Excursions in Computing Science:
Book 8d. Rocket Science.
Part I Propulsion.

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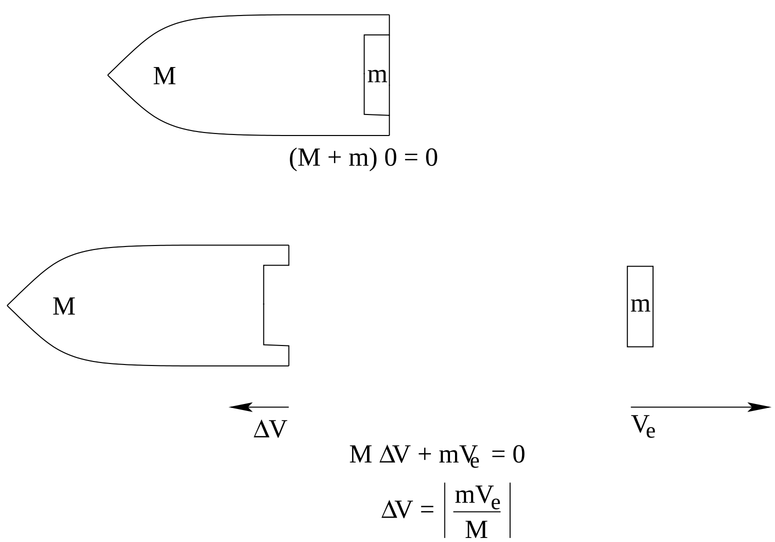
I. Prefatory Notes

1. The rocket equation. When we walk we move ourselves by pushing against the ground with our feet. Bicycles and cars push with their wheels. Boats and airplanes push the water or air with their propellors or maybe with a jet turbine.

In space there is nothing to push against. Rockets use instead a principle from physics, the *conservation of momentum.*

A vehicle of mass M moving at velocity V has momentum MV. Unless some *external* force is applied, this never changes no matter what.

Here is a rocket whose mass without fuel is M and with fuel is $M + m$.



Initially the fuel is in the rocket and both are at rest: velocity 0.

But when we burn the fuel its mass (in the form of hot gases) shoots out the back of the rocket with an "exhaust velocity" V_e which is specific to the fuel and the rocket.

After the burn, the total momentum is still 0, because momentum is conserved.

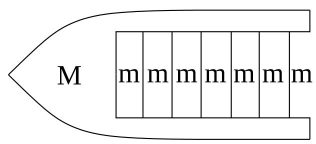
But this momentum has two parts, $M\Delta V$ and mV_e. The empty rocket is now moving with velocity ΔV, which we must figure out. (This "delta-vee" is the traditional rocketry term, so I am using it. The two symbols together stand for only one number, the velocity of the rocket (strictly, the *change* in the velocity of the rocket), and not for two numbers multiplied.)

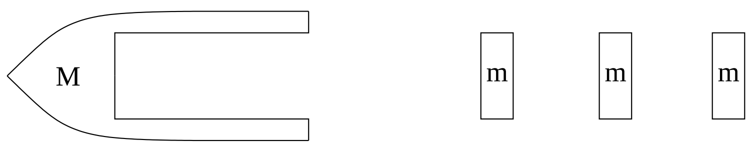
Since the sum of the two parts is zero, we can calculate ΔV.

$$\Delta V = -\frac{mV_e}{M}$$

In the picture, since we don't really care that the two velocities, ΔV and V_e, have opposite signs, I've taken the absolute value, so that ΔV is explicitly positive.

Now really a rocket does not burn and eject all its fuel in one go, so we must refine the picture. Here is the rocket again with more fuel but broken into chunks of mass m each.





$$\Delta V = \frac{mV_e}{M+(n-1)m} + \frac{mV_e}{M+(n-2)m} + \ldots + \frac{mV_e}{M+(0)m}$$

So initially the rocket and fuel mass $M + nm$ (with $n = 7$ in the picture for concreteness).

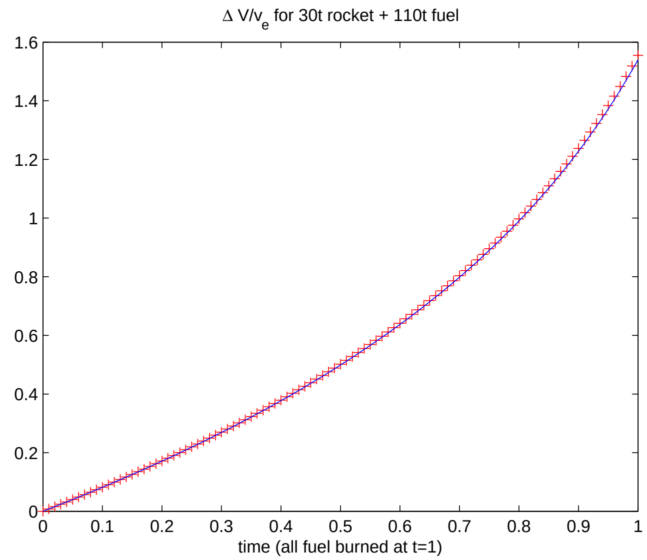
After all the fuel is burned, the ΔV of the rocket is the sum of the individual changes in velocity. Note that these are fractions as before, with mV_e on top, but now different denominators because the rocket gets progressively lighter as each chunk is burned. (And I've been cavalier again about signs.)

(Also the velocity of each chunk, relative to the original frame in which everything was at rest, is complicated to express, but what matters is only that it left the rocket with velocity V_e relative to the rocket at the time.)

To find the final velocity of the rocket, all we have to do is add these pieces up.

What we must do, though, is make the chunks smaller and smaller (and get burned and ejected faster and faster), in order to approximate the continuous burning of the rocket.

Here is the succession of 100 such steps for a rocket with empty mass 30 tonnes and fuel mass 110 tonnes.



