identical.*

Proof. The proof is very similar to that of Lemma 18. Let P denote the set of all paths in G. The necessary and sufficient conditions of Lemma 2 applied to G state that for all q and r:

$$V_q'(d_q) \leq \sum_{p'} \sum_{e \in p \cap p'} \gamma_e d_{qp'} + \sum_{e \in p} \gamma_e f_e \text{ for all } p \in P \text{ such that } d_{qp} > 0 \tag{73}$$

$$V_q'(d_q) \geq \sum_{p'} \sum_{e \in p \cap p'} \gamma_e d_{qp'} + \sum_{e \in p} \gamma_e f_e \text{ for all } p \in P \tag{74}$$

$$C_r'(f_r) \geq \gamma_r f_r \left(2 - \frac{1}{\epsilon f_r}\right) \text{ if } f > 0 \tag{75}$$

$$C_r'(f_r) \leq \gamma_r f_r \left(2 - \frac{1}{\epsilon f_r}\right) \tag{76}$$

For any q, the efficiency conditions for the aggregate game are can be expressed as:

$$V_q'(d_q) \leq \sum_{t'} \sum_{l \in t \cap t'} \mathit{\Gamma}_l d_{qt'} + \sum_{l \in t} \mathit{\Gamma}_l f_l \text{ for all } t \in T \text{ such that } d_{qt} > 0 \tag{77}$$

$$V_q'(d_q) \geq \sum_{t'} \sum_{l \in t \cap t'} \mathit{\Gamma}_l d_{qt'} + \sum_{l \in t} \mathit{\Gamma}_l f_l \text{ for all } t \in T \tag{78}$$

$$C_r'(f_r) \geq \gamma_r f_r \left(2 - \frac{1}{\epsilon f_r}\right) \text{ if } f_r > 0 \tag{79}$$

$$C_r'(f_r) \leq \gamma_r f_r \left(2 - \frac{1}{\epsilon f_r}\right) \tag{80}$$

Consider a vector strategies $(d_{qt})_{t \in T}$ that satisfies conditions (77)-(80) and a route t. The aggregate mechanism will allocate rates d_{qp} on paths $p \in t$ that coincide with t according to the same rules as for parallel-serial graphs: proportionally to the reciprocals of the slopes of the pricing functions. We have to show that conditions (73)-(74) hold for these d_{qp}.

It can be verified that if we take any link $l \in t$ and any edge $e \in l$, the sum of the flows on paths $p \in t$ that pass by e equals

$$\sum_{p \in t, e \in p} d_{qp} = \frac{1/\gamma_e}{\sum_{e' \in l} 1/\gamma_{e'}} d_{qt}.$$

We use this observation to derive condition (73) for any path $p \in P$. Path p must coincide with a unique route $t \in T$ and at every link $l \in t$ take some edge $e_l \in l$. We derive condition (73) as follows.

$$\begin{aligned}
V_q'(d_q) &\leq \sum_{t'} \sum_{l \in t \cap t'} \mathit{\Gamma}_l d_{qt'} + \sum_{l \in t} \mathit{\Gamma}_l f_l \\
&= \sum_{t'} \sum_{l \in t \cap t'} \gamma_{e_l} \frac{1/\gamma_{e_l}}{\sum_{e' \in l} 1/\gamma_{e'}} d_{qt'} + \sum_{l \in t} \gamma_{e_l} \frac{1/\gamma_{e_l}}{\sum_{e' \in l} 1/\gamma_{e'}} f_l \\
&= \sum_{t'} \sum_{l \in t \cap t'} \gamma_{e_l} \sum_{p' \in t', e_l \in p'} d_{qp'} + \sum_{l \in t} \gamma_{e_l} f_{e_l} \\
&= \sum_{t'} \sum_{p' \in t'} \sum_{e \in p \cap p'} \gamma_e d_{qp'} + \sum_{e \in p} \gamma_e f_e \\
&= \sum_{p'} \sum_{e \in p \cap p'} \gamma_e d_{qp'} + \sum_{e \in p} \gamma_e f_e
\end{aligned}$$

Condition (74) follows similarly.

