Excursions in Computing Science: Book 11d. Forces and Invariants Part III. Quantum Electromagnetism

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Part I. Electrostatics and Electromagnetism

1. Central Forces.
2. Gravity vs. Electricity.
3. Energy and momentum scales.
4. Divergence, gradient and $\vec{\mathrm{div}}$ $\vec{\mathrm{grad}}$.
5. Electrodynamics departs from gravitation.
6. Invariants, cross-products and convention.
7. Electromagnetic waves.

Part II. Partial Slope Equations and Quantum Mechanics

8. Partial Slope Equations: Laplace's Equation.
9. The Wave Equation.
10. The Schrödinger Equation I: Physics.
11. The Schrödinger Equation II: Animating in 1D.
12. The Schrödinger Equation III: Animating in 2D.

I. Prefatory Notes

13. The electromagnetic Schrödinger equation. In Note 5 we found that electromagnetism introduces a *potential momentum* M as well as the potential energy P. This suggests that the Schrödinger equation of Note 10

$$i\hbar\partial_t u = -\frac{\hbar^2}{2m}\partial_x^2 u + Vu$$

based on the equation of motion

$$E = \frac{p^2}{2m} + V$$

should become

$$i\hbar\partial_t u_1 = \frac{1}{2m}(-i\hbar\partial_x + M)^2 u_1 + (V+P)u_1$$

by including M with the momentum $-i\hbar\partial_x$ and P with the potential energy V due to non-electromagnetic causes (if any).

This is more difficult to solve than Schrödinger's equation, but there is a trick.

Suppose

$$u = u_1 e^{i\beta}$$

for some phase factor $\beta(t,x,y,z)$ which depends on timespace and thus has slopes

$$\kappa_\alpha = \partial_\alpha\beta \qquad \alpha = t,x,y,z$$

Then

$$\begin{aligned}
-i\hbar\partial_x u &= -i\hbar\partial_x(u_1 e^{i\beta}) \\
&= -i\hbar(e^{i\beta}\partial_x u_1 + ie^{i\beta}u_1\partial_x\beta) \\
&= (-i\hbar\partial_x u_1 + \hbar\kappa_x u_1)e^{i\beta}
\end{aligned}$$

and similarly for $-i\hbar\partial_y u$ and $-i\hbar\partial_z u$, and in particular,

$$i\hbar\partial_t u = (i\hbar\partial_t u_1 - \hbar\kappa_t u_1)e^{i\beta}$$

We need the spatial slopes squared and we can go at these from two directions.

$$\begin{aligned}
(-i\hbar\partial_x)^2 u &= -i\hbar\partial_x(-i\hbar\partial_x u_1 + \hbar\kappa_x u_1)e^{i\beta} \\
&= (-\hbar^2\partial_x^2 u_1 - i\hbar^2\partial_x(\kappa_x u_1) - i\hbar^2\kappa_x(\partial_x u_1) + \hbar^2\kappa_x^2 u_1)e^{i\beta}
\end{aligned}$$

but

$$\begin{aligned}
(-i\hbar\partial_x + \hbar\kappa_x)^2 u_1 &= (-i\hbar\partial_x + \hbar\kappa_x)(-i\hbar\partial_x u_1 + \hbar\kappa_x u_1) \\
&= -\hbar^2\partial_x^2 u_1 - i\hbar^2\partial_x(\kappa_x u_1) - i\hbar^2\kappa_x(\partial_x u_1) + \hbar^2\kappa_x^2 u_1
\end{aligned}$$

so these are almost the same thing

$$(-i\hbar\partial_x)^2 u = (-i\hbar\partial_x + \hbar\kappa_x)^2 u_1 e^{i\beta}$$

Putting them together in Schrödinger's equation, which u satisfies (I've shown only the ∂_x^2 terms: $\partial_y^2$and ∂_z^2 may be added)

$$\begin{aligned}
i\hbar\partial_t u_1 - \hbar\kappa_t u_1 &= (i\hbar\partial_t u)e^{-i\beta} \\
&= -\frac{\hbar^2}{2m}(\partial_x^2 u)e^{-i\beta} + Vue^{-i\beta} \\
&= \frac{1}{2m}(-i\hbar\partial_x + \hbar\kappa_x)^2 u_1 + Vu_1
\end{aligned}$$

or

$$i\hbar\partial_t u_1 = \frac{1}{2m}(-i\hbar\partial_x + \hbar\kappa_x)^2 u_1 + (V + \hbar\kappa_t)u_1$$

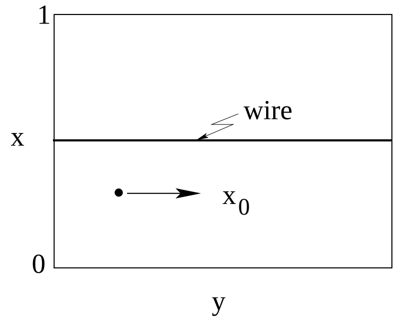
This is the electromagnetic Schrödinger equation if we put

$$\begin{aligned}
\hbar\partial_t\beta &= \hbar\kappa_t = P \\
\hbar\partial_x\beta &= \hbar\kappa_x = M_x \\
\hbar\partial_y\beta &= \hbar\kappa_y = M_y \\
\hbar\partial_z\beta &= \hbar\kappa_z = M_z
\end{aligned}$$

So the solution to the electromagnetic Schrödinger equation is just an ordinary wavefunction u multiplied by a phase factor β whose slopes are the electromagnetic potential energy and potential momentum

$$u_1 = ue^{-i\beta}$$

14. Simulating a charged wavepacket moving near a current. It is easy to modify `free2dSchroeGauss()` from Note 12 to simulate a charged wavepacket moving near a wire.



Here

$$\begin{aligned} P &= 0 \\ M_x &= 0 \\ M_z &= 0 \end{aligned}$$

and from Note 5

$$M_y = \frac{-2E_C qI}{c^2} \ln \frac{r}{a}$$

where $r = \sqrt{x^2 + y^2}$ and where we can set $a = 1$ without changing the effect of the potential momentum.

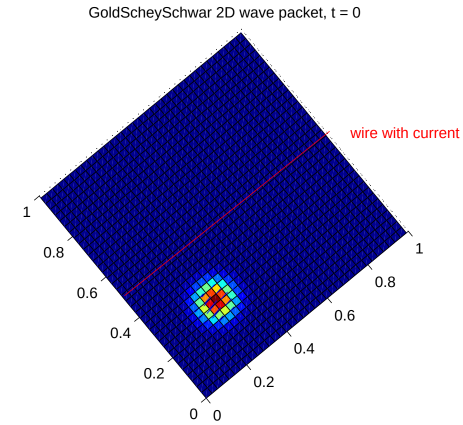
In this two-dimensional simulation $z = 0$, even though the formulas model the 3D physics.

