Dualities in Mathematics: Analysis dressed up as algebra is dual to topology

Part II: Gelfand Duality

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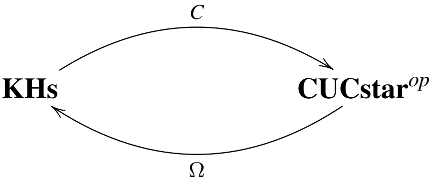
The basic message



KHs: Compact Hausdorff spaces, continuous functions.
CUCstar: Commutative, unital, (complex) C^* algebras,
with $*$-homomorphisms as the morphisms.
$C(\cdot)$: continuous complex-valued functions.
Ω: “characters” $=$ “maximal ideal space”.

Stone-Gelfand

- Strangely, there is also a duality between *real* (commutative, unital) C^*-algebras and compact Hausdorff spaces: Stone-Gelfand.
- Very thoroughly treated in *Stone Spaces* by Johnstone.
- Hence there is an equivalence of categories between the two types of C^* algebras.
- In real C^* algebras the $*$ structure is trivial.
- The complex version uses different mathematics and
- is much more relevant for quantum mechanics.

A word from our sponsor

- In traditional treatments of quantum mechanics the state space is a Hilbert space.
- Most quantities of interest are modelled by (bounded ?) operators on the Hilbert space.
- These form a complex C^* algebra; the $*$ operation is adjoint.
- Observables are self-adjoint, but these do not form a sub-algebra of the C^* algebra of all bounded operators.
- The observables can be viewed as a real C^*-algebra but one loses the essential role played by the complex numbers in quantum mechanics.

Rings

Commutative unital rings

A commutative unital ring R is a set containing two distinguished elements 0 and 1 and two binary operations $+$ and $\times$ satisfying:

- $(R, +, 0)$ forms an abelian group,
- $(R, \times, 1)$ forms a commutative monoid,
- $\times$ distributes over $+$.

- Integers $\mathbb{Z}$, reals $\mathbb{R}$, complex numbers $\mathbb{C}$.
- Polynomials in n variables with coefficients in $\mathbb{Z}$ or, indeed any commutative ring.
- A non-example: matrices with entries in a ring.
- Complex-valued continuous functions from a compact Hausdorff space X to $\mathbb{C}$.

Ideals in a ring

Let R be a fixed commutative ring. Henceforth, *all* rings are assumed to be unital unless otherwise stated.

Ideals

An **ideal** I in R is a subset that is closed under $+$ and if $x \in I$ and $r \in R$ then $r \cdot x \in I$.

Typical example: all multiples of, say, 9 in $\mathbb{Z}$. Write (9) for this ideal; the ideal *generated* by 9.

We can define $\sim_I$ by $r \sim_I r'$ if $r - r' \in I$ and R/I as the set of equivalence classes of $\sim_I$; R/I is also a (commutative) ring.

The ring $Z/(9)$ has an element $[3]$ with $[3] \cdot [3] = [9] = [0]$. Such an element is called **nilpotent**.

Maximal and prime ideals

Maximal ideal

An ideal I of R is called a **maximal ideal** if there are no ideals strictly containing it and strictly contained in R.

The ideal (9) is not maximal, it is contained in (3), which is a maximal ideal.

Prime ideal

An ideal I is a **prime ideal** if whenever $xy \in I$ then $x \in I$ or $y \in I$.

If p is a prime number then (p) is a prime ideal in $\mathbb{Z}$.

Maximal ideals are always prime but not conversely. In $\mathbb{Z}$ (0) is prime but not maximal.

The ring $C(X)$

- Let X be a compact Hausdorff space and let $C(X)$ be the ring of complex-valued continuous functions on X.
- $C(X)$ is clearly a commutative unital ring.
- It has a lot more structure than that.
- Fix $x \in X$ then $M_x := \{f \in C(X) \mid f(x) = 0\}$ is a maximal ideal of $C(X)$.
- It has no nontrivial nilpotent elements.

Points define maximal ideals

- Fix a compact Hausdorff space X.
- In the ring $C(X)$, fix $x \in X$; the set $M_x = \{f \mid f(x) = 0\}$ is a maximal ideal.
- Clearly M_x is an ideal.
- Not hard to see that it is maximal: any attempt to enlarge it will lead to a nowhere vanishing function f in the ideal. Then $\frac{1}{f}$ is a well-defined continuous function so $\lambda x.1$ in the ideal.
- We have a map $\Gamma : X \longrightarrow \mathfrak{M}(C(X))$, where $\mathfrak{M}(R)$ is the set of maximal ideals of a ring R.
- By Urysohn's Lemma, Γ is injective.

Maximal ideals define points

- Given a maximal ideal M there exists $x \in X$ such that $M = M_x$.
- Suppose not, then $\forall x \in X, \exists f_x \in M$ with $f_x(x) \neq 0$.
- Since f_x is continuous there is an open set $O_x \ni x$ where f_x is non-vanishing.
- The $\{O_x | x \in X\}$ form a cover of X, so by compactness, there is a finite subcover: $\{(x_1, f_1, O_1), \ldots, (x_k, f_k, O_k)\}$.
- $\sum_{i=1}^{k} f_i^2$ is nowhere vanishing and in M.
- Γ is bijective.

Getting the topology of X

- Given $f \in C(X)$, define $O_f = \{x \in X | f(x) \neq 0\}$: base for the topology of X.
- Let $U_f = \{M \in \mathfrak{M}(C(X)) | f \notin M\}$: base for a topology on $\mathfrak{M}(C(X))$.
- Easy to see that $\Gamma(O_f) = U_f$, so Γ is a homeomorphism.

Are we there yet?

