On the Reachable Regions of Chains

(extended abstract)

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Abstract

A *chain* is a sequence of rigid rods or links consecutively connected at their endjoints, about which they may rotate freely. A *planar chain* is a chain whose rods lie in the plane, with rods allowed to pass over one another as they move.

This paper studies properties of the reachable regions of the endjoints of an $n-$link chain Γ lying inside a convex polygon P. We show that if the length of the longest link in Γ is sufficiently small, then the reachability of a point $p \in P$ by the endpoint of Γ is independent of the initial configuration of Γ. Then we ask how large the bound on the longest link may be made so that reachability does not depend on initial placement of Γ. Here the bound is a function of P.

1 Introduction

An *n-link chain* is a sequence of n rigid rods consecutively connected together at their endjoints, about which the rods may rotate freely. This paper considers the reachability properties of the endjoints of chains confined inside convex polygons.

Figure 1 illustrates an n-link chain Γ with joints $A_0, \ldots, A_n$. Here l_i denotes the length of link $L_i = [A_{i-1}, A_i]$. Joints A_0 and A_n are called *endjoints* and the others are called *intermediate joints.*

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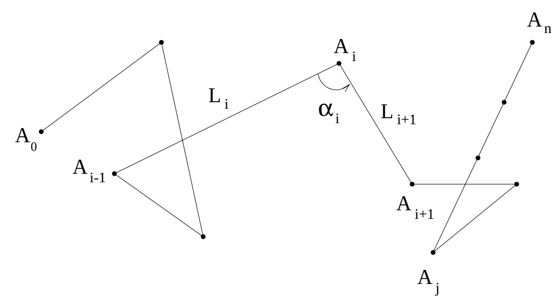

Figure 1: Notation for chains.

We denote $\max_{1 \leq i \leq n}\{l_i\}$ by l_{max} and say that Γ is *bounded* by b, denoted by $\Gamma \prec b$, if $l_{max} < b$.

We propose a strong notion of reachability, namely, we say that a point $p \in P$ is *reachable* by chain Γ provided that Γ can be moved from any *arbitrary* initial configuration to one in which A_n touches p. The set of reachable points of P for a given chain Γ is called the *reachability region* of Γ, denoted $P_\Gamma(A_n)$. This paper studies such reachability regions.

We give conditions under which the reachability region of a chain Γ in a convex polygon P is exactly that of an equivalent, one-link chain Γ^e in P, and we use this equivalence to compute the reachability region in linear time.

We show that the reachability regions of the chains inside a convex polygon P are linearly ordered by set inclusion, provided their links are not too long. This motivates us to propose an even stronger notion of reachability; namely, we say that a point $p \in P$ is *l-reachable* provided that it is a reachable point for *all* chains Γ whose longest link has length at most l. We also develop the notion of a hardest reachable point. A point $p \in P$ is a *hardest reachable point* if being reachable by Γ implies that every other point is

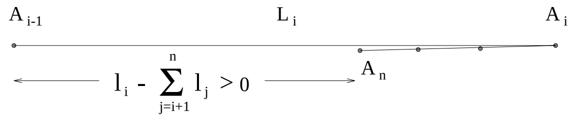

Figure 2: $l_i - \sum_{j=i+1}^{n} l_j$

reachable by Γ. We show such a point exists and is unique; in fact, it is the center of the minimal spanning circle of P.

Our study of reachability is novel in two ways. First, it asks when reachability by a given chain is independent of its initial configuration. Second, it asks for environment dependent conditions that, when satisfied, guarantee reachability by an otherwise arbitrary chain.

1.1 Motivation

A problem closely related to reachability is reconfiguration. Reconfiguration problems ask whether an object or ensemble of objects can be moved from one given configuration to another, often in the presence of obstacles. Reachability and reconfiguration problems in which the number n of degrees of freedom is allowed to vary from instance to instance are often at least NP-hard. Hence it is interesting to find big subfamilies of problems that can be solved quickly.