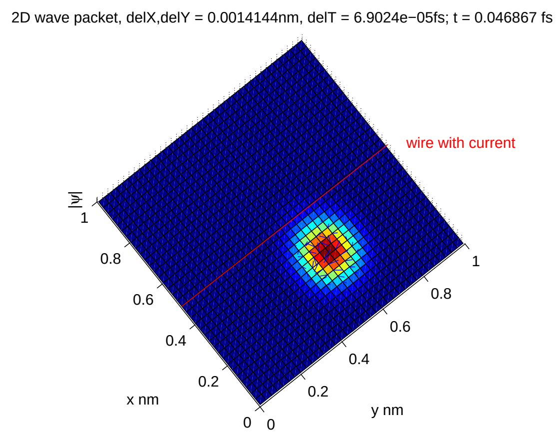
In the simulation $V = 0$ but after each time step we increment the phase

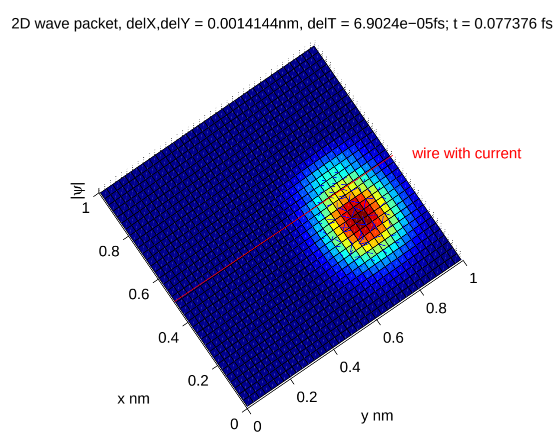
$$\begin{aligned} \beta &\rightarrow \beta + \partial_x \beta \Delta x + \partial_y \beta \Delta y + \partial_z \beta \Delta z \\ &= \beta + \frac{M_y}{\hbar} \Delta y \end{aligned}$$

(and note that β must be an array dependent on x). The starting value for β at time 0 does not matter so we just set it to 0.

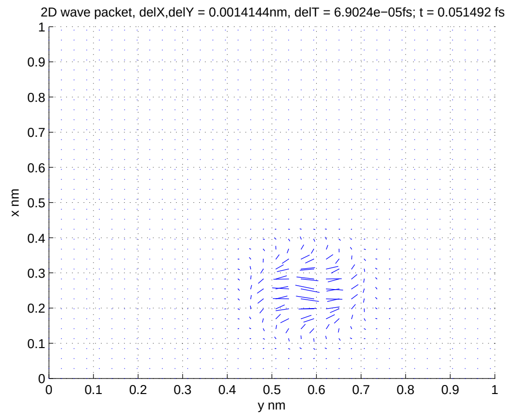
Here are three snapshots of running `phase2dSchroeGauss(E,ws,v,ang,I)` with the parameters `E = 2, v = '3', ang =` $\pi/2$ having the same meaning as for `free2dSchroeGauss()` in Note 12 with `ws = 'w'` meaning we're simulating a wire with a current, and `I = 20` amperes.

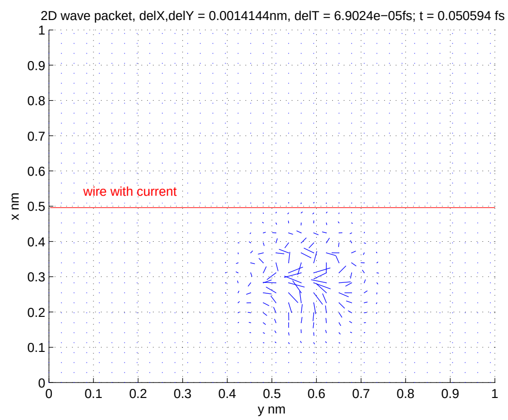






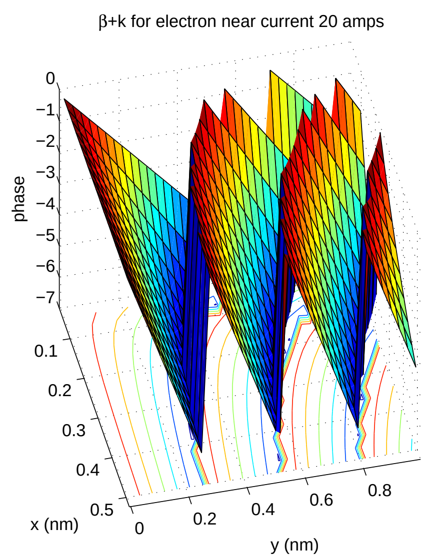
We can see the wavepacket drawn towards the wire at $x = 0.5$ and we can see that it is accelerating. Here is another version, showing just the phase angles





We can see the disturbance caused by the magnetic field of the wire.

A simple program showing the sum of phases, $\beta - k_0 y$, with $\beta = M_y y/\hbar$ (remembering M_y depends on $\ln(x)$) illustrates the bending of the wavefront towards the current in the wire at $x = 0.5$nm



Note that the phase is shown modulo 2π. We can visualize the corresponding wavefront by taking the sine of the phase plot. We can also imagine that the wavefront would be perpendicular to the

y-direction of motion if current $I = 0$.

15. Links with geometry. In the simulation of Note 14 we stepped through the phases β at each position of timespace (actually only the x coordinate matters in this example) in a way analogous to the parallel transport described in Note 13 of Book 11c Part I where we traced out a geodesic in curvilinear coordinates by moving a vector parallel to itself and appending it each step to the current end of the geodesic.

We find another analogy in the electromagnetic Schödinger equation of Note 13. We could abbreviate (and again I've left the x and y terms for you to fill in)

$$i\hbar\partial_t u_1 - \hbar\kappa_t u_1 = \frac{1}{2m}(-i\hbar\partial_x + \hbar\kappa_x)^2 u_1 + V u_1$$

to

$$i\hbar\mathcal{D}_t u_1 = -\frac{\hbar^2}{2m}\mathcal{D}_x^2 u_1 + V u_1$$

with

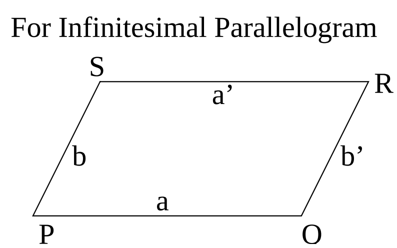
$$\mathcal{D}_t = \partial_t + i\kappa_t$$

and

$$\mathcal{D}_x = \partial_x + i\kappa_x$$

This imitates the absolute slope of Note 14 in Book 11c Part I. The role of the "affine connection", Γ (Note 12 of Book 11c) is here played by the 4-vector $\kappa = M/\hbar$. In this context, we'll call it the *inside connection*: the phase whose slope is κ is *internal* to the particle but perceived as a field which is external to it.

Since we have analogs for parallel transport and for the affine connection, what about curvature? We can modify the argument of Note 17 of Book 11c—it actually becomes simpler. We move the wavefunction $u(t, x, y, z)$ around the same infinitesimal parallelogram, $u(P)$ to $u(Q)$ to $u(R)$ to $u(S)$, to see what the effect is.



This is a parallelogram in ordinary, flat, space: the "curvature" will not be a curvature in this space but another tensor describing the effect of the electromagnetic field on the phase of the wavefunction. So $b' = b$ and $a' = a$, but it is useful to distinguish them for the moment.

To move the wavefunction from P to Q we need the transformation

$$u(Q) = U(a)u(P)$$

where

$$\begin{aligned} U(a) &= e^{-i\Delta\beta} \\ &\approx 1 - i\Delta\beta \end{aligned}$$

with $\Delta\beta$ the change from $\beta(P)$ to $\beta(Q)$, which we make arbitrarily small, along the infinitesimal displacement a. So, summed over the indices α

$$\Delta\beta = \partial_\alpha\beta(P)a_\alpha = \kappa_\alpha(P)a_\alpha$$

and

$$U(a) \approx 1 - i\kappa_\alpha(P)a_\alpha$$

Similarly, to move from P to R

$$u(R) = U(b')u(Q) = U(b')U(a)u(P)$$

with

$$\begin{aligned} U(b') &= 1 - i\kappa_\alpha(Q)b'_\alpha \\ &= 1 - i\kappa_\alpha(P)b'_\alpha - i\partial_\mu\kappa_\alpha(P)b'_\alpha a_\mu \end{aligned}$$

where we use the slope of $\kappa(P)$ to find $\kappa(Q)$ from $\kappa(P)$.