The blue line, hiding among the 100 red '+' symbols, is what we might get if we used "integral calculus" to find the answer instead of adding up all the terms. The calculus also gets for us the answer for an arbitrarily large number of arbtrarily small steps.

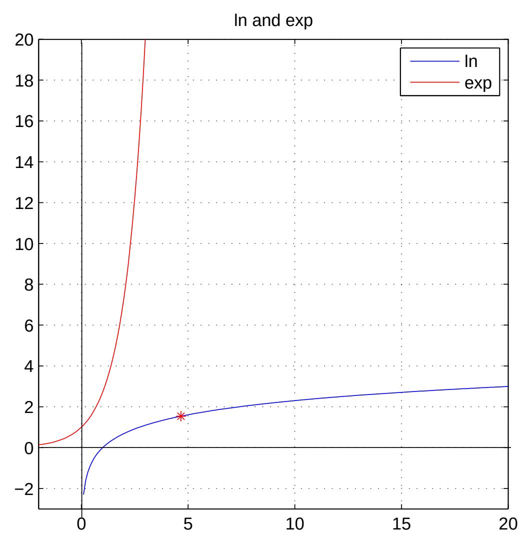
That answer is

$$\Delta V = V_e \ln \frac{M_0}{M_f} = 1.54 V_e$$

where $M_f = M$, the final mass of the rocket, i.e., its empty mass of 30t, and $M_0 = M + nm$, the initial mass of rocket and fuel.

The 1.54 is the highest point on the curve. I plotted the whole curve to show all the intermediate ΔV before all the fuel is spent.

The "ln()", produced above by calculus, is the "natural logarithm", which you may find on your calculator either as "ln" or as "log". It is the function plotted in blue below, and the particular value $\ln(140/30) = 1.54$ is shown as a red asterisk.



This plot also shows the closely related function "exp()", the "exponential". You can see where to put a mirror in this plot which would reflect the blue curve into the red curve and vice-versa.

2. Specific impulse. None of this tells us exactly how fast a rocket might go. We will need to say something about V_e, the exhaust velocity.

This depends on the rocket and especially on the fuel, but a typical figure for kerosene, say, and an oxidizer such as liquid oxygen (LOX), is around 3 Km/s, and liquid hydrogen with liquid oxygen might be over 4 Km/s.

So a kerosene/LOX rocket with the masses used in the previous Note would have a ΔV of four and a half kilometers per second.

(Jumping ahead to Part II, the "Delta-V" table at the end of Note 21 on *Travelling the solar system* (p.25) has very few trips within this range of ΔV. We would have to push it even to get from orbit around Earth to orbit around our Moon.)

The usual way in rocketry to express exhaust velocity is to give it as "specific impulse", $\mathrm{I_{sp}}$, which is measured, strangely, in seconds.

The reason for this originates in the distinction between mass and weight. Mass is a measure of both inertia (resistance to changes in motion) and gravitational attraction. Weight is the force that your bathroom scale measures when you step on it.

Of course, that force is the gravitational attraction of the Earth for your body. Suppose your mass is m kilograms. Then, since $F = ma$ and the acceleration a is just $g = 9.8$ m/s/s, the acceleration due to gravity at the Earth's surface, your scale experiences a force of mg "newtons".

Naturally, a scale which gives the weight of a 50 Kg person as 490 newtons would not sell very well. So your scale divides the weight it records by 9.8 and reports your mass as 50 Kg.

Similarly, the thrust (a force) produced by a rocket by exhausting burned fuel at velocity V_e is conventionally divided by g.

Dividing a velocity V_e (m/s) by an acceleration g (m/s/s) winds up with just seconds (s) as the unit of measurement. This is the specific impulse.

$$\mathrm{I_{sp}} = V_e/g \qquad\qquad V_e = g\times\mathrm{I_{sp}}$$

An important advantage of defining specific impulse in this way is that it is the same in metric (meters-kilograms-seconds) units and in British (feet-pounds-seconds) units (and in British usage, pounds may mean force as well as mass).

Another way of thinking about specific impulse considers the *thrust*. This is a force, which is the rate of change of momentum. Since the ejection of a mass m of propellant changes the momentum of the rocket by mV_e, we can think of $m = \dot{m}t$ where t is time and $\dot{m}$ is the *rate* of ejection of propellant. Then the thrust is $\dot{m}V_e$, which we get by dividing by t. If the t we divided by were the specific impulse V_e/g then the thrust is $mV_e/(V_e/g) = mg$ which is just the "weight" (at the Earth's surface) of that mass m of propellant. It is misleading, though, to think of $\mathrm{I_{sp}}$ as the time it takes to convert m of propellant into mg of thrust.

3. Fuels. Note 1 showed that the speed a rocket can achieve, its ΔV, depends gently on the amount of fuel it carries—the ratio of fuelled to empty masses—and strongly on the velocity with which the fuel is exhausted to provide momentum to the rocket. And, very conveniently, on nothing else—such as the force or "thrust" from the rocket.

So we'll focus on the exhaust velocity in this Note. It depends on the type of fuel we choose, so the ΔV we need will determine the fuel we must use.

This exhaust velocity depends on the energies of the particles in the exhaust.

Specifically it depends on the *kinetic energy* which for a particle of mass m and velocity v is

$$\text{K.E.} = \frac{mv^2}{2}$$

The particles that result from combustion in chemical rockets are molecules or smaller, as are the particles heated by nuclear reactions in other rockets.

The appropriate energy units for such particles are *electron volts* (eV).

I won't try to define these units here but we can look at various numbers of eV and their associated velocities and temperatures. To associate a velocity with an energy, we need to fix the mass m, and I'll choose $m = 1$ AMU (atomic mass unit), the mass of a hydrogen atom.

I'll quantify the energies by prefixes: m ("milli") means 1/1000; K ("kilo") means 1000; M ("mega") means a million; and G ("giga") means a thousand million which we can call a billion although just giga will do.

Thus, 25.9 meV is 25.9 milli electron volts or 25.9/1000 of an electron volt. And 938.28 MeV is 938.28 mega electron volts or 938.28 million electron volts.