In this paper, we ask whether there are simple properties one could require of a confining region and an enclosed chain that, if satisfied, would ensure that point reachability is easy to determine. To this end, denote by $d_{max}(p)$ the distance between $p \in P$ and a point $q \in P$ that is farthest from p; then note that a necessary condition that a point p be reachable by the endpoint A_n of a chain Γ is the following (see Figure 2).

Condition (∗): For all $i \in \{1, \ldots, n\}$

$$l_i - \sum_{j=i+1}^{n} l_j \leq d_{max}(p). \qquad (*)$$

This condition is obviously necessary because, if not satisfied, there is no way to place A_n at p while also keeping in P the joint A_{i-1} of the link L_i associated with the maximum in Condition (∗). However, this condition is so mild that it does not guarantee the existence of a configuration of Γ lying in P and placing A_n at p.

Intuitively, it seems reasonable that if all the links of Γ are very tiny compared to some measure of P, then the initial configuration of Γ will not matter, Condition (∗) will be satisfied by every point $p \in P$, and it will be possible to reconfigure Γ so that A_n is brought to p. Thus Condition (∗) will trivially determines reachability.

Kantabutra [4] has shown that Condition (∗) is not only necessary but also sufficient for determining whether a point in a square can be reached by the endpoint of a chain whose link lengths are bounded by the length of the side of the square. In this case, not every point is reachable, but Condition (∗) determines exactly which points can be reached; furthermore, reachability turns out to be independent of initial configuration and can be tested in linear time.

Inspired by Kantabutra's example, we define for a given confining convex polygon P, a bound b^S on maximum link length l_{max} that guarantees that Condition (∗) is *sufficient* (hence the superscript S on b) as well as necessary for deterimining reachability independent of initial configuration. In terms of this notation, Kantabutra's result can be restated as saying that for a square of side length s, we have $s \leq b^S$: as long as a chain Γ confined inside S has no link that is longer than s, then Condition (∗) tests reachability.

1.2 Results

One of our results is that for a convex polygon P, $b^S \leq w$, where w is the *width* of P. The *width* of P is the minimum possible distance between two parallel lines of support of P. Concerning l-reachability, we show that the largest value l for which a polygon P is l-reachable is $l = min\{r, b^S\}$, where r is the radius of the *minimal spanning circle*. The (unique) *minimal spanning circle* of a convex polygon P is the circle of smallest radius that contains P. We show that the point of P that is "hardest reachable" is the center, o, of the minimal spanning circle.

1.3 Previous Work

Reconfiguration problems for linkages have been investigated by several researchers.

Paper [1] shows that a kind of reachability problem for a planar linkage with certain joints possibly fixed to the plane is PSPACE-hard. Here the problem is to determine, given an initial configuration of the linkage, some joint j of the linkage, and some point p in the plane, whether the linkage can be moved to place j at p. The desired configuration of the linkage is not further specified.

Paper [2] studies reachability problems for n-link chains with one endpoint fixed to the plane and with an initial configuration specified in the input to the problem. It shows that this kind of reachability problem is NP-hard when the environment is polygonal. However, for environments that consist of an enclosing circle, it gives an order $O(n^2)$ algorithm to solve the reachability problem in decision form. When a point is reachable, [2] gives an $O(n^3)$ algorithm to move the chain to some configuration in which the endpoint touches the given point. Kantabutra and Kosaraju [5] improved this running time from $O(n^3)$ to $O(n)$.

Algorithms for fast reconfiguration of n-link chains have been given for very simple confining regions: circles, squares, equilateral triangles, or no confining region at all.

Lenhart and Whitesides [6] gave a linear time algorithm for reconfiguring closed link chains in d-dimensional space, with no confining region. Kantabutra (see [5] and [4]) gave fast algorithms for reconfiguring chains inside squares, where the lengths of the links are bounded by the length of the side of the square. The problem of folding an n-link chain of equal-length links onto one link inside an equilateral triangle of unit side was considered by van Kreveld, Snoeyink and Whitesides [3].

