Dualities in Mathematics: Locally compact abelian groups

Part III: Pontryagin Duality

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Recall Gelfand Duality

$$\begin{tikzcd} \mathbf{KHs} \arrow[r, bend left, "C"] & \mathbf{CUCstar}^{op} \arrow[l, bend left, "\Omega"] \end{tikzcd}$$

$\mathbf{KHs}$: Compact Hausdorff spaces, continuous functions.
$\mathbf{CUCstar}$: Commutative, unital, (complex) C^* algebras, with $*$-homomorphisms as the morphisms.
$C(\cdot)$: continuous complex-valued functions.
Ω: “characters” $=$ “maximal ideal space”.

Pontryagin duality

- A *self-duality* between a certain category of groups and itself.
- Can also be seen through the Gelfand prism as a duality between a certain algebra (the convolution algebra) and a topological space; this time a topological group.
- Plays a key role in understanding the Fourier transform.

The Categorical Picture

$$\begin{tikzcd} \mathbf{LCA} \arrow[rr, bend left, "\Xi"] & & \mathbf{LCA}^{op} \arrow[ll, bend left, "\Xi"] \end{tikzcd}$$

$\mathbf{LCA}$: the category of locally compact abelian groups.
Morphisms are both homomorphisms and continuous maps.

Cuts down to

$$\begin{tikzcd} \mathbf{CAG} \arrow[rr, bend left, "\Xi"] & & \mathbf{DAG}^{op} \arrow[ll, bend left, "\Xi"] \end{tikzcd}$$

$\mathbf{CAG}$: compact abelian groups, $\mathbf{DAG}$: discrete abelian groups.

Canonical example: S^1, the circle and $\mathbb{Z}$, the integers.

Finite abelian groups

- A finite group G with a single element τ with $G = \{e, \tau, \tau^2, \ldots, \tau^{n-1}\}$ is called a *cyclic* group.
- Clearly abelian.
- One can think of them as the groups of roots of unity: $\mathbb{Z}_N = \{1, e^{2\pi\, i/N}, \ldots, e^{2l\pi\, i/N}, \ldots\}$

Structure Theorem

Every finite abelian group is the direct sum of cyclic groups.

Characters I

- $\mathbb{C}^\times$: the non-zero complex numbers with multiplication forms an abelian group.
- $S^1 \subset \mathbb{C}^\times$ is the subgroup of complex numbers of modulus 1.
- G any abelian group: a homomorphism $\chi : G \longrightarrow \mathbb{C}^\times$ is called a *character*.
- Clearly (?) $\forall g \in G,\ |\chi(g)| = 1$.
- So really $\chi : G \longrightarrow S^1$.

Characters II

- Define $(\chi_1 \cdot \chi_2)(g) = \chi_1(g)\chi_2(g)$.
- The characters form a group: $\widehat{G}$; the *dual* group.
- $\widehat{G}$ is also abelian.
- For a cyclic group like $\mathbb{Z}_N$ with generator τ we have N distinct characters $\chi_l(\tau^k) = e^{2\pi\ ikl/N}$ where $l \in \{1, \ldots, N\}$.
- $\widehat{\mathbb{Z}_N} \simeq \mathbb{Z}_N$.
- $\widehat{G_1 \times G_2} \simeq \widehat{G_1} \times \widehat{G_2}$
- Using the structure theorem it follows that $|G| = |\widehat{G}|$ for any finite abelian group.

Pontryagin duality for finite abelian groups

Theorem

If G is a finite abelian group then there is an isomorphism ev : $G \rightarrow \widehat{\widehat{G}}$ given by ev$(g) = (\chi \mapsto \chi(g))$.

- ev is a homomorphism.
- ev is injective.
- $|G| = |\widehat{\widehat{G}}|$, so ev is a bijection.

Fourier transform

- Let G be a finite abelian group.
- Write $l^2(G)$ for the vector space $G \longrightarrow \mathbb{C}$.
- It is actually a Hilbert space with $\langle \alpha, \ \beta \rangle = \sum_{g \in G} \alpha^*(g)\beta(g)$.
- Characters are elements of $l^2(G)$;
- $\langle \chi, \ \eta \rangle = 0$ if $\chi \neq \eta$. $\langle \chi, \ \chi \rangle = |G|$.

$f \in l^2(G), \widehat{f} \in l^2(\widehat{G})$

$$\widehat{f}(\chi) = \frac{1}{\sqrt{|G|}} \sum_{g \in G} \langle f, \ \chi \rangle.$$

The map $f \mapsto \widehat{f}$ is an isomorphism of Hilbert spaces.

Convolution

$\alpha, \beta \in l^2(G)$

$$(\alpha * \beta)(g) = \frac{1}{\sqrt{|G|}} \sum_{a \in G} \alpha(a)\beta(a^{-1}g).$$

Theorem

$$\widehat{\alpha * \beta} = \widehat{\alpha}\widehat{\beta}.$$

So $*$ is associative and commutative and distributes over $+$. In fact, we have a commutative, unital, Banach algebra: the convolution algebra.

Where can we go from finite abelian groups?