Putting these together

$$\begin{aligned} U(b')U(a) &= (1 - i\kappa_\alpha(P)b'_\alpha - i\partial_\mu\kappa_\alpha(P)b'_\alpha a_\mu)(1 - i\kappa_\nu(P)a_\nu) \\ &\approx 1 - i\kappa_\alpha(P)b'_\alpha - i\kappa_\nu(P)a_\nu - i\partial_\mu\kappa_\alpha(P)b'_\alpha a_\mu + \kappa_\alpha(P)\kappa_\nu(P)b'_\alpha a_\nu \end{aligned}$$

keeping only terms at most quadratic in the infinitesimals a and b'.

That gets us from P to R via Q. We can also get from P to R via S, by just replacing $a \to b$, $b' \to a'$.

$$U(a')U(b) = 1 - i\kappa_\alpha(P)a'_\alpha - i\kappa_\nu(P)b_\nu - i\partial_\mu\kappa_\alpha(P)a'_\alpha b_\mu + \kappa_\alpha(P)\kappa_\nu(P)a'_\alpha b_\nu$$

The whole circuit from P back to P is the difference of these two. We can drop the (P) after each κ because they are all $\kappa(P)$. And we can now reduce $a' = a$ and $b' = b$.

$$U(b')U(a) - U(a')U(b) = -i(\partial_\mu\kappa_\alpha - \partial_\alpha\kappa_\mu)a_\mu b_\alpha + (\kappa_\alpha\kappa_\mu - \kappa_\mu\kappa_\alpha)a_\mu b_\alpha$$

If κ_α and κ_μ commute (which they do for the electromagnetic field) this combined transformation of $u(P)$ to $u'(P)$, having cycled around the parallelogram, is

$$\begin{aligned} U(b)U(a) - U(a)U(b) &= -i(\partial_\mu\kappa_\alpha - \partial_\alpha\kappa_\mu)a_\mu b_\alpha \\ &= \frac{-iq}{\hbar}(\partial_\mu A_\alpha - \partial_\alpha A_\mu)a_\mu b_\alpha \end{aligned}$$

where $A = \frac{\hbar}{q}\kappa$ is the specific (charge-independent) potential energentum (for magnetism usually called the vector potential).

The expression $\partial_\mu A_\alpha - \partial_\alpha A_\mu$ looks like a curl, and in Note 6 we defined the magnetic field $\vec{B} = \vec{\bigtriangledown} \times \vec{A}$ and the electric field $\vec{E} = \vec{\bigtriangledown}\phi - \partial_t\vec{A}$ where $\phi = A_t$.

Thus, for electromagnetism, we get

$$U(b)U(a) - U(a)U(b) = \frac{iq}{\hbar}F_{\mu\alpha}a_\mu b_\alpha$$

where the Maxwell field tensor

$$F = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

This tensor is the analog of timespace curvature (general relativity) for the local connections of the phase of the wavefunction (quantum electromagnetism).

We shall be considering more general fields, A, in which A_α and A_μ do not commute

$$[A_\alpha, A_\mu] \neq 0$$

Then the Maxwell tensor generalizes to

$$F_{\mu\alpha} = \partial_\mu A_\alpha - \partial_\alpha A_\mu - iq[A_\mu, A_\alpha]$$

16. Local action versus action-at-a-distance. The important insight of the derivation, in the previous Note, of the Maxwell field tensor is that the argument is *local*, based on an *infinitesimal* parallelogram. If we can keep our arguments local then we avoid action at a distance, just as general relativity avoids Newtonian action at a distance for gravity.

The phase space of a wavefunction shows an important symmetry: we can add any constant phase we like, provided that we do so everywhere in timespace.

$$u(t, x, y, z) \to e^{ib} u(t, x, y, z)$$

for any angle b independent of t, x, y or z.

This is because $uu^* \to e^{ib} uu^* e^{-ib} = uu^*$ and there is no difference in the physics, which always depends on the probability uu^* rather than the probability amplitude u.

But suppose we have a wavefunction stretching from here to Alpha Centari (which might happen in some entangled communication system of the future). Adding b to the phase here requires us to add b to the phase there and the simpleminded way of doing so must be action at a distance.

We must be more subtle. The *slope* of b is zero, because of the constancy. So if we connect *here* to *here-plus-an-infinitesimal* using

$$b(\text{here } + \Delta s) = b(\text{here}) + \Delta s * \text{slope } b = b(\text{here})$$

then we have the first step. Many steps later we have

$$b(\text{Alpha Centauri }) = b(\text{here})$$

which is what we want, but propagated locally.

In the presence of an electromagnetic field A the slope of the phase $\beta(t, x, y, z)$ is no longer zero. We can still add any angle we like to the phase here but this angle will be propagated to neighbouring points by the electromagnetic field

$$\text{slope } \beta = \kappa = \frac{q}{\hbar} A$$

It is not hard to see, using the argument of Note 13, that if

$$u(t, x, y, z) \to e^{iq\lambda} u(t, x, y, z)$$

where $\lambda(t, x, y, z)$ is a function of timespace, then

$$A_\mu \to A_\mu - \partial_\mu \lambda$$

This is a classical condition on the vector potential: we can add to it the gradient of any field without changing the physics at all.

The above argument can be reversed: if we make the wavefunction invariant under *position-dependent* phase changes $\lambda(t, x, y, z)$, then we get an electromagnetic field, which appears as the "inside connection" in the "absolute slope"

$$\mathcal{D}_\mu = \partial_\mu + \frac{iq}{\hbar} A_\mu$$

and which itself transforms under the phase change as

$$A_\mu \to A_\mu - \partial_\mu \lambda(t,x,y,z)$$

in this way, we derive a force, electromagnetism, from the condition of invariance under local (position-dependent) phase changes.

Because the angle β in the slope equation

$$\text{slope } \beta = \frac{q}{\hbar} A$$

allows an arbitrary constant, we can still add a constant phase angle everywhere to the wave function. This invariance under what is thus an arbitrary rotation in the internal (phase) space of the wavefunction constitutes a *symmetry* of the type described in Note 26 of Book 8c (Part III).

(The invariance under a position-dependent rotation, which gives rise to the electromagnetic field, is called *local*. The invariance under position-independent rotation is called *global*. These terms are misleading because the global invariance, arising from the arbitrary constant, is in a sense a special case of local invariance. But see [FLS64a, Sect.27.a] and [FLS64b, Sect.21.2].)

17. Other symmetries, other forces. The symmetry of the "inside connection" of Notes 13 to 16, with its local, timespace-dependent phase factor, $\beta(t,x,y,z)$, is known as $U(1)$—1-dimensional unitary—because $e^{-i\beta}$ describes the circumference of a unit circle in the space of 2D numbers. This led in Note 16 to a variant of absolute slope which captures both external (the usual slope) and internal (the slope of β) slopes

$$D_\mu = \partial_\mu - ig_1 \frac{Y}{2} B_\mu$$

I have changed notation to be more conventional: I've changed the sign of the phase β; I've changed the name of the field from the electromagnetic A_μ to B_μ for reasons which will soon be clear; I've changed the charge q to a combination of "hypercharge generator" Y and coupling constant g_1; and I've buried the $\hbar$ but exposed a factor 1/2.

In Notes 32 and 33 of Book 8c (Part IV) we discussed more complicated symmetry groups, $SU(2)$ and $SU(3)$, respectively. These are the two- and three-dimensional *special unitary* groups, based respectively on 2-by-2 and 3-by-3 matrices with "special" referring to the restriction that the determinants of those matrices are all +1.