Pei and Whitesides [7] solved the reachability problem for n-link planar chains confined within convex obtuse polygons. A *convex obtuse polygon* is a convex polygon whose internal angles each measure $\pi/2$ or more. In particular, [7] gives a polynomial time algorithm that decides whether a given endjoint of a chain confined within a convex obtuse polygon P can reach a given point $p \in P$ and that produces a sequence of moves that bring the endjoint to p when p is reachable.

1.4 Additional Preliminaries

The angle at intermediate joint A_i, denoted by α_i, is that determined by rotating L_i about A_i counterclockwise to bring L_i to L_{i+1}. We use $[A, B]$ to denote a single link chain having joints A and B. We denote the distance between two points x, y by $d(x, y)$.

We use V to denote the set of vertices of a polygon P. We regard polygons as 2-dimensional closed sets and denote the boundary of P by ∂P. We denote the length of the shortest side of P by s_{min}. We denote the circle centered at o with radius r by $C(o, r)$, or simply by C if o, r are clear from the context.

For a closed polygonal region P, $v_{max}(p)$ denotes a point of P farthest from p, and $d_{max}(p)$ denotes $d(p, v_{max}(p))$. Obviously, $v_{max}(p)$ is a vertex of P.

For an n-link chain Γ confined by a region P, the set of points of P that are reachable by A_n, independent of the initial configuration of Γ, is denoted by $P_\Gamma(A_n)$ and called the *reachable region* of A_n. The complement with respect to P of $P_\Gamma(A_n)$, denoted by $\overline{P_\Gamma(A_n)}$, is called the *unreachable region* of A_n.

In terms of this notation, a point $p \in P$ is called an *l-reachable* point if $p \in P_\Gamma(A_n)$ for every $\Gamma \prec l$ no matter where Γ initially lies. The set of l-reachable points in P, denoted by P_l, is called the *l-reachable region* of P, and if $P_l = P$, P is said to be *l-reachable.*

Clearly these concepts remain valid for non-polygonal confining regions as well.

2 $b^S \leq w$

To show $b^S \leq w$, the width of the confining convex polygon P, we first establish the following.

Lemma 2.1 *If $p \in \partial P$, then $d_{max}(p) \geq w$. Furthermore, if $v \in V$, then $d_{max}(v) > w$.*

Theorem 2.1 $b^S \leq w$.

proof sketch: Figure 3 illustrates the general idea of the proof, which goes by contradiction. If for some convex polygon P, the bound b^S were greater than the width w, then the situation illustrated in the figure could arise. Here the chain consists of a single link $[A, B]$, initially placed so that A lies at vertex v_2 and $[A, B]$ points, as a first case, upward and leftward toward a vertex v_2' farthest from v_2. By the previous lemma, $w < d_{max}(v_2)$. Suppose further that the line l_2 through edge (v_1, v_2) is a support line of P that, together with a parallel support line l_1, achieves the width w of P. Suppose the length of $[A, B]$ is greater than w but less than $d_{max}(v_2)$, $d_{max}(v)$ and b^S. Such a length is always possible under the assumption that $b^S > w$. Then in order to bring B to the rightmost vertex v of P, where possibly v is v_2 or lies on l, it is necessary to turn the link around, making it parallel to l at some moment. But this is impossible since the link is longer than the distance between l_1 and l_2. Nevertheless, vertex v satisfies Condition $(*)$ and so should be reachable, since $l_{max} < b^S$, a contradiction. □

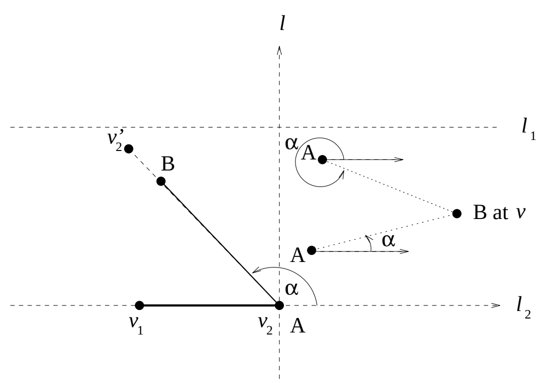

Figure 3: B cannot reach v.