To associate a temperature with an energy we need *Boltzmann's constant*

$$k_B = 0.0862 \text{ meV}/^o\text{K}$$

(Book 9c Part II, Note 17, footnote to Note 12)

So a warm summer temperature of 27^{o}C (Celcius), which is 300^{o}K (Kelvin)—just add 273, (or 81^{o}Fahrenheit), imparts a kinetic energy of 25.9 meV (on the average) to all molecules.

Energy	1meV ("thermal")	1eV (chemical)	1KeV	1MeV (nuclear)	1GeV (annihilation)
Temperature	12^{o}K	12,000^{o}K	12M^{o}K	12G^{o}K	12T^{o}K
Proton velocity	440 m/s	14 Km/s	440 Km/s	14 Mm/s	?? Mm/s

The temperature row introduces one more prefix, T ("tera") meaning a million million. I've called the milli-electron-volt range "thermal", although I have associated temperatures with all energies, because only this range gives temperatures we would call normal.

I've added a row "Proton velocity" to indicate exhaust speeds (for protons): protons mass 1.67_{10}-27 Kg and 1 eV = 1.6_{10}-19 joules. For heavier exhaust particles, these velocities must be divided by the square root of the ratio of masses, e.g., by 4 for oxygen ions.

The "chemical" energy range is that of the electrons orbiting the nucleii of atoms. It corresponds somewhat to the energy of visible light. When an electron "falls" from one orbit to an orbit closer to the nucleus the difference in energy appears as photons—quantities of electromagnetic radiation, including light. The formula for the frequency f of these electromagnetic waves is (Week 7a Note 4)

$$f = \frac{E}{h}$$

where $h = 4.136_{10}{-}15$ eV-seconds is *Planck's constant*, and, for the wavelength λ, from $f\lambda = c$ where c is the speed of the light, the formula is

$$\lambda = \frac{ch}{E}$$

Visible light has λ in the range of 750 nm (nanometers: a nano is the inverse of a giga, one billionth) to 380 nm (the most energetic), corresponding to energies of 1.65 eV for red light and 3.26 eV for violet.

The "nuclear" range has energies some million times that of the "chemical". When uranium 235 "fissions" into two pieces averaging half the size, the cesium 141 flies off at 10 megameters per second and the rubidium 45 at 15 megameters per second in the opposite direction. Notice that the numbers I gave don't add up: some mass has been converted into this energy of motion (124MeV), from $E = mc^2$,

When two deuterons "fuse" to form helium, as happens in the Sun, the relevant velocity is 34 megameters per second, and fusion works out to be some eight times more effective at converting mass to energy than fission. (Week 7a Notes 14 and 13.)

The last entry in the velocity row would have been 440 megameters per second. This exceeds lightspeed at 300 megameters per second. The $mv^2/2$ formula breaks down in that the mass m is no longer independent of velocity. Much of the energy goes into increasing the mass, from $m = E/c^2$ ($E = mc^2$), so the velocity v cannot exceed lightspeed c. So I guess that entry should be 299. The 14 Mm/s entry, being 5% of c, is also a little iffy.

Exhaust velocities achievable with fuels actually used in current rockets are naturally somewhat less than these ideal numbers.

Here are some potential rocket fuels and typical possible V_e and $\mathrm{I_{sp}}$. (Adapted from https://www.sciencedirect.com/topics/earth-and-planetary-sciences/specific-impulse)

Fuel	V_e(m/s)	$\mathrm{I_{sp}}$ (s)
Chemical, solid	2800	290
Chemical, liquid	3000–4500	300–460
Fission, thermal	9000–10,000	900–1000
Fission, electric	50,000	5000
Ion thrusters	25,000–100,000	2500–10,000
Fusion	up to 1,000,000	up to 100,000
Antimatter	up to 20,000,000	up to 2,000,000

(These numbers are not precise and we can often get away with treating g as just 10.)

Chemical rockets.

For the chemical rockets, the combustion products are usually heavier than hydrogen atoms and so move much more slowly for a given energy. A kerosene and liquid oxygen (LOX) rocket produces carbon dioxide, CO_2, with mass $12+2\times16 = 44$ AMU, so for a given energy its velocity is $\sqrt{44} = 6.6$ times lower than that of the hydrogen atom: 2.1 Km/s at 1 eV, for instance.

A rocket burning liquid hydrogen and liquid oxygen exhausts steam, H_2O, with mass $2\times1+16 = 18$, which will thus move $4\frac{1}{4}$ times slower, giving an exhaust velocity of 3.3 Km/s at 1 eV. (So the liquid chemical entry above corresponds to an energy of about 1.5 eV per particle.)

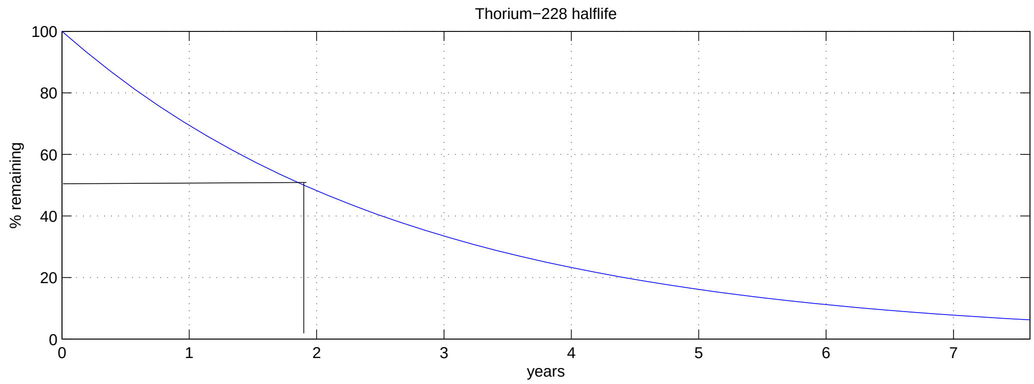
Note that chemical rockets, to be able to burn their fuel in the vacuum of space, must carry their own *oxidizer*, such as LOX, as well as the substance it combines with.