$SU(2)$ consists of all 2-by-2 unitary matrices with that determinant and can be generated by $2 \times 2 - 1 = 3$ basic matrices, which we can take to be the Pauli matrices—more precisely, $1/2\times$ the Pauli matrices—making $SU(2)$ closely related to the rotation group, in fact a generalization of it. (Here I anticipate Note 21 in Part IV of these Notes: you can come back to this discussion if you like after reading Part IV.) The -1 in $2 \times 2 - 1 = 3$ reflects the loss of freedom imposed by the restriction on the determinant.

$SU(3)$ correspondingly has $3 \times 3 - 1 = 8$ generators which in Note 33 of Book 8c we took to be the Gell'mann matrices—also with a factor 1/2.

Taking the electromagnetic inside connection to its empirical[1] limit, we get two new types of force-field, the "weak force" governed by $SU(2)$ and the "strong force" governed by $SU(3)$.

$$D_\mu = \partial_\mu - ig_1 \frac{Y}{2} B_\mu - ig_2 \frac{f_j}{2} W_\mu^j - ig_3 \frac{\lambda_k}{2} G_\mu^k$$

Here the g_d are coupling constants to give the strengths of the fields, the numerators (over the denominator 2) are the generators, and the final factor in each term is the field component. The superscripts j and k appear in pairs and are summed over.

The g_2 term contains three generators, which in Note 19 we will call $f_j, j = 1,2,3$ for reflection, and which are just the Pauli matrices.

[1]Not *logical* limit: the Standard Model does not know why these symmetries apply nor even if they are all.

$$
\begin{array}{ccc}
f_1(f_z) & f_2(f_x) & f_3(f_y) \\
\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} & \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} & \begin{pmatrix} & -i \\ i & \end{pmatrix}
\end{array}
$$

And in Note 21 we will see that the 2-by-2 rotation generators are half of these: $J_x = f_x/2$, $J_y = f_y/2$ and $J_z = f_z/2$.

These 2-by-2 matrices imply a 2-state system analogous to spins: the electron and its neutrino are examples of the two "states" of a more fundamental underlying "particle", the electron "state" corresponding, say, to spin-up, and the neutrino to spin-down. This explains the "spin" part of the name "isospin" used for weak-force states; explaining the "iso" part is left as an Excursion.

So the weak force has three fields, W^1_μ, W^2_μ and W^3_μ, each a 4-vector just as the electromagnetic field A_μ is a 4-vector. We will see in Note 25 that fields are particles: the weak-force fields are spin-1 bosons, as too are the EM and strong-force fields. The W-bosons frequently appear in the linear combinations

$$
\begin{aligned}
W^+_\mu &= -\frac{1}{\sqrt{2}}(W^1_\mu - iW^2_\mu) \\
W^-_\mu &= -\frac{1}{\sqrt{2}}(W^1_\mu + iW^2_\mu) \\
W^0_\mu &= W^3_\mu
\end{aligned}
$$

In the "electroweak" unified theory of electromagnetic and weak interactions, the EM field A_μ and the Z_μ boson are further linear combinations of B_μ and W^0_μ.

The strong force has eight fields, the "gluons" $G^k_\mu, k = 1, ...8$.

The thinking I have collapsed into this Note so far, which unites the electromagnetic and nuclear forces into a consequence of "inside connections", took a good half-century (from 1919 into the 1970s) to develop, and deserves a name. It is known, unhelpfully, as gauge theory for historical reasons. I thought of calling it WYMH theory after its primary developers, Hermann Weyl, C N Yang, R Mills and Peter Higgs, but instead I'll take Kane's suggestion [Kan93, p.36] that it really describes symmetries and invariants of the *phase* of the wave-function or quantum field, and call it *faze* theory. The variant in spelling is to distinguish the theory from other important technical uses of the word "phase".

In principle, we need only extend ∂_μ in the Lagrangian of a free particle with the inside connections in order to arrive at the "Standard Model" of the three forces of particle physics, In practice there are complications. The Standard Model is less a theory than a committee of theories,

An obvious complication is that each term in the expanded absolute slope is in a different mathematical category. The g_d term invokes a d-by-d matrix, so it must be reduced to the single number that is the phase by pre- and post-multiplying by a d-dimensional vector. For the weak force, for instance, the electron, e, and its neutrino, ν_e, can be regarded as components of a single, 2-vector, state, (e, ν_e), and this and its conjugate would be the 2-component vectors.

A more serious practical complication has to do with the *ranges* of the weak and strong forces. Both are short-range, for different reasons, as opposed to the long (infinite) ranges of gravity and electromagnetism.

In Note 25 below, quantum field theory will teach us that the range of a force is governed by the mass of its carrier particle. Electromagnetism, for instance, is carried by the photon, which is massless and gives it infinite range.

To give the short range of the weak force, the carriers—the W bosons, for instance—must have mass (and quite a lot of mass). Unfortunately, faze-theory bosons are obliged to be massless, so this explanation stalled until the mechanism of the Higgs field was proposed to lend mass to the bosons, essentially by slowing them down. (We know from $E^2 - p^2 = m^2$ that massless particles must travel at lightspeed, and vice-versa, so slowing them down effectively imparts mass.)

The symmetry of the inside connection must also be broken[2], and the Higgs mechanism does this, too, by an effect discovered in the explanation of superconductivity—a totally different field. (The symmetry is broken by the *ground state* of a Lagrangian (Hamiltonian, potential energy) which is itself perfectly symmetrical but with off-center minima.)

The apparent short range of the strong force is explained quite differently. The three dimensions of $SU(3)$ are due to three different types of "charge", which are called r, g and b. They are named after colours because they have some properties which are analogous to colours. For instance, the combination of all three, rgb, is neutral, like white light, as is the combination of a colour and its complementary, or anti-, colour, e.g., $r\overline{r}$. (These two kinds of combination give the baryons (three differently-coloured quarks) and mesons (quark-antiquark pairs), respectively.)

Unlike the photon, which carries no electric charge, the eight "gluons" (the $SU(3)$ bosons, corresponding to the eight generators) carry colour charge. Thought of as fields, this means that the "lines of force" carried by gluons between two colour charges do not spread to infinity the way electromagnetic (photons) lines of force do, but bunch into a tube between the two charges. The strong force thus acts like a spring, getting stronger the greater the separation. Pulling two colour charges apart, then, requires putting energy into the system in increasing amounts. Enough stretch and the spring will break, with the energy going into the production of a meson—a quark-antiquark pair. This can happen multiple times and these "meson jets" account for many of the multiple particles produced in high-energy accelerator collisions. And the lines of force break in this way at

[2]It is not surprising that the symmetry is not perfect: for example, the electon and its neutrino have widely different masses, half an MeV for the one and almost zero for the other. (Indeed, the Standard Model requires the neutrino masses be zero: the discovery that neutrinos "oscillate" among their three types and thereby must have mass is one of the refutations of the Standard Model which physics in now addressing, despite its miraculous success in calculating almost everything else.)

very short ranges.

Part IV. Quantum Field Theory: Matrix Quantum Mechanics

18. Introduction to Quantum Fields.
19. Small matrices.
20. Tensor products.
21. Spin.
22. Vectors and spinors,
23. Multiple and independent systems.
24. A simple field.
25. The Yukawa potential.
26. Perturbation approximations.
27. Fermions.
28. Slopes and antislopes of 2D numbers, etc.
29. Charge conservation and antimatter.
30. Relativistic quantum field theory redux, so far.