We remark that $b^S < w$ may be considerably less than w, as illustrated in Figure 4. In this figure, the polygon P is constructed by cutting off three congruent tiny right triangles of an equilateral triangle $\triangle$ with unit side. Thus $w = \sqrt{3}/2 - \epsilon$ for some small ϵ. We claim that $b^S < w$ in P.

To see this, consider a 3-link chain Γ having joints A, B, C, D whose initial configuration is as follows: B, C are at the midpoints of two sides of $\triangle$, respectively; A, D are at two right corners of $\triangle$, respectively. Then $\Gamma \prec 1/2$. If $b^S = w$, then the top vertex of $\triangle$ is reachable by D. But this is impossible as Γ is completely stuck.

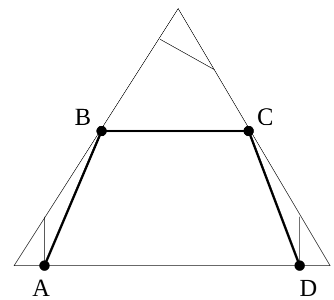

Figure 4: $b^S < w$.

3 Reachable Regions

Definition 3.1 *Let Γ be confined inside convex polygon P, where $l_{max} < b^S$, and let $l_1, \ldots, l_n$ be the lengths of the links of Γ. We call the chain Γ^e consisting of a single link of length l^e $= \max_{0<i\leq n}\{l_i - \sum_{j=i+1}^{n} l_j\}$ the equivalent chain of Γ, and we call l^e the* equivalent length *of Γ.*

The notion of an equivalent chain provides a simple way to compute, for (Γ and P such that $\Gamma \prec b^S$, the *unreachable regions* or P, i.e., the points of P not reachable by Γ.

notation: From now on, Γ^e consisting of a single link $[A_0, A_1]$ denotes the equivalent chain of Γ, and l^e denotes the equivalent length of Γ.

Observation: Note that $\Gamma \prec b^S$ implies that $\Gamma^e \prec b^S$. In this case, a point p is reachable A_n of Γ if and only if it is reachable by point B of Γ^e. This enables us to establish the following theorem.

Theorem 3.1 *Let $\Gamma \prec b^S$. Then $\overline{P_\Gamma(A_n)}$ is either empty or has boundary composed of at most m circular arcs centered at certain vertices of P and all having radius l^e.*

We remark that the number of circular arcs bounding the unreachable region for different chains sharing the same P may change, as shown in Figure 5. This figure shows a convex 5-gon that is nearly regular. The solid and dashed arcs show the construction lines for the boundary of the unreachable region of a longer and a shorter single link chain, respectively. Note that the boundary of the unreachable region of the longer link chain is composed of five circular arcs, whereas the boundary of the unreachable region of the shorter link chain is composed of three arcs.

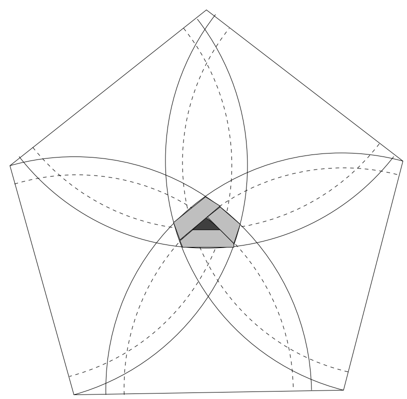

Figure 5: The circular arc number of unreachable regions may change

Theorem 3.2 *All $P_\Gamma(A_n)$ for $\Gamma \prec b^S$ are linearly ordered by set inclusion.*