Rockets using radioactive decay.

Nuclear rockets get their heat from fission or just the heat of radioactive decay. So they do not need to carry oxidizer. And they can work with something lighter than those molecules—such as hydrogen—for their "reaction mass". Transferring the heat to the exhaust particles is tricky so the table does not show the million-fold increase over chemical that the principles would lead us to expect.

A simple nuclear-powered rocket which does not require a fission reactor just uses radioactive decay to heat a propellant. Thorium-228 decays in seven steps to lead-208 with a "half-life" of 1.9 years, emitting five *alpha particles* (helium nucleii of two protons and two neutrons) and two *beta particles* (electrons), with a total energy release of 37.32 MeV.

Here is a picture of the decay of Th-228. It never disappears completely so the way to give a time for how fast it decays is to say how long it takes for half of it to be gone: its *halflife*.



I've shown the first half life. But you can see that after another 1.9 years, half again of what remained is gone, leaving a quarter after 3.8 years. And so on.

The thing about a "cascade" decay is that the intermediate products also decay radioactively, and with halflives shorter than that of the parent. So the seven steps, emitting four alphas then a beta then an alpha then a beta (actually there is an alternative at the end, with the last two emissions exchanged), can be thought of as one decay from thorium to stable lead.

The energy released, 37.32 MeV for the 228 AMU of the thorium atom, can be expressed as joules per gram.[1]

$$\frac{37\text{ MeV}}{228\text{ AMU}} \rightarrow \frac{37\text{ MeV} \times 0.16_{10}{-}12\text{ J/MeV}}{228\text{ AMU} \times 1.66_{10}{-}24\text{ g/AMU}} = 15.6\text{ GJ/g}$$

In a half life this is 7.8 GJ/g. In a more familiar measure of energy, it is 2180 kilowatt-hours per gram in the 1.9 years, and that is a power output of 130 watts per gram. So a gram of thorium-228 can light eight or ten LED bulbs, a bright roomful of light.

At the end of Note 21 of Part II, the energy required to move from Earth's orbit to Mars' orbit is given as 0.3 GJ per kilogram of spacecraft. So a 100-tonne ship would need 30TJ which would completely consume about 2Kg of Th-228. But in the 0.7 years (259 days) which another table in that Note says it would take to get from Earth to Mars, that 30TJ would need some 10Kg of Th-228.

The temperature associated with 37 MeV would be 429 G^{o}K, and protons at that temperature would leave the rocket nozzle at lightspeed. This is misleading. Such temperatures would burn the engine up or at least melt it and the whole spacecraft.

We must distinguish fuel from propellant. The 10 Kg of thorium-228 is not the propellant but serves only to heat it.

We can consider hydrogen atoms to be the propellant and ask what temperatures would be practical for a rocket engine. Thorium dioxide melts at 3600^{o}K and so the stable isotope of thorium is used in this form in hot devices such as gas lamps. So we might take our working temperature to be 3000^{o}K or 3300^{o}K.

How many protons must a decaying thorium nucleus share its energy with to result in these temperatures? The table above says that 3000^{o}K corresponds to a quarter eV. So $4 \times 37_{10}6 = 150$

[1] I'll use $_{10}x$ instead of $\times 10^x$ so that $0.16_{10}{-}12$ means 0.16×10^{-12}. Of course, in numbers K means $_{10}3$, M means $_{10}6$, G means $_{10}9$, T means $_{10}12$ and so on.

million is the answer. These protons will leave the nozzle at 7 Km/s for a specific impulse I_{sp} = 700 sec.

That 150 million says that the propellant hydrogen must mass 150/228 = 2/3 of a million times the thorium fuel, or 6700 tonnes for the 10 Kg of thorium. Or else we will run out of propellant, which is also the coolant. For our 100 tonne spacecraft this gives $\Delta V = \ln(68)V_e = 30$ Km/s. (This messes up the potential-energy calculations of Part II.)

Perhaps we should not stop at Mars but keep going to Saturn.

This decay-based engine has two important drawbacks. First, we can't stop it. The fuel will decay, giving off heat, whether we are using it for propulsion or not. Second, thorium-228 and its decay products also give off *gamma rays* (electromagnetic radiation, like X-rays only harder), as well as the alpha and beta particles. These take away energy without adding to ΔV, and must be kept away from human crew, bystanders, and even the electronics.

Electric propulsion.

NASA's NEXT engine uses a voltage gradient to accelerate xenon propellant to a specific impulse of 4190 seconds, which gives an exhaust velocity of 41 Km/s. Since xenon masses 131 AMU this corresponds to an energy of

$$10.36 \times \frac{131 \times 41^2}{2} = 1.14MeV$$

We need not concern ourselves with the corresponding temperature because the xenon ions are highly directed by the electric field and need not come into contact with the engine.

The thrust developed is 236 millinewtons, so the spacecraft must be launched by other means for missions to explore the solar system. (A precursor was NSTAR at 3100 seconds, which flew spacecraft Deep Space 1 past asteroid Braille and Jovian comet Borelly in 1999 and 2001 respectively, and was powered by solar panels. A spacecraft planned for travel further away from the Sun would be nuclear powered.) This thrust is equivalent to a weight of 24 grams and would accelerate a craft such as Deep Space 1 (486 Kg) at 50 micro-gees.

At 0.236 N thrust NEXT consumes propellant at the rate

$$\dot{m} = \frac{\text{thrust}}{V_e} = \frac{0.236}{41000} = 5.7 \text{ mg/s}$$

and so, if the NEXT were to run for 16,000 hours (which NSTAR did), then it would need only 331 Kg of propellant.

Fusion rockets.

There are as yet no fusion rockets. So the entries for them and for antimatter are only suggestive. Note that the 20-fold improvement of antimatter over fusion brings the 14 Mm/s exhaust velocity for nuclear in our first table up to almost lightspeed.