Part V. Functional Integrals

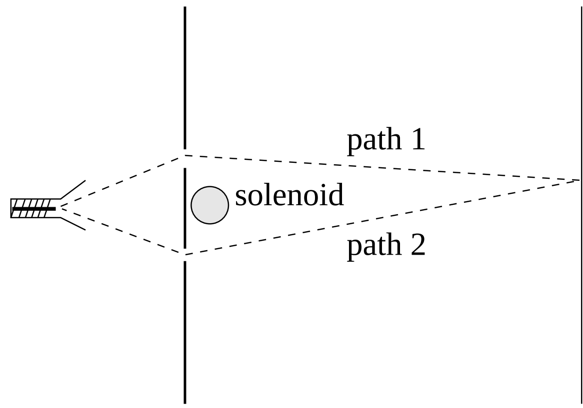
31. Path amplitudes.
32. Functionals.
33. Gaussian integrals.
33. Gaussian integrals.
34. Diagrams and QED.
35. Chirality and electroweak.
36. Green's functions.
37. Propagators.
38. Quantum Computing.
39. Binary Fourier transform.
40. Quantum Fourier transform.
41. Finding periods.
42. Quantum key distribution.
43. No cloning.
44. Database search.
45. Detecting and correcting errors.
46. Nonlocality: Einstein-Podolsky-Rosen.
47. Building a quantum computer.

II. **The Excursions**

You've seen lots of ideas. Now *do* something with them!

1. **Aharanov-Bohm effect.** In Notes 5 and 6 of Part I we introduced "potential momentum" and its conventional name "magnetic vector potential". Pre-quantum electromagnetism considered this quantity to be only a handy conceptual, and calculational, tool, not something

observable. Only the magnetic field (Note 6) was observed—although without the sign that appears in the math as I noted at the end of Note 6. In Note 13 we saw that the potential momentum is central to quantum electromagnetism and contributes to the phase of the wavefunction. Aharonov and Bohm pointed out that this is observable in the following modification to the electron two-slit experiment.



Here a solenoid is placed behind the screen and between the two slits in it. As we saw in Excursion *Visualizing magnetic fields* (Part I), the field $\vec{B}$ is essentially contained in the solenoid and we can ensure that it is zero at the two paths followed by the electron. So the electron encounters only $\vec{A}$. Nevertheless, the phase of the electron wavefunction is affected and the phase difference between the two paths has been calculated and measured to be proportional to the "magnetic flux" in the solenoid. See [FLS64a, pp.15-11,15-12].

2. Modify your MATLAB simulation `free2dSchroeGauss()` in Note 14 to `phase2dSchroeGauss()`. Build in code which reports on the trajectory of the peak of the wavefunction (x vs. t), its velocity and acceleration, and compare the results with electromagnetic theory (see Note 5).

3. **Maxwell's tensor and equations.** Referring to Note 15 and using Note 6 (equations EM1 and EM2 with $c = 1$), show that

$$(\partial_t, \partial_z, \partial_y, \partial_z)\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} = 4\pi E_C(\rho, j_x, j_y, j_z)$$

and hence that Maxwell's equations become

$$\partial_\mu(\partial_\mu A_\lambda - \partial_\lambda A_\mu) = 4\pi E_C j_\lambda$$

4. **Gauge theory.** The connection between electromagnetism and general relativity was first made by Hermann Weyl in 1919. Look up Weyl and find his mis-step. Ten years later Weyl introduced the German term "Eichinvarianz" for the symmetry we found in Note 16. "Eichen" translates to "gauge" or "calibrate", and the theory Weyl started has become known as gauge theory.
The word "gauge" is used often to describe different conventions for the A field in electromagnetism: for example, the Coulomb gauge is $\vec{\bigtriangledown} \cdot \vec{A} = 0$; the Lorentz gauge is $\vec{\bigtriangledown} \cdot \vec{A} = -\partial_t\phi/c^2$ (Think of the "gauge" conventions in railway building, meaning the distances separating the rails.) What is the relationship between this invariance of A and the 2-D-number rotation symmetry of the phase of the wavefunction?

5. Kane [Kan93, p.82] calculates the g_2 term contribution to the intersection Lagrangian we discuss in Note 17, Work through this example. On pp.44–46 he does an analogous calculation for protons and neutrons: why did Heisenberg invent the concept of "isospin" and why did Yang and Mills' theory based on this get initially rejected?

6. **Abelian and non-Abelian symmetries.** (This excursion looks ahead to Parts IV and V.) Ryder [Ryd85, §§3.3, 3.5, 3.6] elaborates the discussion of Note 17. Here is an outline, in parallel, of the discussions there of an Abelian faze theory (electromagnetism) and of a non-Abelian faze theory (hinting at the Weak force).
(In an Abelian group all elements commute; in non-Abelian groups they do not. The term honours Niels Henrik Abel, 1802–29.)

$$\begin{aligned}
\phi_1 + i\phi_2 &\longrightarrow (c - is)(\phi_1 + i\phi_2) \\
\phi_1 - i\phi_2 &\longrightarrow (c + is)(\phi_1 - i\phi_2) \\
\phi_1 &\longrightarrow c\phi_1 + s\phi_2 \approx \phi_1 + \Lambda_3\phi_2 \\
\phi_2 &\longrightarrow -s\phi_1 + c\phi_2 \approx -\Lambda_3\phi_1 + \phi_2
\end{aligned}$$

where c is $\cos(\Lambda_3)$ and s is $\sin(\Lambda_3)$ and in 3D we can consider the angle $\vec{\Lambda}$ to have three components of which so far we are discussing only the z-component, i.e., for a rotation in the x-y plane.

$$\begin{aligned}
\phi = (\phi_1 + i\phi_2)/\sqrt{2} &\longrightarrow e^{-i\Lambda}\phi \approx \phi - i\Lambda\phi \\
\phi^* = (\phi_1 - i\phi_2)/\sqrt{2} &\longrightarrow e^{i\Lambda}\phi \approx \phi + i\Lambda\phi
\end{aligned}$$

The equations remind us of a "cross-product" (see Note 22 of Part IV), except that in 2D only component Λ_3 in nonzero.

$$\vec{\Lambda} \times \vec{\phi} = \frac{1}{2}[\not{\Lambda}\ .\not{\phi}\ ]_-$$

$$\begin{aligned}
\phi_j &\longrightarrow \phi_j - (\vec{\Lambda} \times \vec{\phi})_j \\
\phi_1 &\longrightarrow \phi_1 - (\Lambda_2\phi_3 - \Lambda_3\phi_2) \\
\phi_2 &\longrightarrow \phi_2 - (\Lambda_3\phi_1 - \Lambda_1\phi_3) \\
\phi_3 &\longrightarrow \phi_3 - (\Lambda_1\phi_2 - \Lambda_2\phi_1)
\end{aligned}$$

We focus now on $\Delta\phi$ and $\Delta(\partial_\mu\phi)$ and how they transform under faze invariance.