Definition 3.2 *Let $\Gamma \prec b^S$ be an n-link chain confined within P. We say that Γ is* covering for P *or* covers P, *denoted by $\Gamma \vdash P$, if $R_\Gamma(A_n) = P$.*

By the previous theorem, the unreachable regions of the noncovering chains are also linearly ordered by set inclusion. The supremum of these regions, $\bigcup_{\Gamma \prec b^S, \Gamma \not\vdash P} \overline{R_\Gamma(A_n)}$, is clearly the complement of P_{b^S}. In the next section, we will show that the infimum of these regions, $\bigcap_{\Gamma \prec b^S, \Gamma \not\vdash P} \overline{R_\Gamma(A_n)}$, is a unique point that is "hardest reachable" (to be defined), which coincides with the center of the minimal spanning circle of P.

4 l-Reachability

First we recall some facts about minimal spanning circles. See [8].

Let P be a convex polygon. A *spanning circle* of P is a circle C such that each vertex of P lies either inside or on C. A *minimal spanning circle*of P is a spanning circle of P having minimum radius. Every convex polygon P has a minimal spanning circle, which is unique. Furthermore, a spanning circle C of a convex polygon P is the minimal spanning circle of P if and only if C passes through two diametrically opposite vertices (i.e. the line segment between the two vertices defines a diameter of C) or through three vertices that define an acute triangle. Consequences of these facts include the following.

Facts. Suppose P is a convex polygon, and let $C(o, r)$ be its minimal spanning circle. Then $o \in P$ and $r = d_{max}(o)$. Furthermore, a point $o \in P$ is the center of C if and only if for any $p \in P$, $d_{max}(p) \geq d_{max}(o)$.

The above facts are used in the proof of our main result on l-reachability, which is as follows.

Theorem 4.1 *Let P be a convex polygon and let r be the radius of its minimal spanning circle. Then $sup\{l \mid P \text{ is } l\text{-reachable } \} = min\{r\,,\, b^S\}$.*

We now illustrate applications of the above in the next three theorems. Our results suggest that the shape of P determines its l-reachability.

Theorem 4.2 *Let $\triangle$ be a triangle with an interior angle $\geq \pi/2$. Then $\triangle$ is b^S-reachable.*

Theorem 4.3 *Let P be a rectangle having sides a, b with $a \geq b$. Then P is w-reachable if and only if $a/b \geq \sqrt{3}$.*

Theorem 4.4 *Let P be a convex obtuse m-gon. If $m > 5$, then P is s_{min}-reachable.*

We conclude by stating a result that says the center of the minimal spanning circle is the unique, *hardest reachable* point of a convex polygon.

Definition 4.1 *Suppose P is not b^S-reachable. A point $o \in P$ is the* hardest reachable point *if for any n-link chain Γ, $o \in R_\Gamma(A_n)$ implies that $\Gamma \vdash P$.*

Theorem 4.5 *Let P be a convex polygon that is not b^S-reachable and let C be its minimal spanning circle with radius r. Then the following are equivalent.*

(1) o is the center of C;

(2) o is the hardest reachable point;

(3) o is the infimum of non-empty unreachable regions, i.e., $o = \bigcap_{\Gamma \prec b^S, \Gamma \not\vdash P} \overline{R_\Gamma(A_n)}$.

5 Conclusion

We have proposed a novel study of reachability in which we ask for conditions that guarantee that testing a point for reachability, independent of initial configuration, by the endpoint of a chain Γ is just a matter of checking whether Condition $(*)$ holds. We proposed bounding the link lengths by some best possible number b^S depending on P. For convex polygons, we obtained the result that b^S is at most w. We also defined a very strong notion of reachability, called l-reachability, and showed for convex polygons P that P is l-reachable for every l up to the minimum of the radius of the minimal spanning circle and b^S. We also showed that the center of the minimal spanning circle is the hardest point to reach of a convex polygon, and we gave several special applications of our results.

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