The British Interplanetary Society's Project Icarus aims to design a rocket to travel to our nearest neighbour star, Proxima Centauri, and go into orbit, all within a duration of 100 years. Since that is a distance of 4 light years (it takes light 4 years to reach us from Proxima Centauri), the average travel speed must be 4% of lightspeed, or 12 Mm/s.

The delta-V must be twice that, since the ship will accelerate to that speed then decelerate from it. That means the exhaust velocity from the rocket engine must be tens of megameters per second. This requires fusion power.

A way of building such a fusion engine might be to use magnetic "Z-pinch". When an electric current flows it induces a magnetic force which can "pinch" the conductor quite violently. Here is a copper tube through which a bolt of lightning passed in New South Wales in 1905. The magnetic force from the electric currents squeezed it. (Source: http://deathpix.com:8000/wikipedia_en_all_07_2014/A/Pinch_(plasma_physics).html last accessed 2021 Oct 15.)

A lightning bolt is an electric current of about 30 kilo-amperes. This current multiplied 150 times, in a *plasma* of charged *ions* (atoms stripped of their orbiting electrons), could induce fusion. Such a fusion reaction between two *deuterons* ("heavy hydrogen" ions of one proton and one neutron, instead of just one proton) is

$$
\begin{aligned}
D^2 + D^2 &\overset{\rightarrow}{50\%} & He^3\ (0.82\ \text{MeV}) + n^1\ (2.45\ \text{MeV}) \\
&\overset{\rightarrow}{50\%} & T^3\ (1.01\ \text{MeV}) + p^1\ (3.02\ \text{MeV}) \\
D^2 + T^3 &\rightarrow & He^4\ (3.49\ \text{MeV}) + n^1\ (14.1\ \text{MeV}) \\
D^2 + He^3 &\rightarrow & He^4\ (3.6\ \text{MeV}) + p^1\ (14.7\ \text{MeV})
\end{aligned}
$$

Here, D stands for deuterium, n for neutron, p for proton, T for tritium (hydrogen with two neutrons and one proton) and He for helium, either the regular, stable isotope, He^4, with two protons and two neutrons, or the rarer variant, He^3, with two protons and one neutron. The superscript numbers shown are just the numbers of protons and neutrons combined.

The fusion reaction $D^2 + D^2$ has a 50% chance of going either way.

The last two reactions above involve the byproducts of the main reaction, which also fuse with deuterium.

The neutrons produced have no electric charge and so cannot be controlled. They must be considered waste energy as far as propulsion is concerned.

The charged particles, especially the protons, contribute to the exhaust velocity, which works out to be 12 Mm/s (protons of just under 1 MeV according to our table above: we don't have to worry about melting the engine because the parts that might melt are not material but magnetic fields).

If the "wet" (fully fuelled) to "dry" (rocket without fuel) mass ratio is 232:22,

$$\Delta V = 12 \ln(232/22) = 28 \text{ Mm/s}$$

which is 9.4% of lightspeed (c).

This permits us to accelerate the rocket for ten years to 4.7%c, coast for 85 years with the engine turned off, then decelerate for five years (the rocket is much lighter then) so as to go into orbit around Proxima Centauri.

This proposal towards the goals of Project Icarus has the delightful name, "Firefly", and consists of a rocket of 2200 tonnes (2.2 gigagrams) dry mass, mostly in heat radiators but including a 150-tonne payload of scientific instruments (no crew), and 21 kilotonnes of deuterium fuel which can be separated out of seawater on Earth. The electric current of 5 mega-amperes at 235 kilovolts requires 1170 gigawatts of power, which is a third of the total electric power generated in the U.S. in 2005. The fusion generates four times that, so 3/4 of the fusion power would go into propulsion.

Firefly would be 3/4 of a kilometer long. It would be assembled in space and launched from behind the Moon "so the Z-pinch engine could be ignited without alarming the world population". It was designed to minimize mass.

4. Multistage rockets. In Note 2 I implied that chemical rockets could hardly get to the Moon. But they do. Here's how.

Let's imagine a *two-stage* rocket. The first stage carries the second stage at the beginning, then falls away and lets the second stage carry on.

	Initial mass, T	Final mass, T	Exhaust vel, Km/s	ΔV Km/s
Stage 1	600	170	3.3	4.2
Stage 2	140	30	3.3	5.1

The delta-Vs add up so we have a final $\Delta V = 9.2$ Km/s. That will get us almost to Mars orbit from Earth orbit. It gives plenty for the Moon and sublunar space, including takeoff from Earth, which I haven't factored in.

5. Thrust. The delta-V of the stage 2 rocket in the previous Note worked out to be 5.1 Km/s. To change speed by this much we must accelerate. If we did the whole delta-V in 1 second, the acceleration would be 5100 meters per second per second, which is about 510 gee, where one *gee* is the acceleration due to gravity at the Earth's surface, which we are all accustomed to, of 9.8 m/s/s, or about 10 m/s^2.

That would be a crushing acceleration. To keep it to 1 gee we would need to make the velocity change over 510 seconds, or $8\frac{1}{2}$ minutes.

The point is that our ΔV calculations have not said anything about the acceleration. But acceleration is clearly an important consideration.

Acceleration is caused by *thrust*, the second quantity we need to assess our rocket engines. The relationship is given by Newton's famous formula $F = ma$ or

$$a = \frac{F}{m}$$

So to accelerate a mass of 140 tonnes by 1 gee we need a thrust of

$$140,000 \times 9.8 = 1.372 \text{ meganewtons}$$

(a *newton* being the force that accelerates 1 Kg by 1 m/s^2).

At the end of its fuel, our stage-2 rocket of the prevous Note masses only 30 tonnes. So if its thrust is constant, the acceleration will have increased to

$$1.372_{10}6/30_{10}3 = 45.7 \text{ m/s}^2 = 4.7 \text{ gee}$$

This would be quite uncomfortable for a human crew, at the end of seven minutes of increasing acceleration. Liquid-fuelled chemical rockets, at least, can be throttled down to reduce the acceleration at the end of the burn.

