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Degeneracy in Binary Trees

1. Ideal \((h = \log_2 N)\)

2. Degenerate \((h = N)\)

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3. Actual ($h = 2\ln N$, expected)

We need balance.

Can we satisfy these rules? Given a maximum fanout, $f$:

- The root has at least two subtrees (unless it is a leaf).

- Every node (apart from root or leaf) has $s$ subtrees, $f/2 \leq s \leq f$.

- All leaves are on the same level.
B-Tree

1. 0 [A B]

2. 2 [B D]
   0 [A C] 1 [C D]

3. 0 [A C] 1 [C D] 2 [B D] 3 [E F]

4. 2 [B D] 6 [D F]
   0 [A C] 1 [C D] 3 [E F] 4 [G H]

5. 2 [B D] 6 [D F] 3 [F H]

Java

RandomAccessFile btreed = new RandomAccessFile("bTree","r");
btreed.seek(loc*(f*4 + (f-1)*recordSize));

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B-Tree with Full Records

4 \( \text{PR H} \)

2 \( \text{NYC N} \)

1 \( \text{GNS E} \)

3 \( \text{L&S E} \)

5 \( \text{NYC H} \)

6 \( \text{B&O H} \)

7 \( \text{GTRC N} \)

8 \( \text{GNS E} \)

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Splitting a Node of a B-tree

The diagram illustrates a B-tree node being split. The node contains values from $p_0, k_{p}, p_1, \ldots, k_{[f/2]-1}, p_{[f/2]-1}$ to $k_{[f/2]}^t, k_{[f/2]+1}^t, \ldots, k_f^t, p_f$. When the node is split, $p$ moves to the parent node, the original node becomes the left node, and a new right node is created with the split values.

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B-Tree Improvements

1. B*-Tree \((2 \rightarrow 3\) splits: at least \(\sim 2/3\) full) 

\(2f\) keys in 2 nodes and parent; put in 3 nodes, moving up keys \(\lfloor 2f/3\rfloor\) and \(\lceil 4f/3\rceil\)  \(f = 3\)

1. \(0\) \(A\) \(B\)

2. \(2\) \(B\)

\(0\) \(A\) \(1\) \(C\) \(D\)

3. \(2\) \(C\)

\(0\) \(A\) \(B\) \(1\) \(D\) \(E\)

4. \(2\) \(B\) \(D\)

\(0\) \(A\) \(1\) \(C\) \(3\) \(E\) \(F\)

5. \(2\) \(B\) \(E\)

\(0\) \(A\) \(1\) \(C\) \(D\) \(3\) \(F\) \(G\)

6,7,8.

\(6\) \(D\)

\(0\) \(A\) \(1\) \(C\) \(3\) \(E\) \(4\) \(G\) \(7\) \(I\) \(J\)

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2. Inhomogeneous B-Tree
Decrease height (hence cost) by increasing fanout (but without changing bytes/block), for a given number of records.

\[ f = 5 \]

\( a \) (3 bytes) is the key for record \( A \) (10 bytes) etc.
B-Trees: Activity, Volatility, and Symmetry

Volatility   lo   hi
Symmetry     √   ×
Activity     √   √:

If $|\text{RAM}| = O(\log N)$ then retrieve each needed page only once for any level of activity. E.g., above, 4 buffers can fit into RAM.

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Cost Analysis of B-Trees

<table>
<thead>
<tr>
<th>Level</th>
<th>No. Nodes</th>
<th>No. Records</th>
<th>No. Nodes</th>
<th>No. Records</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>$f - 1$</td>
<td>1</td>
<td>$[\frac{f}{2}] - 1$</td>
</tr>
<tr>
<td>1</td>
<td>$f$</td>
<td>$f - 1$</td>
<td>2</td>
<td>$[\frac{f}{2}] - 1$</td>
</tr>
<tr>
<td>2</td>
<td>$f^2$</td>
<td>$f - 1$</td>
<td>2$[\frac{f}{2}]$</td>
<td>$[\frac{f}{2}] - 1$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$h - 1$</td>
<td>$f^{h-1}$</td>
<td>$f - 1$</td>
<td>2$[\frac{f}{2}]^{h-2}$</td>
<td>$[\frac{f}{2}] - 1$</td>
</tr>
</tbody>
</table>

(best case)

$$N = (f - 1)(1 + f + f^2 + ... + f^{h-1}) = f^h - 1$$

(worst case)

$$N = 1 + 2(f_2 - 1)(1 + f_2 + ... + f_2^{h-2}) = 2 * f_2^{h-1} - 1$$

where $f_2 = [\frac{f}{2}]$.

$$\log_f N + 1 \leq h \leq 1 + \log_{[\frac{f}{2}]}(N + 1)/2$$

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## Cost Analysis of φ-ary Trees

<table>
<thead>
<tr>
<th>Level</th>
<th>No. Nodes</th>
<th>No. Accesses</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>φ</td>
<td>2φ</td>
</tr>
<tr>
<td>2</td>
<td>φ²</td>
<td>3φ²</td>
</tr>
<tr>
<td>:</td>
<td>:</td>
<td>:</td>
</tr>
<tr>
<td>h − 1</td>
<td>φ^{h−1}</td>
<td>hφ^{h−1}</td>
</tr>
</tbody>
</table>

total nodes = \( 1 + φ + φ^2 + \ldots + φ^{h−1} = \frac{φ^{h−1}}{φ−1} \)

total accesses = \( 1 + 2φ + 3φ^2 + \ldots + hφ^{h−1} \)

Note that \( (k + 1)φ^k = \frac{d}{dφ}φ^{k+1} \)

So total accesses = \( \frac{d}{dφ}φ^{h−1} = \frac{hφ^{h+1}−(h+1)φ^{h}+1}{(φ−1)^2} \)

average accesses = total accesses / total nodes

Special cases:

- binary tree (φ = 2)
  
  average accesses = \( \frac{(h−1)2^h+1}{2^{h−1}} \approx h − 1 \)

- large φ
  
  average accesses \( \approx \frac{hφ^{h−1}}{φ^{h−1}} = h \)

So expected depth of search is \( h − 1 \) for binary tree; and \( h \), effectively, for anything else.

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Text and PATRICIA

1) Truncated Trie  2) PATRICIA Trie

Sample “text”:
mocha : 1110110101110111011000111110100011100001
with “starts” every eight bits.

Note that pointers alone, to each byte of a 100 Mb text, would occupy 4 times the size of the text: text indexes are big.

Using PATRICIA tries, stored 1 bit per node, and simple tricks, a byte-by-byte text index can be reduced to 2.7 times the text size. (Word-by-word: 0.7 times.)

Not only any substring but also arbitrary regular expressions can be found.

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Volatility—Updating Tries

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Updating Tries

Adding 10001010

N.B. $t$ node levels per page level; $\geq 2^t - 1$ nodes per page

Adding 01011010

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Updating Tries

Deleting 10001000

T = 0
B = 0
Symmetry

kd-Trees (2d-Tree)

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**kd-Tries (2d-Trie)**

Bit interleaving, e.g.: $(6,5) = (110,101) \Rightarrow 111001$

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Orthogonal Range Queries

b) partial range; c) exact match; d) partial match

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ORQ with Z-Order
(also applies to kd-tries)

a) 1 probe, 1 scan
b,c) 2 probes, 2 scans;
or 1 probe, 1 scan with extraneous reads
d) 4 probes, 4 scans; or ..

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