This shows that for any $\boldsymbol{\gamma}$, the demand at each edge remains the same in the aggregate game. Thus the function $f_r(\boldsymbol{\gamma})$ is unchanged, and the equilibrium prices $\boldsymbol{\gamma}$ of the aggregate game also form a supply-side equilibrium in the original game. □

We can now establish our main claim. However, we will need the following assumption on the elasticity. How to derive this assumption from basic principles in an open question for now.

Assumption 3 *Let q be a user and let k be a link. Let T_q be a set of routes in the "aggregate" graph (V,L) available to q, and let $t \in T_q$ be such that $l \notin t$. We assume that restricting the users routes to $P_q - t$ does not make the demand of q and link l more elastic.*

Lemma 22. *Suppose that users have monomial valuation functions $\alpha_q f^x$ with the same degree x. Let k be a link, and let f_k be the flow across the link. Then, under Assumption 3, we have:*

$$\epsilon_{\Gamma_k} f_k \leq -\frac{1}{2-x} \min_q \min_{t \in T_q, k \in t} \epsilon_{\Gamma_k} \sum_{l \in t} \Gamma_l.$$

Proof. Until the end of the proof, fix a user q and a route t that goes through k. Differentiating the utility of user q with respect to d_{qt}, we obtain the following Nash equilibrium condition.

$$\begin{aligned}\alpha_q x(\sum_{t' \in T_q} d_{qt'})^{x-1} &= \sum_{l \in t} \Gamma_l (f_l + d_{qt}) + \sum_{t' \in T_q; t' \neq t;} \sum_{l \in t' \cap t} \Gamma_l d_{qt'} \\ &= \sum_{l \in t} \Gamma_l (f_l + d_{qt}) + \sum_{l \in t} \Gamma_l \sum_{t' \neq t, l \in t'} d_{qt} \\ &= \sum_{l \in t} \Gamma_l (f_{l-q} + 2 f_{ql})\end{aligned}$$

where $f_{ql} = \sum_{t' \in T_q; l \in q'} d_{qt'}$ and $f_{l-q} = \sum_{q' \neq q} f_{q'l}$. Also, given $t' \in T_q$ for some q, let $S_{t'} = \sum_{l \in t' \cap t} \Gamma_l$. We now divide both sides of the above equation by $2 \sum_{l \in t} \Gamma_l$, combine the Γ into the $S_{t'}$ we just defined, and separate the routes that avoid k into different sums:

$$\frac{V'(\sum_{t' \in T_q} d_{qt'})}{2 \sum_{l \in t} \Gamma_l} = \frac{2 \sum_{t' \in T_q} S_{t'} d_{qt'} + \sum_{q' \neq q} \sum_{t' \in T_{q'}} S_{t'} d_{q't'}}{2 \sum_{l \in t} \Gamma_l} \tag{81}$$

$$= \frac{2 \sum_{t' \in T_q; k \in t'} S_{t'} d_{qt'} + 2 \sum_{t' \in T_q; k \notin t'} S_{t'} d_{qt'} + \sum_{q' \neq q} \left(\sum_{t' \in T_{q'}; k \in t'} S_{t'} d_{q't'} + \sum_{t' \in T_{q'}; k \notin t'} S_{t'} d_{qt'} \right)}{2 \sum_{l \in t} \Gamma_l} \tag{82}$$

Now this is where we will use Assumption 3. We assume that user q does not have any routes avoiding k. Thus

$$\epsilon_k \frac{V'(\sum_{t' \in T_q} d_{qt'})}{2\sum_{l\in t} \Gamma_l} = \epsilon_k \frac{2\sum\limits_{t'\in T_q; k\in t'} S_{t'} d_{qt'} + 2\sum\limits_{t'\in T_q; k\notin t'} S_{t'} d_{qt'} + \sum\limits_{q'\neq q}\left(\sum\limits_{t'\in T_{q'}; k\in t'} S_{t'} d_{q't'} + \sum\limits_{t'\in T_{q'}; k\notin t'} S_{t'} d_{qt'}\right)}{2\sum\limits_{l\in t} \Gamma_l}$$

$$= \epsilon_k \frac{2\sum\limits_{t'\in T_q; k\in t'} S_{t'} d_{qt'} + \sum\limits_{q'\neq q}\sum\limits_{t'\in T_{q'}; k\in t'} S_{t'} d_{q't'}}{2\sum\limits_{l\in t} \Gamma_l} \tag{83}$$