The thrust is the rate of change of momentum of the rocket. In Note 1 we used the product of the exhaust velocity V_e with the mass m to give a change of momentum during whatever period of time it takes to eject that mass of fuel at that velocity. So if m is *all* the fuel and it is ejected in time t, then the thrust is

$$\frac{mV_e}{t}$$

This gives the thrust only crudely. We'll bypass the smaller additional term that depends on the thermodynamics of the gases in the nozzle.

Chemical rockets.

Suppose the rocket we've been using, with $V_e = 3.3$ Km/s, and 110 tonnes of fuel, burns it all up in 7 minutes. Then rate of fuel ejection is

$$\dot{m} = \frac{110_{10}3}{7 \times 60} = 262 \text{ Kg/s}$$

Our crude calculation of the thrust gives

$$\text{Thrust} = \dot{m}V_e = 3300 \times 262 = 0.864 \text{ meganewtons}$$

That will accelerate our 140-tonne rocket at 0.63 gee. (Note that this wouldn't get it off the ground; this rocket would have to be the second stage to a much larger first stage with, say, nine times the thrust.)

Fusion rockets.

For the Z-pinch fusion Firefly, you can work out the acceleration in your head, if you happen to know that a gigasecond is about 32 years (and a kilosecond is a quarter of an hour, a megasecond is two weeks). The ten years of acceleration to 5% lightspeed is a third of a gigasecond to get to 5% of a third of a gigameter per second, so the acceleration is 0.05 m/s^2 or half a percent of a gee. The five-year deceleration would be twice this or one percent gee.

More precisely,

$$0.047\frac{0.2998_{10}9}{10 \times 35.25 \times 24 \times 3600} = 0.045 \text{ gee}$$

This very slight acceleration would need to be augmented by artificial gravity if there were a human crew.

The thrust for the 23.2 kilotonne Firefly must then be 1.04 meganewtons.

This thrust is comparable to that of the smallish chemical rocket we've been examining, yet it will take Firefly to the stars. We see how the roles of thrust and specific impulse differ.

Rockets using radioactive decay.

If our thorium rocket, above, on the other hand, ejects 6700 tonnes of propellant in getting to Mars in 259 days (and of course it will be less because we're accelerating all the way), our crude thrust calculation gives

$$\dot{m}V_e = \frac{6.7_{10}6}{259 \times 24 \times 3600}7000 = 2.1 \text{ KN}$$

which is a lot smaller.

It accelerates the 100-tonne spacecraft at only 21 milligee.

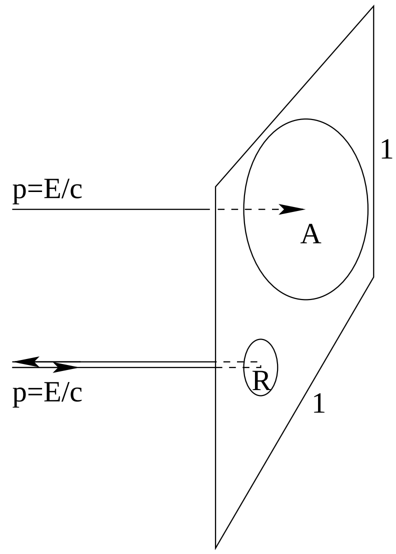
6. Photon sails. We can get away from fuel altogether, and hence rockets, by using a *photon sail.* This uses an external source of photons—a powerful laser, or the Sun—to push it, much as the wind pushes a sailboat[2].

Just as photons have energy, $E = hf$ (see Note 4 of Week 7a), they also have momentum, $p = h/\lambda$, where f is the frequency of the light and λ is its wavelength. h is "Planck's constant", but we don't need it here because we need only relate the momentum to the energy.

Since $\lambda = c/f$ where c is the speed of light, $p = E/c$. We'll find out about E in a minute.

First, let's see how a photon sail works.

[2]The "solar wind" refers, not to photon pressure, but to the particles, mainly charged, that stream from the Sun and cause the aurora, and is significantly less effective on sails; so the analogy with sailboats must be expressed carefully.



It is made of very thin, strong material, such as aluminized mylar (areal density $\sigma = 7$ grams per square meter) or, ten thousand times better, graphene (a carbon monolayer with areal density $\sigma = 0.7$ milligrams / m^2). It will either absorb, reflect or transmit a photon, in the respective proportions A, R or $T = 1 - A - R$.

For aluminized mylar we might have $R = 0.9$ and $A = 0.1$. For graphene $A = 0.4$, $R = 0.05$ and $T = 0.55$. The transmitted photons do nothing for us. The absorbed photons contribute their full momentum to the sail. The reflected photons contribute twice their momentum since they recoil with momentum opposite to what they had originally.

The force on the sail is the rate of change of momentum, and its acceleration is this divided by its mass. We can relate the rate of change of momentum to the rate of change of energy by the same photon equation

$$\dot{p} = \frac{\dot{E}}{c}$$

and so the acceleration of each unit area (1 meter by 1 meter) of the sail is

$$a_c = \frac{\dot{E}(A + 2R)}{c\sigma}$$

Now let's focus on the Sun as the source of the light—a *solar sail*. The solar energy has been measured at the Earth's orbit of 1 AU (an "astronomical unit" is discussed starting in Note 13 of Part II as 0.15 terameters) to be 1366 watts per square meter. A watt is a rate of energy production and is a joule per second. So $\dot{E} = 1366$ at 1 AU.

This light energy falls off as the square of the distance from the Sun. (This is plausible because the surface areas of imaginary spheres around the Sun increase as the square of their radii, and solar light energy does not increase or decrease as it expands from one sphere to the next.)