$$\begin{aligned}
\phi &\longrightarrow \phi - i\Lambda\phi \qquad \Delta\phi = -i\Lambda\phi \\
\partial_\mu\phi &\longrightarrow \partial_\mu\phi - i(\partial_\mu\Lambda)\phi - i\Lambda\partial_\mu\phi
\end{aligned}$$

$$\Delta(\partial_\mu\phi)$$

$$= \begin{cases} -i\Lambda\partial_\mu\phi & \text{if } \Lambda \text{ const. : "global"} \\ -i\Lambda\partial_\mu\phi - i(\partial_\mu\Lambda)\phi & \text{if } \Lambda(t,x,y,z)\text{: "local"} \end{cases}$$

$$\begin{aligned}
\vec{\phi} &\longrightarrow \vec{\phi} - \vec{\Lambda} \times \vec{\phi} \qquad \Delta\vec{\phi} = -\vec{\Lambda} \times \vec{\phi} \\
\partial_\mu\vec{\phi} &\longrightarrow \partial_\mu\vec{\phi} - (\partial_\mu\vec{\Lambda}) \times \vec{\phi} - \vec{\gamma} \times \partial_\mu\vec{\phi} \\
\Delta(\partial_\mu\vec{\phi}) &= -(\partial_\mu\vec{\Lambda}) \times \vec{\phi} - \vec{\Lambda} \times \partial_\mu\vec{\phi}
\end{aligned}$$

Change ∂_μ to "covariant" slope: add field A_μ or $\vec{W}_\mu$. The first lines show the transformations A_μ and $\vec{W}_\mu$ must undergo to support faze invariance. In the electromagnetic case, this says that A_μ can change by adding a divergence without affecting the physics: the original "gauge" invariance.
The second lines give the covariant slopes in each case, choices that must be justified by the

following.

$$
\begin{aligned}
A_\mu &\longrightarrow A_\mu + \frac{1}{e}\partial_\mu \Lambda \\
D_\mu \phi &= (\partial_\mu + ieA_\mu)\phi \\
\Delta(D_\mu \phi) &= \Delta(\partial_\mu \phi) + ie(\Delta A_\mu)\phi \\
&\quad + ieA_\mu \Delta\phi \\
&= -i\Lambda\partial_\mu\phi - i(\partial_\mu\Lambda)\phi \\
&\quad + i(\partial_\mu\Lambda)\phi - iei\Lambda A_\mu\phi \\
&= -i\Lambda(\partial_\mu\phi + ieA_\mu\phi) \\
&= -i\Lambda(D_\mu\phi)
\end{aligned}
$$

which we can compare with $\Delta\phi = -i\Lambda\phi$: the covariant slope transforms in the same way as the original field, to preserve faze invariance. We interpret the covariant slope as follows. As well as the field ϕ changing from point to point in timespace, so does the internal angle Λ. The covariant slope isolates the change to that in ϕ.

$$
\begin{aligned}
\vec{W}_\mu &\longrightarrow \vec{W}_\mu - \vec{\Lambda} \times \vec{W}_\mu + \frac{1}{g}\partial_\mu \vec{\Lambda} \\
D_\mu \vec{\phi} &= \partial_\mu \vec{\phi} + g\vec{W}_\mu \times \vec{\phi} \\
\Delta(D_\mu \vec{\phi}) &= \Delta(\partial_\mu \vec{\phi}) + g(\Delta \vec{W}_\mu) \times \vec{\phi} \\
&\quad + g\vec{W}_\mu \times \Delta\vec{\phi} \\
&= -\vec{\Lambda} \times \partial_\mu \vec{\phi} - g(\partial\vec{\Lambda}_\mu) \times \vec{\phi} \\
&\quad - g(\vec{\Lambda} \times \vec{W}_\mu) \times \vec{\phi} + g(\partial\vec{\Lambda}_\mu) \times \vec{\phi} \\
&\quad - g\vec{W}_\mu \times (\vec{\Lambda} \times \vec{\phi}) \\
&= -\vec{\Lambda} \times \partial_\mu \vec{\phi} \\
&\quad - g[(\vec{\Lambda} \times \vec{W}_\mu) \times \vec{\phi} + \vec{W}_\mu \times (\vec{\Lambda} \times \vec{\phi})] \\
&= -\vec{\Lambda} \times \partial_\mu \vec{\phi} - g\vec{\Lambda} \times (\vec{W}_\mu \times \vec{\phi}) \\
&= -\vec{\Lambda} \times (\partial_\mu \vec{\phi} + g\vec{W}_\mu \times \vec{\phi}) \\
&= -\vec{\Lambda} \times D_\mu \vec{\phi}
\end{aligned}
$$

which we can compare to $\Delta\vec{\phi} = -\vec{\Lambda} \times \vec{\phi}$.
The fourth line above uses the identity

$$(\vec{A} \times \vec{B}) \times \vec{C} + (\vec{B} \times \vec{C}) \times \vec{A} + (\vec{C} \times \vec{A}) \times \vec{B} = 0$$

(check this from the definition of $\times$), from which

$$(\vec{A} \times \vec{B}) \times \vec{C} + \vec{B} \times (\vec{A} \times \vec{C}) = \vec{A} \times (\vec{B} \times \vec{C})$$

From the "potential" fields A_μ and $\vec{W}_\mu$ we get the EM (electromagnetic) field $F_{\mu\nu}$ and a Y-M (Yang-Mills) field $\vec{W}_{\mu\nu}$ respectively.
The Lagrangians $\mathcal{L}$ that follow are in the end justified by giving the right equations, which we know to be Maxwell's in the EM case (see Note 35 in Part V), and which are analogous in the Y-M case.
Both times, the mass m (in the Klein-Gordon part of the Lagrangian) of the carrier particle must be 0 by an argument we give below. For EM this is the photon, whose mass has never been found to exceed 0. For Y-M, the electroweak carrier bosons, W_1, W_2, W_3, do not have 0 mass, so the theory must be refined to match observation.

$$
\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\
\mathcal{L} &= (D_\mu \phi)(D^\mu \phi) - m^2 \phi^* \phi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}
\end{aligned}
$$

From this Lagrangian the Euler-Lagrange equations give Maxwell's equations (see Note 35 of Part V)

$$\partial_\mu F^{\mu\nu} = e j^\nu$$

with the "4-current"

$$j^\nu = i(\phi^* D^\nu \phi - \phi D^\nu \phi^*)$$

$$
\begin{aligned}
\vec{W_{\mu\nu}} &= \partial_\mu \vec{W_\nu} - \partial_\nu \vec{W_\mu} + g\vec{W_\mu} \times \vec{W_\nu} \\
\Delta \vec{W_{\mu\nu}} &= -\vec{\Lambda} \times \vec{W_{\mu\nu}} \\
\mathcal{L} &= (D_\mu \vec{\phi}) \cdot (D^\mu \vec{\phi}) - m^2 \vec{\phi} \cdot \vec{\phi} - \frac{1}{4} \vec{W^{\mu\nu}} \cdot \vec{W_{\mu\nu}}
\end{aligned}
$$

You can confirm the second line above by again using the cross-product identity,
The extension of Maxwell is

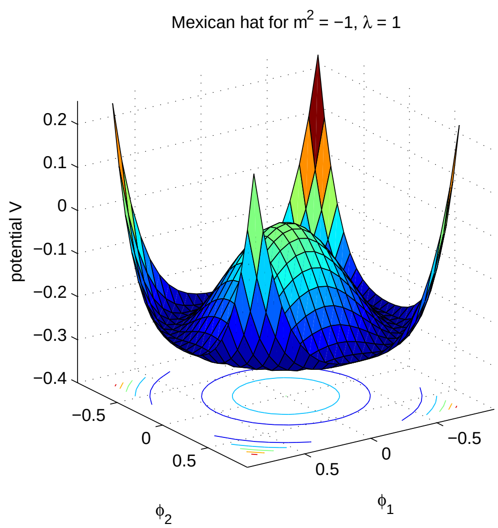
$$D_\mu \vec{W^{\mu\nu}} = g \vec{j^\nu}$$

with the source term defined

$$\vec{j^\nu} = (D^\mu \vec{\phi}) \times \vec{\phi}$$

An important consequence of the non-commutativity of the generators of the Y-M group is the nonlinearity of the equations for $\vec{W^{\mu\nu}}$ with the result that even if the source $j^\nu = 0$, the $\vec{W}$ field is its own source. $D_\mu \vec{W^{\mu\nu}} = 0$ gives $\partial_\mu W^{\mu\nu} = -g\vec{W_\mu} \times \vec{W^{\mu\nu}}$
Note also that $\mathcal{L}$ has terms cubic and quartic in $\vec{W}$. We discuss cubic terms, and also mention quartic terms, in Note 33 of Part V.
Finally a non-zero mass in the Lagrangian, say $m^2 \vec{W_\mu} \cdot \vec{W^\mu}$, will give $D^\mu \vec{W_{\mu\nu}} = g j_\nu + m^2 \vec{W_\nu}$ but this is not faze invariant. So faze invariance requires massless carrier particles, something which is not true for the weak force.