I now claim that the elasticity (83) is greater than the elasticity of f_k. First observe that the function in (83) is less that f_k (simple calculation). Second, taking the derivative of the function in (83), we obtain

$$\begin{aligned}
&\frac{\partial}{\partial \Gamma_k}\frac{2\sum_{t'\in T_q;k\in t'} S_{t'}d_{qt'} + \sum_{q'\neq q}\sum_{t'\in T_{q'};k\in t'} S_{t'}d_{q't'}}{2\sum_{l\in t}\Gamma_l} \\
&\quad = \frac{2\sum_{t'\in T_q;k\in t'}(d_{qt'} + S_{t'}d'_{qt'}) + \sum_{q'\neq q}\sum_{t'\in T_{q'};k\in t'}(d_{q't'} + S_{t'}d'_{q't'})}{2\sum_{l\in t}\Gamma_l} \\
&\qquad - \frac{2\sum_{t'\in T_q;k\in t'} S_{t'}d_{qt'} + \sum_{q'\neq q}\sum_{t'\in T_{q'};k\in t'} S_{t'}d_{q't'}}{2(\sum_{l\in t}\Gamma_l)^2} \\
&\quad = \frac{2\sum_{t'\in T_q;k\in t'} d_{qt'} + \sum_{q'\neq q}\sum_{t'\in T_{q'};k\in t'} d_{q't'}}{2\sum_{l\in t}\Gamma_l} + \frac{2\sum_{t'\in T_q;k\in t'} S_{t'}d'_{qt'} + \sum_{q'\neq q}\sum_{t'\in T_{q'};k\in t'} S_{t'}d'_{q't'}}{2\sum_{l\in t}\Gamma_l} \\
&\qquad - \frac{2\sum_{t'\in T_q;k\in t'} S_{t'}d_{qt'} + \sum_{q'\neq q}\sum_{t'\in T_{q'};k\in t'} S_{t'}d_{q't'}}{2(\sum_{l\in t}\Gamma_l)^2} \\
&\quad \geq \frac{2\sum_{t'\in T_q} S_{t'}d'_{qt'} + \sum_{q'\neq q}\sum_{t'\in T_{q'};} S_{t'}d'_{q't'}}{2\sum_{l\in t}\Gamma_l} \\
&\quad \geq \sum_{q'}\sum_{t'\in T_{q'};k\in t} d'_{q'p'} = \frac{\partial f_k}{\partial \Gamma_k}
\end{aligned}$$

where $d'_{qt'} = \partial d_{qt'}/\partial \Gamma_k$ is the negative derivative of $d_{qt'}$. Then from the definition of elasticity it follows that (83) $\geq \epsilon_k f_k$.

Now consider the marginal utility of user i.

$$\begin{aligned}
\epsilon_k \frac{\alpha_q x (\sum_{t' \in T_q} d_{qt'})^{x-1}}{2 \sum_{l \in p} \Gamma_l} &= (x-1)\epsilon_k (\sum_{t' \in T_q} d_{qt'}) - \epsilon_k \sum_{l \in p} \Gamma_l \\
&= (x-1) \frac{f_{ik}}{\sum_{t' \in P_q} d_{qt'}} \epsilon_k (f_{qk}) + (x-1) \frac{\sum_{t' \in T_q; k \notin t'} d_{qt'}}{\sum_{t' \in P_q} d_{qt'}} \epsilon_k (\sum_{t' \in T_q; k \notin t'} d_{qt'}) - \epsilon_k \sum_{l \in p} \Gamma_l \\
&\leq (x-1)\epsilon_k (f_{qk}) - \epsilon_k \sum_{l \in p} \Gamma_l
\end{aligned}$$

Combining this with our previous results, we get that for any user q,

$$\epsilon_k f_k \leq (x-1)\epsilon_k (f_{qk}) - \epsilon_k \sum_{l \in t'} \Gamma_l$$

Using part 4 of Lemma 1, we deduce that

$$\epsilon_k f_k \leq (x-1)\epsilon_k f_k - \min_{t' \in T_q} \epsilon_k \sum_{l \in t'} \Gamma_l$$

Solving for $\epsilon_k f_k$ we get that

$$\epsilon_k f_k \leq -\frac{1}{2-x} \min_{t' \in T_q} \epsilon_k \sum_{l \in t'} \Gamma_l$$

The $\epsilon_k f_k$ above assumes there are no routes that avoid k. But if there are such routes, by our assumption, that makes f_k even more elastic, and so the bound still holds. □

The main theorem now follows from the lemma.