So $\dot{E}$ will be 100 times at 0.1 AU what it is at 1 AU, or generally

$$\frac{1366}{r^2} \text{ watts/m}^2$$

at distance r AU from the Sun.

Let's work out the accelerations at 1 AU and at 0.1 AU. We'll suppose a graphene sail but with several layers resulting in $\sigma = 5_{10}$–6 Kg/m^2, and the coefficients A and R above, and we'll add a payload which increases this by something over half again so finally

$$\sigma = 7.73 \times 10^{-6} \text{ Kg/m}^2$$

distance (AU)	acceleration (m/s^2)	acceleration (gee)
0.1	29.45	3.0
1.0	0.29	0.03

So if a spacecraft were launched in a parabola toward the Sun, approaching it at 0.1 AU at its closest point ("perihelion"), and unfurled its sail at that point (folding the sail is a nice problem in origami), its 3 gee acceleration away from the Sun might be tolerable for a human crew.

(What about temperature this close to the Sun, one quarter of Mercury's orbit? Analysis which I will not reproduce says that the peak temperature is 1022^{o}K (about 750^{o}C or 1380^{o}F) and that this will not melt the graphene.)

How fast will the spacecraft eventually be going? Let's suppose it is intended to leave the solar system on an interstellar mission. We'll figure this out by adding up the energy it gets from the Sun as its distance increases from perihelion to infinity, and interpreting the total as kinetic energy $mv^2/2$.

We'll do this for a square meter of sail, augmented as above with a payload, so that m is the same as σ, the mass per square meter.

The energy gain as the craft travels a distance d is d times the force, and the force depends on distance r from the Sun. So we must add up a whole lot of

$$d \times \frac{1366(A+2R)}{cr^2}$$

as we did for increments of velocity in Note 1.

Or we can ask what calculus gets for this calculation. The answer is

$$\frac{1366(A+2R)}{cr_p}$$

where r_p is the distance at perihelion, namely 0.1 AU. (Without calculus skills we might wonder why only the closest distance, r_p, comes in, but apart from this the result is perhaps more plausible than the calculus answer we got in Note 1.)

With all this energy going into velocity we get

$$v_\infty = \sqrt{\frac{2 \times 1366(A+2R)}{cr_p\sigma}}$$

If we work it out with $r_p = 0.1$ AU we get 937 Km/s. This is rather better than any rocket except fusion. It will get us to Proxima Centauri in 1375 years. Since we don't know how to make a graphene sail any more than we know how to make a fusion rocket, I won't elaborate further.

The result for v_∞ can be shortened into something neat, if we anticipate the *vis viva* equation of Note 16 of Part II. This says that the velocity of a planet in elliptical orbit around the Sun, whose mass is M, is

$$v_{\text{orbit}} = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)}$$

where r is distance from the Sun and a is half the long axis of the ellipse. (G is a constant which we'll leave to Part II, Newton's gravitational constant.)

Note that $a = r$ for a circular orbit and $a = \infty$ for a parabolic orbit, which will escape the solar system altogether. So the relationship between the two is

$$v_{\text{escape}} = \sqrt{2}\, v_{\text{circular}}$$

And the above final velocity for the solar sail is

$$v_\infty = \sqrt{\eta}\, v_{\rm escape}$$

where the orbit we're escaping from is the perihelion orbit and where we have a handy dimensionless constant, the ratio of the accelerations due to photon pressure and due to gravity, both of which vary as the inverse square of distance from the Sun.

$$\begin{aligned}\eta &= \frac{1366(A+2R)}{cr^2\sigma}\frac{r^2}{GM} \\ v_\infty &= \sqrt{\eta}\sqrt{\frac{2GM}{r_p}}\end{aligned}$$

Graphene is about as light and strong as we can get, so solar sailing does not seem the best way to visit the stars. We could use powerful lasers to get more light pressure. A gigawatt laser would generate $10^9(A+2R)/(0.3\times10^9) \approx 1.5$ newtons, which could whisk a miniaturized automatic probe very far if it could be kept in the beam and the beam kept narrow enough to stay within the sail. But note that power generation for the whole U.S. was only 3000 gigawatts in 2019.

Project Breakthrough StarShot proposes an array of 10 kilowatt lasers for up to 100 gigawatts focussed on 1000 "StarChips" (wafer-sized robots holding cameras, processors, diode laser photon thrusters and a laser for communicating back to Earth, all powered by a plutonium-238 or americium-240 atomic battery).

If each StarChip receives 1 terajoule of energy we can calculate what mass it must have to wind up travelling at 20% of lightspeed (and so be able to get to the potentially habitable planet Proxima Centauri b, 4 lightyears away, in 20 years).

$$m = \frac{1\text{TJ}}{v^2/2} = \frac{2_{10}12}{(0.2\times300_{10}6)^2} = \frac{1}{1800}\,\text{Kg}$$

or half a gram. Including the sail, which might be 4m by 4m and must withstand gigawatts of lightray.

If the lasers must be focussed on the StarChip for 10 minutes to deliver this much energy (1 TJ $\div$ 600 sec = 1.7 Gw), over what distance must this focus be maintained? The average velocity during the steady acceleration is half of the final velocity.

$$v_{\rm avg}\times 600\,\text{sec} = 0.1\times300_{10}6\times600 = 18_{10}9\,\text{m}$$

which is 0.12 A.U. or 47 times the distance of the Moon.

7. Solar wind. In a footnote to Note 6 I mentioned the solar wind to distinguish it from photon pressure. It consists of charged particles from the Sun, mostly protons, having rather variable density and speed.

Speed and density representative of the "slow" solar wind, emanating from the corona at the Sun's equator, are

$$\text{speed}\quad v = 400\ \text{Km/sec}$$

$$\text{density}\quad 7.3\ \text{ions/cm}^3$$

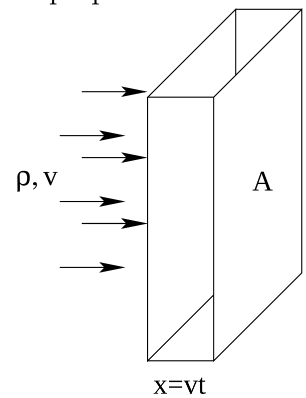
The speed is rather fast but the density, of a few particles per cubic centimeter, is very low and accounts for the feebleness of the solar wind as far as propulsion is concerned.

