Dualities in Mathematics:
Analysis dressed up as algebra is dual to topology
Part II: Gelfand Duality

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The basic message

KHs: Compact Hausdorff spaces, continuous functions.
CUCstar: Commutative, unital, (complex) $C^*$ algebras, with $*$-homomorphisms as the morphisms.
C(·): continuous complex-valued functions.
Ω: “characters” = “maximal ideal space”.
Strangely, there is also a duality between real (commutative, unital) \( C^* \)-algebras and compact Hausdorff spaces: Stone-Gelfand. 

Very thoroughly treated in *Stone Spaces* by Johnstone. 

Hence there is an equivalence of categories between the two types of \( C^* \) algebras. 

In real \( C^* \) algebras the \( * \) structure is trivial. 

The complex version uses different mathematics and is much more relevant for quantum mechanics.
A word from our sponsor

- In traditional treatments of quantum mechanics the state space is a Hilbert space.
- Most quantities of interest are modelled by (bounded ?) operators on the Hilbert space.
- These form a complex $C^*$ algebra; the $*$ operation is adjoint.
- Observables are self-adjoint, but these do not form a sub-algebra of the $C^*$ algebra of all bounded operators.
- The observables can be viewed as a real $C^*$-algebra but one loses the essential role played by the complex numbers in quantum mechanics.
Rings

Commutative unital rings

A commutative unital ring \( R \) is a set containing two distinguished elements 0 and 1 and two binary operations + and \( \times \) satisfying:

- \((R, +, 0)\) forms an abelian group,
- \((R, \times, 1)\) forms a commutative monoid,
- \(\times\) distributes over +.

- Integers \( \mathbb{Z} \), reals \( \mathbb{R} \), complex numbers \( \mathbb{C} \).
- Polynomials in \( n \) variables with coefficients in \( \mathbb{Z} \) or, indeed any commutative ring.
- A non-example: matrices with entries in a ring.
- Complex-valued continuous functions from a compact Hausdorff space \( X \) to \( \mathbb{C} \).
Ideals in a ring

Let $R$ be a fixed commutative ring. Henceforth, all rings are assumed to be unital unless otherwise stated.

Ideals
An ideal $I$ in $R$ is a subset that is closed under $+$ and if $x \in I$ and $r \in R$ then $r \cdot x \in I$.

Typical example: all multiples of, say, 9 in $\mathbb{Z}$. Write $(9)$ for this ideal; the ideal generated by 9.

We can define $\sim_I$ by $r \sim_I r'$ if $r - r' \in I$ and $R/I$ as the set of equivalence classes of $\sim_I$; $R/I$ is also a (commutative) ring.

Maximal and prime ideals

Maximal ideal
An ideal $I$ of $R$ is called a **maximal ideal** if there are no ideals strictly containing it and strictly contained in $R$.

The ideal $(9)$ is not maximal, it is contained in $(3)$, which is a maximal ideal.

Prime ideal
An ideal $I$ is a **prime ideal** if whenever $xy \in I$ then $x \in I$ or $y \in I$.

If $p$ is a prime number then $(p)$ is a prime ideal in $\mathbb{Z}$.
Maximal ideals are always prime but not conversely. In $\mathbb{Z}$ $(0)$ is prime but not maximal.
Let $X$ be a compact Hausdorff space and let $C(X)$ be the ring of complex-valued continuous functions on $X$.

$C(X)$ is clearly a commutative unital ring.

It has a lot more structure than that.

Fix $x \in X$ then $M_x := \{f \in C(X) \mid f(x) = 0\}$ is a maximal ideal of $C(X)$.

It has no nontrivial nilpotent elements.
Points define maximal ideals

- Fix a compact Hausdorff space $X$.
- In the ring $C(X)$, fix $x \in X$; the set $M_x = \{ f \mid f(x) = 0 \}$ is a maximal ideal.
- Clearly $M_x$ is an ideal.
- Not hard to see that it is maximal: any attempt to enlarge it will lead to a nowhere vanishing function $f$ in the ideal. Then $\frac{1}{f}$ is a well-defined continuous function so $\lambda x.1$ in the ideal.
- We have a map $\Gamma : X \to \mathcal{M}(C(X))$, where $\mathcal{M}(R)$ is the set of maximal ideals of a ring $R$.
- By Urysohn’s Lemma, $\Gamma$ is injective.
Maximal ideals define points

