The Shannon Capacity of Graphs
CG Seminar

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A very elusive graph parameter.

- Originally studied by Shannon in 1956.
- Breakthrough work by Lovász in 1979.
- Some work by Alon and hungarians et al.
- ...
Introduction

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**Definition**
Two letters $u, v$ can be *confused* if they are adjacent in $G$.

At most $\alpha(G)$ distinct 1-letter messages can be sent without confusion. What about longer messages?
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Two $k$-letter words can be confused if each $i^{\text{th}}$ letter are confused or equal.
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The strong product $G \cdot H$ is as follows:

- $V(G \cdot H) = V(G) \times V(H)$
- $(u_1, v_1) \sim_{G \cdot H} (u_2, v_2)$ if $u_1 \sim_G u_2$ and $v_1 \sim_H v_2$.

Denote by $G^k$ the $k$-fold strong product of $G$ with itself.

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Denote by $G^k$ the $k$-fold strong product of $G$ with itself.

Two $k$-letter words $u, v$ are confused iff $u \sim_{G^k} v$.

At most $\alpha(G^k)$ distinct $k$-letter messages can be sent without confusion.
Definition
Define the *Shannon capacity* of $G$ to be:

$$c(G) = \sup_k \sqrt[k]{\alpha(G^k)}$$

The sequence $(\sqrt[k]{\alpha(G^k)})$ is the *independence sequence*.
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Clearly, $c(G) \geq \alpha(G)$. What about upper bounds?
Consider the following ILP:
- variables $x_u \geq 0$ for each $u \in V(G)$
- constraints $x_u + x_v \leq 1$ for every pair $u \sim v \in G$
- maximizing $\sum_{u \in V(G)} x_u$

This computes $\alpha(G)$. The fractional relaxation computes $\alpha^*(G)$. 

Theorem (Shannon) $c(G) \leq \alpha^*(G)$

Theorem (Lovász) If $G$ is perfect, then $\alpha(G) = \alpha^*(G)$. 
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**Theorem (Shannon)**

$$c(G) \leq \alpha^*(G)$$

**Theorem (Lovász)**

*If $G$ is perfect, then $\alpha(G) = \alpha^*(G)$.***
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Remark

Let $X$ be a maximal independent set. Then $X^k = \{(u_1, \ldots, u_k) : u_i \in X\}$ is independent in $G^k$:

$$|X|^k = |X^k| \Rightarrow \alpha(G) \leq k\sqrt[2]{\alpha(G^k)}$$
Corollary

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If $c(G) = \alpha(G)$, then $\alpha(G) = k^{\sqrt[k]{\alpha(G^k)}}$ for every $k$.

Vertices cannot be “packed” in independent sets in powers of $G$. 
In a sense, $C_{2k+3}$ and $\overline{C}_{2k+3}$ are the simplest imperfect graphs.

Remark

$$c(\overline{C}_{2k+3}) \geq \alpha(\overline{C}_{2k+3}) = 2$$
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**Remark**

$$c(\overline{C}_{2k+3}) \geq \alpha(\overline{C}_{2k+3}) = 2$$

**Theorem (Bohman, Holzman)**

$$\alpha(\overline{C}_{2k+3}^{2^k}) \geq 2^{2^k} + 1$$

We “pack” one vertex into the trivial independent set in $\overline{C}_{2k+3}^{2^k}$!

**Corollary**

$$c(\overline{C}_{2k+3}) > 2$$
We can use algebraic tools to bound $c(G)$ from above.

- Haemers: matrix representation $A \Rightarrow c(G) \leq \text{rank}(A)$
- Alon: polynomial representation over $\mathcal{F} \Rightarrow c(G) \leq \text{dim}(\mathcal{F})$.
- Lovász: orthonormal representation of vectors...

We will focus on Lovász’ technique, which was the first to appear.
Definition
A $d$-dimensional orthonormal representation of a graph $G$ is a set of unit vectors $\{x_u \in \mathbb{R}^d : u \in V(G)\}$, such that if $u \not\sim v$, then $x_u \cdot x_v = 0$.

The value of such a representation is:

$$\min_{\|c\|=1} \max_{u \in V(G)} \frac{1}{(c \cdot x_u)^2}$$
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Definition
Define \(\vartheta(G)\) to be the minimum value of any orthonormal representation of \(G\).

By the extreme value theorem, \(\vartheta(G)\) is always attained.
Lemma

\[ \vartheta(G \cdot H) \leq \vartheta(G) \vartheta(H) \]
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Lemma

\[ \alpha(G) \leq \vartheta(G) \]
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Lemma

\[ \alpha(G) \leq \vartheta(G) \]

Theorem

\[ c(G) \leq \vartheta(G) \]

Proof.
From the previous two lemmas, we have for any \( k \geq 0 \):

\[ \alpha(G^k) \leq \vartheta(G^k) \leq (\vartheta(G))^k \]
The 5-cycle

Order the vertices of $C_5$ as $u_1, u_2, \ldots, u_5$.

Theorem

$$c(C_5) = \sqrt{5}$$
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Proof.