We will use kilograms per cubic meter, so the density must be multiplied by a million and by the mass of the proton.

$$\text{density}\quad \rho = 1.67_{10}{-}27\times7.3_{10}6\ \text{Kg/m}^3$$

Using these we can calculate some quantities important for propulsion.

Volume	$V = Ax = Avt$
Mass	$m = \rho V = \rho Avt$
Mass rate	$m/t = \rho Av$
Momentum	$mv = \rho Av^2 t$
Force	$F = mv/t = \rho Av^2$
Pressure	$P = F/A = \rho v^2$



First we can see that the pressure is much lower than that of solar photons.

$$P = \rho v^2 = 1.67_{10}{-27} \times 7.3_{10}6 \times 0.4_{10}6^2 = 1.95_{10}{-9}\ \mathrm{N/m^2}$$

compared with the solar pressure at 1 A.U. which we can calculate from Note 6 as

$$7.73_{10}{-6} \times 0.29 = 2.24_{10}{-6}\ \mathrm{N/m^2}$$

a factor of a thousand. Well, half that, because a charged "electric sail" would reverse the momentum of the protons, an advantage we already allowed for photon sails.

A compensation is that the solar wind is not the same in all directions and falls off more slowly with distance than photon pressure. And the numbers for its speed and density are highly variable with time (as are the winds for sailboating), leading to the appropriate term "space weather".

But since the solar wind is made up of already charged particles, perhaps it can be captured and used as propellant.

In Note 3 we arrived at a propellant consumption of $5.7_{10}{-6}$ Kg/sec of the xenon propellant for the NEXT ion engine. How much solar wind can a spacecraft scoop up in order to use the particles as propellant?

From the table the mass rate is

$$\rho Av = 1.67_{10}{-27} \times 7.3_{10}6 \times 0.4_{10}6\ A = 4.9_{10}{-15}\ A$$

and if we equate this to $5.7_{10}{-6}$ Kg/sec for argument's sake, we need a scoop of cross-sectional area

$$A = 5.7_{10}{-6}/4.9_{10}{-15} = 1.16_{10}9\ \mathrm{Km^2}$$

and if we equate that to πr^2 for a scoop of circular cross-section we get a radius of about 20 Km.

Such a ship would provide ion-engine levels of thrust a long way from the Sun and could be a good candidate for interplanetary travel.

Part II Orbits.

8. Ideology.
9. Circles and ellipses.
10. Pythagoras.
11. Circular orbits.
12. Finding and using GM_{Earth}.
13. Weighing Earth and Sun.
14. The Solar planets.
15. Transfer orbits.

16. Velocities in elliptical orbits.
17. Momentum and kinetic energy.
18. Potential energy.
19. Delta-V for orbit changes.
20. Surface to LEO.
21. Travelling the solar system.
22. Launch windows.
23. Conic sections.
24. Gravity assist.
25. Resonances.
26. Tides.
27. The Lagrange points.

Part III The Space Adventure.

28. Economics.
29. Microgravity.
30. Radiation.
31. Space debris.
32. Space elevator.
33. Ecology.
34. Population.
35. Genetics.
36. History.
37. Self-reproducing probes.
38. "Where Are They?"

Part IV Spaceship Earth.

39. Speeds.
40. Extinctions.
41. Herd science.
42. Climate.

Appendix. Trigonometry and calculus.

43. Trigonometry.
44. Integral calculus.
45. Differential calculus.

II. The Excursions

You've seen lots of ideas. Now *do* something with them!

1. The thorium nuclear rocket discussed in Note 3 is due to [Bih20]. Check the differences between my numbers and his. (I've used his numbers, more or less, up to where I calculate the propellant mass. My 6700 tonnes seems crazy so I've perhaps not understood something; Bihari says "tens of metric tonnes".)
   The NEXT ion-propulsion engine is described in [Cen20].
   The Z-pinch fusion star probe is described in [FL15].
   I follow the discussion of a graphene solar sail in [Mat14]

2. The solar wind discussed in Note 7 has been monitored in detail since 1998 by the Advanced Composition Explorer (ACE) and, since 2016, by the Deep Space Climate Observatory (DSCOVR) satellites at the L1 Earth-Sun Lagrange point (see Note 27 of Part II). The data are reported at https://www.swpc.noaa.gov/products/real-time-solar-wind.
   My calculations in this Note are very crude. They are improved by my predecessors [GYM21].

3. Any part of the Preliminary Notes that needs working through.

References

[Bih20] Gábor Bihari. Thermal thorium rocket (THOR): A new concept for radioactive decay heated thermal rocket engine. *J.Brit.Interplanetary Soc.*, 73(5):170–9, May 2020.

[Cen20] NASA Glenn Research Center. Gridded ion thrusters (next-c). URL https://www1.grc.nasa.gov/space/sep/ gridded-ion-thrusters-next-c/; Last updated: October 26, 2020; Last accessed Dec 31 2020, 2020.

[FL15] Robert M Freeland II and Michel Lamontagne. Firefly Icarus: An unmanned interstellar probe using Z-pinch fusion propulsion. *J.Brit.Interplanetary Soc.*, 68(3/4):68–79, March/April 2015.

[GYM21] Thomas R Gemmer, Christopher D Yoder, and Andre P Mazzolini. Performance analysis and parametric studies of the solar wind ion focusing thruster (SWIFT) for interplanetary travel. *J.Brit.Interplanetary Soc.*, 74(1):30–40, Jan. 2021.

[Mat14] Gregory L Matloff. Graphene solar photon sails and interstellar arks. *J.Brit.Interplanetary Soc.*, 67(6):237–48, June 2014.