• Given a maximal ideal \( M \) there exists \( x \in X \) such that \( M = M_x \).
• Suppose not, then \( \forall x \in X, \exists f_x \in M \) with \( f_x(x) \neq 0 \).
• Since \( f_x \) is continuous there is an open set \( O_x \ni x \) where \( f_x \) is non-vanishing.
• The \( \{ O_x \mid x \in X \} \) form a cover of \( X \), so by compactness, there is a finite subcover: \( \{(x_1,f_1,O_1), \ldots, (x_k,f_k,O_k)\} \).
• \( \sum_{i=1}^{k} f_i^2 \) is nowhere vanishing and in \( M \).
• \( \Gamma \) is bijective.
Getting the topology of $X$

- Given $f \in C(X)$, define $O_f = \{x \in X | f(x) \neq 0\}$: base for the topology of $X$.
- Let $U_f = \{M \in M(C(X)) | f \notin M\}$: base for a topology on $M(C(X))$.
- Easy to see that $\Gamma(O_f) = U_f$, so $\Gamma$ is a homeomorphism.
Are we there yet?

- No! The inverse image of a maximal ideal is not necessarily a maximal ideal so we cannot make these constructions functorial.
- That’s why algebraic geometers use the prime ideals and define the spectrum of a ring in terms of prime ideals.
- Before I get mired in scheme theory let me beat a hasty retreat!
- $C(X)$ cannot possibly produce arbitrary commutative rings: there will never be nilpotent elements.
- So what kind of rings do arise as $C(X)$ for some compact Hausdorff space $X$?
- Answer: $C^*$-algebras. This is Gelfand duality.
Algebras

All vector spaces are assumed to be over the field of complex numbers.

Algebras

An algebra is a vector space equipped with an associative multiplication operation \( \cdot \), that is bilinear in its arguments.

\[ \text{Mat}_n: n \times n \text{ matrices with entries in } \mathbb{C}; \text{ a noncommutative example.} \]

Bounded linear operators on a Hilbert space \( \mathcal{H} \); written as \( \mathcal{B}(\mathcal{H}) \). The multiplication is composition. This is also noncommutative.

The space \( C(X) \) with pointwise multiplication; a commutative algebra.
Banach algebras

Norm and Banach space

A **norm** on a vector space $V$ is a function $\| \cdot \| : V \rightarrow \mathbb{R}$ satisfying:

- $\| \alpha v \| = |\alpha| \| v \|$
- $\| u + v \| \leq \| u \| + \| v \|$
- $\| v \| = 0$ iff $v = 0$.

A vector space with a norm is called a **normed space** and a normed space that is **complete** in the metric induced by the norm is called a **Banach space**.

A **Banach algebra** $A$ is an algebra and a Banach space with a norm $\| \cdot \| : A \rightarrow \mathbb{R}^+$ such that $\| ab \| \leq \| a \| \| b \|$. It is easy to see that the multiplication operation is jointly continuous in the topology induced by the norm.
Examples of Banach algebras

- If $X$ is any set then $l^\infty(X)$ the set of bounded complex-valued functions with pointwise operations and the $\sup$ norm is a unital Banach algebra.
- If $X$ is a topological space then $C_b(X)$ the space of all bounded continuous complex-valued functions is a unital Banach algebra, in fact a closed subalgebra of $l^\infty(X)$.
- If $X$ is compact, then the space of all continuous complex-valued functions of $X$ (written $C(X)$) is a unital Banach algebra, being the same as $C_b(X)$.
- If $X$ is a locally compact Hausdorff space we say that a function $f : X \to \mathbb{C}$ vanishes at infinity if $\forall \varepsilon > 0$, $\{x \in X \mid |f(x)| \geq \varepsilon\}$ is compact.
- The set of continuous functions that vanish at infinity is written $C_0(X)$.
- $C_0(X)$ is a closed subalgebra of $C_b(X)$ and is unital if and only if $X$ is compact.
Star algebras

- An **involution** on an algebra $A$ is a map $*: A \to A$ such that:
  1. $\forall a \in A, \alpha \in \mathbb{C}, \ (\alpha a)^* = \overline{\alpha} a^*$,
  2. $\forall a \in A, \ (a^*)^* = a$ and
  3. $\forall a, b \in A, \ (ab)^* = b^* a^*$.

- An algebra with an involution is called a **$\ast$-algebra**.
- An element $a \in A$ is called **self-adjoint** or **hermitian** if $a = a^*$.
- Every element $a$ in a $\ast$-algebra can be written as $a = b + ic$ where $b, c$ are hermitian.
- A self-adjoint element $p$ is called a **projection** if $p^2 = p$.
- Note that $aa^*$ and $a^*a$ are always self-adjoint; they are called **positive elements**.
A **$C^*$-algebra** $A$ is a Banach algebra with an involution satisfying

$$\forall a \in A, \|a^* a\| = \|a\|^2: \text{ the } C^* \text{ identity.}$$

It follows that $\|a\| = \|a^*\|$.