7. **Goldstone and Higgs mechanisms.** When the lowest-energy, or vacuum, state is less symmetric than the Lagrangian, we will want to calculate based on the vacuum state, and this changes things when we have either a global or a local faze symmetry as well.
Let's look at a potential $V(\phi, \phi^*) = m^2 \phi^* \phi + \lambda(\phi^* \phi)^2$ where m^2 is usually the mass but here we're going to let m^2 be negative, and where the quartic $(\phi^* \phi)^2$ term is something we'll mention in Note 33 of Part V. Here is a plot of this potential



where

$$
\begin{aligned}
\phi &= \phi_1 + i\phi_2 \\
\phi^* &= \phi_1 - i\phi_2
\end{aligned}
$$

and where m^2 is negative: if m^2 were positive there would be no bump in the middle.
(The shape is called "Mexican hat" because, if it were plotted using a circular grid for ϕ_1 and ϕ_2 instead of a square one, it would look like an exaggerated sombrero.)
This potential has circular symmetry, but an object, say a ball-bearing, placed at $(\phi_1, \phi_2) = (0, 0)$ would not stay there but would fall to somewhere on the circle, on the inside part of the brim, that marks the lowest part. But it can fall only at one point of that circle, not everywhere, so once the system is in that one of many lowest states, the symmetry has been broken. It is said to be spontaneously broken: the system is still symmetrical but now its state is not.
We must review our vocabulary here. Notice that I've discussed and shown V in terms of ϕ_1 and ϕ_2, not x and y. V is a function of *fields*, not of coordinates. That is because relativity does not distinguish space from time, yet the traditional potential $V(x, y, z)$ is a function of space only. Nonetheless, because $V(\phi_1, \phi_2)$ gives energy, we call it a "potential".
And so the minimum value of V identifies not a location in space but a state of *field*s called the vacuum state: the state of lowest energy.
So let's find the radius a for all the possible vacuum states. Since

$$\phi^*\phi = \phi_1^2 + \phi_2^2$$

we can express the potential in terms of

$$\mid \phi \mid = \sqrt{\phi^*\phi} = \sqrt{\phi_1^2 + \phi_2^2}$$

as

$$V(\mid \phi \mid) = m^2 \mid \phi \mid^2 + \lambda \mid \phi \mid^4$$

and take the slope of this to find its maximum and minima

$$0 = \partial_{|\phi|} V = 2m^2 \mid \phi \mid + 4\lambda \mid \phi \mid^3$$

giving the maximum at $\mid \phi \mid = 0$ and the minima at

$$\mid \phi \mid = a = \sqrt{\frac{-m^2}{2\lambda}}$$

What we will do is to adjust the fields to be based on the vacuum, making this the state of lowest energy: $\phi \longrightarrow \phi - a$.
Now we take parallel paths for global faze symmetry and for local faze symmetry. In the following we will work with Lagrangians of the form $\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi^*) - V(\phi, \phi^*)$. We'll consider the coefficients of quadratic terms such as $\phi^*\phi$ to be squares of masses. Each field with such a term will be understood to have excitations which are carriers having that mass. Any field without such a term will be understood to have massless excitations which are carriers. The justification for force carriers being particles with or without mass is in Note 25 of Part IV. We won't pay any attention to higher powers of $\phi^*\phi$ for this discussion, which is about carrier masses. They come up in Note 33 of Part V.

Global faze: Goldstone bosons

Historically these ideas were initiated by Y Nambu in 1960 and J Goldstone in 1961.
The Lagrangian

$$\mathcal{L} = (\partial_\mu\phi)(\partial^\mu\phi^*) - m^2\phi^*\phi - \lambda(\phi^*\phi)^2$$

is invariant under global faze

$$\phi \longrightarrow e^\Lambda\phi$$

with vacuum at

$$\phi_0 = a = \sqrt{\frac{-m^2}{2\lambda}}$$

(a being real, we have selected the ϕ_1 direction for the ball-bearing to have fallen in) so the new state

$$\eta + i\xi = \phi - a$$

or

$$\phi = a + \eta + i\xi$$

and we can rewrite

$$\begin{aligned}\mathcal{L} &= (\partial_\mu\eta - i\partial_\mu\xi)(\partial^\mu\eta_i\partial^\mu\xi)\\ &\quad -m^2(a+\eta-i\xi)(a+\eta+i\xi)\\ &\quad -\lambda((a+\eta-i\xi)(a+\eta+i\xi))^2\\ &= (\partial_\mu\eta)(\partial^\mu\eta) + (\partial_\mu\xi)(\partial^\mu\xi)\\ &\quad -m^2(a^2+2a|eta+\eta^2+\xi^2)\\ &\quad -\lambda(a^4+4a^3\eta+6a^2\eta^2\\ &\quad +4a\eta^3+4a\eta\xi^2+2\eta^2\xi^2+\eta^4+\xi^4)\end{aligned}$$

Using $a^2 = -m^2/(2\lambda)$ and ignoring all constant terms because changing the energy by a constant amount has no physical effect, we get

$$\begin{aligned}\mathcal{L} &= (\partial_\mu\eta)(\partial^\mu\eta) + (\partial_\mu\xi)(\partial^\mu\xi) - 4\lambda a^2\eta^2\\ &\quad -4\lambda a\eta(\eta^2+\xi^2) - \lambda(\eta^2+\xi^2)^2\end{aligned}$$

In this result, η^2 appears with coefficient $-4\lambda a^2$, so the field η has mass $2a\sqrt{\lambda}$.
On the other hand, ξ^2 does not appear, so the field ξ has mass 0.
So, under global faze symmetry, going to the vacuum state (and breaking the circular symmetry the Lagrangian has) turns two massive scalar fields, ϕ_1 and ϕ_2 into a massive scalar field η and a massless scalar field ξ. The latter is called a *Goldstone boson.*

Local faze: Higgs bosons

Historically these ideas were initiated by F. Englert and R. Braut then P. Higgs then G S Guralnik, C R Hagen and T W R Kibble, all in 1964.
Looking at two scalar fields as in the Global faze example, plus a photon field from Excursion *Abelian and non-Abelian symmetries*, we have a term in the Lagrangian involving $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and we must switch to the covariant slope $D_\mu = \partial_\mu + ieA_\mu$. The Lagrangian is

$$\mathcal{L} = (D_\mu\phi)(D^\mu\phi) - m^2\phi^*\phi - \lambda(\phi^*\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

It is invariant under the local faze transformation

$$\begin{aligned}\phi &\longrightarrow e^{i\Lambda(t,x,y,z)\phi}\\ A_\mu &\longrightarrow A_\mu + e\partial_\mu\Lambda(t,x,y,z)\end{aligned}$$