Theorem 15. *Under Assumption 3, the supply-side price of anarchy ρ in an arbitrary graph is bounded by the supply-side price of anarchy of the worst route.*

Proof. Let G be an arbitrary graph. Consider any edge e that is part of a link l. Let t be the route that minimizes the left-hand term in Lemma 22: $\epsilon_{\Gamma_k} \sum_{l \in t} \Gamma_l$.

Observe that $f_e = (1/\gamma_e)/(\sum_{e' \in l} 1/\gamma_{e'}) f_l$. If the entire graph consisted only of route t, then the elasticity of the flow at edge e would equal.

$$\begin{aligned}
(\epsilon_e f_e)_{G=t} &= \epsilon_e \frac{1/\gamma_e}{\sum_{e' \in l} 1/\gamma_{e'}} f_l \\
&= \epsilon_e \frac{1/\gamma_e}{\sum_{e' \in l} 1/\gamma_{e'}} - \frac{1}{2-x} \epsilon_{\Gamma_l} \left(\sum_{l \in t} \Gamma_l \right) \epsilon_e \Gamma_l.
\end{aligned}$$

In reality, the elasticity of the flow at edge e equals

$$\epsilon_e f_e = \epsilon_e \frac{1/\gamma_e}{\sum_{e' \in l} 1/\gamma_{e'}} + \epsilon_{\Gamma_l} f_l \epsilon_e \Gamma_l.$$

However, by Lemma 22,

$$\begin{aligned}\epsilon_e f_e &= \epsilon_e \frac{1/\gamma_e}{\sum_{e' \in l} 1/\gamma_{e'}} - \epsilon_{\Gamma_l} f_l \epsilon_e \Gamma_l \\ &\leq \epsilon_e \frac{1/\gamma_e}{\sum_{e' \in l} 1/\gamma_{e'}} - \frac{1}{2-x} \epsilon_{\Gamma_l} \left(\sum_{l \in t} \Gamma_l \right) \epsilon_e \Gamma_l \\ &= (\epsilon_e f_e)_{G=t}.\end{aligned}$$

Thus as a function, $\epsilon_e f_e$ is always pointwise further away from 0 than $(\epsilon_e f_e)_{G=t}$. Now choose e to be the edge at which supply is the least elastic. Lemma 20 yields a closed-form expression for the elasticity at edge e. Combining that expression with the ratio β_e/γ_e, we can after doing some algebra derive an expression that is identical to the last ratio in Theorems 6 and 9. This yields a price of anarchy bound for the entire graph G that is equal to that within route t. □

11 Existence of Nash equilibria

We conclude by briefly discussing the existence of equilibria within the mechanism. This is an area in which the paper still needs to be expanded.

Nash equilibria have been extensively studied by Correa et al. (2007, 2010) in the supply-side market that faces a fixed demand. In the single-resource setting (2007), they make the same assumptions on the costs as we do. In the parallel-serial graph setting (2010), their model is the same except they assume that providers have quadratic costs. We suspect their results and techniques can be carried over to establish that equilibria exist in our game.

With the techniques that will be developed in this paper, we are already able to prove the following result that is comparable to that of Correa et al.

Theorem 16. *Let G be a parallel-serial graph with at least two providers per link. Suppose providers' costs are quadratic. Then a unique subgame perfect Nash equilibrium exists in the game played on G and best-responses converge on both the demand and the supply side. If we assume that the providers' prices are bounded, we can obtain precise rates of convergence to the equilibrium.*

Proof. Follows by applying Brower's fixed point theorem to the best-response function defined in Lemma 20 within the compact set $[0, \max_r \beta_r]^R$.

We conjecture that we can adapt the above result, as well as the extensive analysis of Correa et al. (2007, 2010) to show the existence of a Nash equilibrium within the mechanism for any type of user utility and any graph.

Conjecture 1. There is a Nash equilibrium in the two-sided game.

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