$(u_1, u_1), (u_2, u_3), (u_3, u_5), (u_4, u_2), (u_5, u_4)$ is independent in $C_5^2$, so:

$$c(G) \geq \sqrt{\alpha(C_5^2)} \geq \sqrt{5}$$
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Unfortunately, $\alpha^*(C_5) = 5/2 > \sqrt{5}$. We need more.
The 5-cycle (cont.)

Proof.
For the upper bound, we give an orthonormal representation:

\[ \cos(\gamma) = \cos(\alpha) \cos(\beta) + \sin(\alpha) \sin(\beta) \cos(\Gamma) \cdot x_i = 5 - \frac{1}{4} \Rightarrow c(C_5) \leq \vartheta(C_5) = \sqrt{5} \]

This was the first specified imperfect graph capacity!
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A generalization

Definition
 A graph $G$ is *vertex-transitive* if for any $u, v \in V(G)$, there is an automorphism $\sigma$ such that $\sigma(u) = v$.

Example
 $C_n$, hypercubes, Petersen graph, prisms, Cayley graphs, etc.
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Theorem (Lovász)
If $G$ is vertex-transitive and self-complementary, then:

$$c(G) = \vartheta(G) = \sqrt{n}$$
Odd cycles

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If $G$ is regular with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$, then:

$$\vartheta(G) \leq \frac{-n\lambda_n}{\lambda_1 - \lambda_n}$$

with equality if $G$ is edge-transitive.
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**Corollary**

*If $n$ is odd, then:*

$$\vartheta(C_n) = \frac{n}{1 + \sec(\pi/n)}$$
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Is $c(G) = \vartheta(G)$ always?
Kneser graphs

Definition
The *Kneser graph* $KG_{n,r}$ has vertex set the $r$-sized subsets of \( \{1, \ldots, n\} \), two subsets being adjacent if they are disjoint.
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Corollary (Lovász)

$$c(KG_{n,r}) = \vartheta(KG_{n,r}) = \binom{n-1}{r-1}$$

Corollary (Erdős-Ko-Rado)

$$\alpha(KG_{n,r}) = \binom{n-1}{r-1}$$
In every case encountered so far, \( c(G) = \vartheta(G) = \sqrt[k]{\alpha(G^k)} \) for some finite \( k > 0 \). Is this always the case?
In every case encountered so far, $c(G) = \vartheta(G) = k\sqrt{\alpha(G^k)}$ for some finite $k > 0$. Is this always the case?

- if $G$ is perfect, $c(G) = \alpha(G)$.
- if $G$ is vertex-transitive self-complementary, $c(G) = \sqrt{\alpha(G^2)}$.
- if $G$ is a Kneser graph, $c(G) = \alpha(G)$.
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If \(c(C_7) = \vartheta(C_7) = \frac{7}{1 + \sec(\pi/7)}\), then \(c(C_7) \neq k\sqrt{\alpha(G^k)}\) for any finite \(k > 0\). Is this possible?
Theorem (Shannon)

In general:

\[ c(G \cup H) \geq c(G) + c(H) \]

with equality if \( G \) or \( H \) is perfect.
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So \( c(C_5 \cup K_1) = \sqrt{5} + 1 \), but no power of \( \sqrt{5} + 1 \) is an integer, so:
\[ c(C_5 \cup K_1) \neq k\sqrt{\alpha((C_5 \cup K_1)^k)} \]

for any finite \( k > 0 \).
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This indicates that it may still be possible that \( c(C_7) = \vartheta(C_7) \).
Theorem (Haemers)

There is a graph $G$ with $c(G) \leq 7 < \vartheta(G) = 9$. 

Remark

$G$ is the Schlafli graph: the intersection graph of 27 lines on a cubic surface.
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$\overline{G}$ is the Schl"afli graph: the intersection graph of 27 lines on a cubic surface.

Figure: The Schl"afli graph...
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Remark

$\overline{G}$ is the Schläfli graph: the intersection graph of 27 lines on a cubic surface.

Figure: The Schläfli graph...

... so, perhaps $c(C_7) < \vartheta(C_7)$. No one knows.
Perhaps the probabilistic method can help. Let $G \sim G_{n,1/2}$. 

**Theorem (Erdős)**

*With high probability, $\alpha(G) = (2 + o(1)) \log n$.**
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**Theorem (Juhász)**

*With high probability, $\vartheta(G) = \Theta(\sqrt{n})$.***

So w.h.p. $\Omega(\log n) \leq c(G) \leq O(\sqrt{n})$. 
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No one currently understands the behaviour of $\alpha(G^k)$.

**Conjecture (Alon)**

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Computational considerations

- Each $\sqrt[k]{\alpha(G^k)}$ is NP-hard to compute.
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- Every graph whose capacity is known has either:
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  - $c(G) = \sqrt[\text{ } k]{\alpha(G)}$
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- Even when $c(G)$ is not attained in $k\sqrt[\text{ } k]{\alpha(G^k)}$, this sequence may behave badly; it may not well-approximate $c(G)$. (Alon)
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- Every graph whose capacity is known has either:
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Seems obvious that the decision problem “$c(G) \geq k$?” is NP-hard.
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Seems obvious that the decision problem “$c(G) \geq k$?” is NP-hard.

First things first: is “$c(G) \geq k$?” decidable?
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