The minimum a and the vacuum fields η and ξ are identical to the Goldstone case, so expanding the Lagrangian

$$\begin{aligned}\mathcal{L} &= (\partial_\mu + ieA_\mu)(a+\eta-i\xi)\\ &\quad +(\partial^\mu - ieA^\mu)(a+\eta+i\xi)\\ &\quad -m^2(a+\eta-i\xi)(a+\eta+i\xi)\\ &\quad -\lambda((a+\eta-i\xi)(a+\eta+i\xi))^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\\ &= (\partial_\mu\eta)(\partial^\mu\eta) + (\partial_\mu\xi)(\partial^\mu\xi) - e^2a^2A_\mu A^\mu\\ &\quad -4\lambda a^2\eta^2 - ea(A_\mu\partial^\mu\xi + A^\mu\partial_\mu\xi)A + \cdots\\ &\quad -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\end{aligned}$$

as in the Goldstone case, where I've left out constant terms, cubic terms such as $-4\lambda a\eta(\eta^2+\xi^2)$, $-e^2aA_\mu A^\mu\eta$, etc., and quartic terms such as $-e^2A_\mu A^\mu\eta^2$.
The two quadratic terms assign mass ea to the photon and mass $2a\sqrt{\lambda}$ to the η field.
Ryder [Ryd85, p.302] uses the faze transformation

$$\begin{aligned}\eta &\longrightarrow \eta - \Lambda\xi\\ \xi &\longrightarrow \xi + \Lambda\eta + \Lambda a\end{aligned}$$

to set $\xi = 0$ by a suitable choice of Λ, so the $A_\mu\partial^\mu\eta$ terms no longer bother us.

Under local faze symmetry, going to the vacuum state turns two massive scalar fields, ϕ_1 and

ϕ_2 (1 degree of freedom each) and one massless photon field A_μ (two degrees of freedom) into a massive scalar field η and a massive "photon" (3 degrees of freedom).
The "degrees of freedom" of a photon are the number of directions it can be polarized in: two transverse polarizations plus, only if it has mass and so travels at less than lightspeed, one longitudinal polarization. (See Note 13 of Week 6.)
We don't observe massive photons, so evidently we have further to go with this theory.

We take the Higgs case one step further, to non-Abelian faze. This gets us closer, but still not all the way, to the unification of electromagnetism and the weak force in "electroweak theory".
The Lagrangian is summed over multiple fields ϕ_j (which are not 2D (complex) fields).

$$\mathcal{L} = \frac{1}{2}(D_\mu\phi_j)(D^\mu\phi_j) - \frac{m^2}{2}\phi_j\phi_j - \lambda(\phi_j\phi_j)^2 = \frac{1}{4}W_{j\mu\nu}W_j^{\mu\nu}$$

with the covariant slope

$$D_\mu\phi_j = \partial_\mu\phi_j + g\varepsilon_{jk\ell}W_{j\mu}\phi_\ell$$

and where

$$W_{j\mu\nu} = \partial_\mu W_{j\nu} - \partial_\nu W_{j\mu} + g\varepsilon_{jk\ell}W_{k\mu}W_{\ell\nu}$$

The ε symbol appears in Excursion *Levi-Civita symbol, alternating tensors* of Book 11c (Part I) and is a more general way of writing the cross-product of the previous Excursion. In the special case of three fields ϕ_1, ϕ_2, ϕ_3 and hence $\vec{W}_1$, $\vec{W}_2$, $\vec{W}_3$ we can expand these to

$$\begin{aligned}
D_\mu\phi_1 &= \partial_\mu\phi_1 + g(W_{2\mu}W_{3\nu} - W_{3\mu}W_{2\nu}) \\
D_\mu\phi_2 &= \partial_\mu\phi_2 + g(W_{3\mu}W_{1\nu} - W_{1\mu}W_{3\nu}) \\
D_\mu\phi_3 &= \partial_\mu\phi_3 + g(W_{1\mu}W_{2\nu} - W_{2\mu}W_{1\nu})
\end{aligned}$$

and

$$\begin{aligned}
W_{1\mu\nu} &= \partial_\mu W_{1\nu} - \partial_\nu W_{1\mu} + g(W_{2\mu}W_{3\nu} - W_{3\mu}W_{2\nu}) \\
W_{2\mu\nu} &= \partial_\mu W_{2\nu} - \partial_\nu W_{2\mu} + g(W_{3\mu}W_{1\nu} - W_{1\mu}W_{3\nu}) \\
W_{3\mu\nu} &= \partial_\mu W_{3\nu} - \partial_\nu W_{3\mu} + g(W_{1\mu}W_{2\nu} - W_{2\mu}W_{1\nu})
\end{aligned}$$

We move ϕ_3 to the vacuum state, and use a faze transformation to remove ϕ_1 and ϕ_2 [Ryd85, p.304]:

$$\phi_1, \phi_2, \phi_3 \longrightarrow 0, 0, a + \chi$$

simplifying

$$\begin{aligned}
D_\mu\phi_1 &= g(a+\chi)W_{2\mu} \\
D_\mu\phi_2 &= -g(a+\chi)W_{1\mu} \\
D_\mu\phi_3 &= \partial_\mu\chi
\end{aligned}$$

and giving

$$\begin{aligned}
(D_\mu\phi_j)(D^\mu\phi_j) &= (g(a+\chi)W_{2\mu})^2 + (g(a+\chi)W_{1\mu})^2 + (\partial_\mu\chi)^2 \\
&= g^2(a^2 + 2a\chi + \chi^2)((W_{2\mu})^2 + (W_{1\mu})^2) + (\partial_\mu\chi)^2
\end{aligned}$$

So

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\chi)^2 - \frac{1}{2}a^2g^2((W_{2\mu})^2 + (W_{1\mu})^2) - \frac{1}{4}(\partial_\mu W_{j\nu} - \partial_\nu W_{j\mu})^2 - 4a^2\lambda\chi^2 + \text{cubic} + \text{quartic}$$

Thus, for this $O(3)$ symmetry, 3 massive scalar fields (ϕ_j: 3 DoF) plus 3 massless vector fields ($\vec{W}_j, j = 1,2,3$: 6 DoF) have become 1 massive scalar field (χ: 1 DoF) plus 2 massive vector fields ($\vec{W}_j, j = 1,2$: 6 DoF) plus 1 massless vector field ($\vec{W}_3$: 2 DoF).
This Excursion is based on [Ryd85, §§8.1, 8.3] with support from [LP01, §§13.4–6] and [Kan93, §§8.1–3]. Kane [Kan93, pp.93,91] and Moriyasu [Mor83, pp.28,51f.] discuss the obligation mentioned in Note 17 for faze bosons to be massless. Ryder in Ch.8 discusses the Higgs mechanism and Moriyasu does this in Ch.7, both linking it to superconductivity.

8. **Quarks and vacuum.** Kane [Kan93, Ch.15] describes the meson-jet phenomenon of Note 17 and elaborates on "quark confinement" and other aspects of the strong force. Moriyasu [Mor83, pp.129ff.] adds a discussion of "vacuum polarization" which explains the infinities suffered by perturbation-based calculations in all the field theories from $U(1)$ QED (quantum electrodynamics) to $SU(3)$ QCD ("quantum chromodynamics"—for the colours, of course). "Renormalization" is used, which modestly admits that the field theory does not apply above certain energies, thereby effectively discretising timespace (the uncertainty principle says that large energentum corresponds to small timespace intervals, so upper limits on the one correspond to lower limits on the other). See [Zee10, p.146]. Why does faze symmetry guarantee renormalizability?

9. Any part of the Prefatory Notes that needs working through.

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