- Finite $\longrightarrow$ compact
- or even locally compact.
- Need a topology on the group.
- Need to replace summation by integration.

Defining topological groups

Definition

A topological group is a topological space that is also a group and the group operations: multiplication $\cdot : G \times G \longrightarrow G$ and inversion $(\cdot)^{-1} : G \longrightarrow G$, are continuous.

Finite groups are regarded as topological groups with the discrete topology.

Morphisms

We define the category of topological groups by defining the morphisms to be continuous homomorphisms.

Examples

- $\mathbb{C}^\times$.
- $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$.
- $SL(n,\mathbb{C})$ is a closed subgroup of $GL(n,\mathbb{C})$.
- The circle S^1 (or $U(1)$) is an abelian topological group.

Basic facts about topological groups

- The topology is *translation invariant*: U is open iff gU is open iff Ug is open.
- U is open iff $U^{-1} := \{x | x^{-1} \in U\}$ is open.
- If H is a sungroup of G so is its closure.
- Every open subgroup is clopen.
- If G is T_1 it is automatically Hausdorff which is the case iff $\{e\}$ is closed.
- We assume that being Hausdorff is *part of the definition* of being a compact group or a locally compact group.

Haar measure

- If you don't know measure theory, just note that a sensible notion of integration can be defined on (certain) topological groups.
- A Borel measure μ is *outer regular* if for any Borel set E we have $\mu(E) = \inf\{\mu(U) | U$ open, $E \subseteq U\}$.
- A Borel measure μ is *inner regular* if for any Borel set E we have $\mu(E) = \sup\{\mu(K) | K$ compact, $K \subseteq E\}$.
- A Radon measure is a Borel measure that is finite on compact sets, inner regular on all open sets and outer regular on all Borel sets.

Definition of Haar measure

Let G be a topological group and μ a Radon measure. We say that μ is a (left) *Haar* measure if all $g \in G$ and E Borel subsets of G we have $\mu(gE) = \mu(E)$.

Haar measure on locally compact groups

Existence and Uniqueness

Let G be a locally compact group. Then G admits a left Haar measure. This measure is unique up to an overall factor.

- G has a left Haar measure iff it has a right Haar measure.
- A non-zero Haar measure is positive on all open sets.
- $\mu(G)$ is finite iff G is compact.

Characters

- A character of a locally compact abelian group G is a *continuous* group homomorphism from G to S^1.
- The characters form a group $\widehat{G}$ under pointwise multiplication just as for finite abelian groups.
- We make this into a topological space by using the compact-open topology.
- If X, Y are topological spaces the compact-open topology on
  $[X \longrightarrow Y]$ is generated by the sets of the form
  $F(K, U) = \{f : X \longrightarrow Y | f(K) \subset U\}$ where K is compact and U is open.
- We get a dual *topological* group $\widehat{G}$.

Example I

The dual of $(\mathbb{Z}, +, 0)$ with the discrete topology is S^1 with the Euclidean topology.

Any χ is determined by $\chi(1)$. So any choice of a complex number in S^1 gives a character. The compact-open topology in this case is the topology of pointwise convergence which is the topology inherited from the complex numbers.

Example II

The dual of S^1 is $(\mathbb{Z}, +, 0)$ with the discrete topology.

Let $\chi \in \widehat{S^1}$. By definition: $\chi(S^1) \subset S^1$.

We know $\chi(e^{i\theta}) = e^{i\,f(\theta)}$ for some continuous f.

Now $\chi(e^{i(\alpha+\beta)}) = e^{if(\alpha+\beta)} = \chi(e^{i\alpha})\chi(e^{i\beta}) = e^{i(f(\alpha)+f(\beta))}$
so $f(\theta) = \lambda\theta$ for some real constant λ
so $\chi(z) = z^\lambda$.

If we take ω to be an nth root of unity we have
$\chi(\omega^n) = \chi(1) = 1 = \chi(\omega)^n = (\omega^\lambda)^n$.So ω^λ is also a root of unity,
thus λ is an integer.

More generally

The dual of a discrete group is a compact group and the dual of a compact group is a discrete group.

Pontryagin duality theorem

There is a *natural isomorphism* between a locally compact abelian group G and its double dual $\widehat{\widehat{G}}$ given by

$$\text{ev} : G \to \widehat{\widehat{G}} \text{ where } \text{ev}(g)(\chi) = \chi(g).$$

The proof requires a lot of analysis but, I hope it is clear that the ingredients that we used in the finite case generalize nicely to the locally compact abelian case.

In particular, Fourier theory generalizes to arbitrary locally compact abelian groups.

Tannaka-Krein

- What can we do if the group is not abelian?
- One cannot work with the characters alone.
- One needs the *category* of representations.
- Tannaka showed how to reconstruct a compact group from the monoidal category of its representations.
- Krein characterized the monoidal categories that arise as the category of representations of a compact group.
- In 1989 Doplicher and Roberts gave a new duality theory for compact groups using the notion of C^* categories.
- These type of duality theorems have been generalized to quantum groups and many other settings.
- With Costin Badescu, Tobias Fritz and Robert Furber, we are developing such theorems for certain kinds of operator algebras.