A $^*$-homomorphism is a map preserving multiplication and the involution.

They are *automatically* contractive: $\|\phi(a)\| \leq \|a\|$ (hence continuous) and,

if $\phi$ is injective then $\|\phi(a)\| = \|a\|$.

$C^*$-algebras may or may not be unital, if they are $\|1\| = 1$.

There is a unique norm on a $C^*$-algebra!

More precisely, given a $^*$-algebra, there is at most one way of endowing it with a norm satisfying the $C^*$ identity.
• We fix a unital $C^*$-algebra $A$ with unit $1$.
• An element $a \in A$ is said to be invertible if $\exists b$ such that $ab = ba = 1$, we write $a^{-1}$ for $b$.
• Write $\text{Inv}(A)$ for the set of invertible elements of $A$. Note that it is a group.

The spectrum of $a$ is

$$\sigma(a) := \{ \lambda \in \mathbb{C} | \lambda 1 - a \notin \text{Inv}(A) \}.$$
Examples of spectra

- If $A$ is the algebra of $n \times n$ matrices then $\sigma(a)$ is the set of eigenvalues of $a$.
- If $X$ is a compact Hausdorff space and $A$ is the algebra $C(X)$ then $\sigma(f) = \text{range}(f)$.
- Thus the notion of spectrum generalizes the notion of range of a function as well as eigenvalues of a matrix.
- For operators on infinite-dimensional spaces the spectrum is not just the set of eigenvalues. In fact there may be no eigenvalues.
- Consider $L^2(\mathbb{R})$ and the bounded linear map $f \mapsto (\frac{1}{1+x^2} \cdot f)$. This has no eigenvalues.
Gelfand
If $A$ is a unital Banach algebra then $\sigma(a)$ is non-empty. The proof uses some basic complex analysis.

Gelfand-Mazur
If a Banach algebra is a field then it is isomorphic to $\mathbb{C}$. This is an immediate corollary.
Characters

Definition
A **character** on an a commutative algebra $A$ is a non-zero homomorphism $\tau : A \rightarrow \mathbb{C}$. We write $\Omega(A)$ for the set of characters on $A$.

Just as we moved from ultrafilters to boolean algebra homomorphisms in Stone duality, we have

**Proposition**
For a commutative unital Banach algebra (CUBA) $\tau \mapsto \ker(\tau)$ is a bijection between $\Omega(A)$ and the set of maximal ideals.

**Proposition**
For a CUBA $A$, $\forall a \in A$, $\sigma(a) = \{ \tau(a) | \tau \in \Omega(A) \}$. 
A general strategy for defining topologies
Choose a set of functions $\mathcal{F}$ from a set $X$ to a topological space $Y$. Define the *weakest* (fewest open sets) topology that makes every function in $\mathcal{F}$ continuous: $\sigma(X, \mathcal{F})$.

The Gelfand topology
Weakest topology that makes all the maps

$$\Omega(A) \to \mathbb{C} : \text{eval}_a(\tau) = \tau(a)$$

continuous.
Theorem

$A$ is a CUBA if and only if $\Omega(A)$ is a compact Hausdorff space. If $A$ is not unital then $\Omega(A)$ is locally compact. Adding a unit to $A$ is the same as the “one-point compactification” of $\Omega(A)$. 
**Functoriality**

$\Omega(\cdot)$ as a functor

If $h : A \rightarrow B$ is a Banach algebra map then $\Omega(h) : \Omega(B) \rightarrow \Omega(A)$ is $f \mapsto h \circ f$.

$C(\cdot)$ as a functor

If $f : X \rightarrow Y$ is a continuous function between compact Hausdorff spaces then $C(f) : C(Y) \rightarrow C(X)$ given by $C(f)(g) = g \circ f$ is a Banach algebra map, i.e. a norm-decreasing homomorphism.

But wouldn’t we like it to be an isometry?
Finally! Gelfand duality

Theorem
For a commutative unital $C^*$-algebra $A$ the map $a \mapsto \text{eval}_a : A \rightarrow C(\Omega(A))$ is an isometric $\ast$-isomorphism.
This is even true for non-unital algebras if one uses $C_0$.

Corollary
For two commutative $C^*$ algebras $A$ and $B$, $\Omega(A)$ and $\Omega(B)$ are homeomorphic iff $A$ and $B$ are isometrically $\ast$-isomorphic.