- No! The inverse image of a maximal ideal is not necessarily a maximal ideal so we cannot make these constructions functorial.
- That's why algebraic geometers use the *prime* ideals and define the *spectrum* of a ring in terms of prime ideals.
- Before I get mired in scheme theory let me beat a hasty retreat!
- $C(X)$ cannot possibly produce arbitrary commutative rings: there will never be nilpotent elements.
- So what kind of rings do arise as $C(X)$ for some compact Hausdorff space X?
- Answer: C^*-algebras. This is Gelfand duality.

Algebras

All vector spaces are assumed to be over the field of complex numbers.

Algebras

An *algebra* is a vector space equipped with an associative multiplication operation $\cdot$, that is bilinear in its arguments.

Mat_n: $n \times n$ matrices with entries in $\mathbb{C}$; a noncommutative example.

Bounded linear operators on a Hilbert space $\mathcal{H}$; written as $\mathcal{B}(\mathcal{H})$. The multiplication is composition. This is also noncommutative.

The space $C(X)$ with pointwise multiplication; a commutative algebra.

Banach algebras

Norm and Banach space

A **norm** on a vector space V is a function $\|\cdot\| : V \longrightarrow \mathbb{R}$ satisfying:

- $\|\alpha v\| = |\alpha| \, \|v\|$
- $\|u + v\| \leq \|u\| + \|v\|$
- $\|v\| = 0$ iff $v = 0$.

A vector space with a norm is called a **normed space** and a normed space that is *complete* in the metric induced by the norm is called a **Banach space**.

A **Banach algebra** A is an algebra and a Banach space with a norm $\|\cdot\| : A \longrightarrow \mathbb{R}^+$ such that $\|ab\| \leq \|a\| \, \|b\|$.

It is easy to see that the multiplication operation is jointly continuous in the topology induced by the norm.

Examples of Banach algebras

- If X is any set then $l^\infty(X)$ the set of *bounded* complex-valued functions with pointwise operations and the *sup* norm is a *unital* Banach algebra.
- If X is a topological space then $C_b(X)$ the space of all bounded *continuous* complex-valued functions is a unital Banach algebra, in fact a *closed* subalgebra of $l^\infty(X)$.
- If X is compact, then the space of all continuous complex-valued functions of X (written $C(X)$) is a unital Banach algebra, being the same as $C_b(X)$.
- If X is a locally compact Hausdorff space we say that a function $f : X \longrightarrow \mathbb{C}$ **vanishes at infinity** if $\forall \varepsilon > 0,\ \{x \in X| \mid f(x) \mid \geq \varepsilon\}$ is compact.
- The set of continuous functions that vanish at infinity is written $C_0(X)$.
- $C_0(X)$ is a closed subalgebra of $C_b(X)$ and is unital if and only if X is compact.

Star algebras

- An **involution** on an algebra A is a map $* : A \longrightarrow A$ such that
  1. $\forall a \in A, \alpha \in \mathbb{C},\ (\alpha a)^* = \overline{\alpha} a^*$,
  2. $\forall a \in A,\ (a^*)^* = a$ and
  3. $\forall a, b \in A,\ (ab)^* = b^* a^*$.
- An algebra with an involution is called a $*$-**algebra**.
- An element $a \in A$ is called **self-adjoint** or **hermitian** if $a = a^*$.
- Every element a in a $*$-algebra can be written as $a = b + ic$ where b, c are hermitian.
- A self-adjoint element p is called a **projection** if $p^2 = p$.
- Note that aa^* and a^*a are always self-adjoint; they are called **positive elements**.

C^*-algebras

- A $\mathbf{C}^*$-**algebra** A is a Banach algebra with an involution satisfying
- $\forall a \in A,\ \|a^*a\| = \|a\|^2$: the C^* identity.
- It follows that $\|a\| = \|a^*\|$.
- A $*$-homomorphism is a map preserving multiplication and the involution.
- They are *automatically* contractive: $\|\phi(a)\| \leq \|a\|$ (hence continuous) and,
- if ϕ is injective then $\|\phi(a)\| = \|a\|$.
- C^*-algebras may or may not be unital, if they are $\|1\| = 1$.
- There is a unique norm on a C^*-algebra!
- More precisely, given a $*$-algebra, there is at most one way of endowing it with a norm satisfying the C^* identity.

Spectrum

- We fix a unital C^*-algebra A with unit 1.
- An element $a \in A$ is said to be *invertible* if $\exists b$ such that $ab = ba = 1$, we write a^{-1} for b.
- Write Inv(A) for the set of invertible elements of A. Note that it is a group.

Spectrum

The **spectrum** of a is

$$\sigma(a) := \{\lambda \in \mathbb{C} | \lambda 1 - a \notin \text{Inv}(A)\}.$$

Examples of spectra

- If A is the algebra of $n \times n$ matrices then $\sigma(a)$ is the set of eigenvalues of a.
- If X is a compact Hausdorff space and A is the algebra $C(X)$ then $\sigma(f) = \text{range}(f)$.
- Thus the notion of spectrum generalizes the notion of range of a function as well as eigenvalues of a matrix.
- For operators on infinite-dimensional spaces the spectrum is not just the set of eigenvalues. In fact there may be no eigenvalues.
- Consider $L^2(\mathbb{R})$ and the bounded linear map $f \mapsto (\frac{1}{1+x^2} \cdot f)$. This has no eigenvalues.

Spectrum is non-empty

Gelfand

If A is a unital Banach algebra then $\sigma(a)$ is non-empty.

The proof uses some basic complex analysis.

Gelfand-Mazur

If a Banach algebra is a field then it is isomorphic to $\mathbb{C}$.

This is an immediate corollary.

Characters

Definition

A **character** on an a commutative algebra A is a non-zero homomorphism $\tau : A \longrightarrow \mathbb{C}$. We write $\Omega(A)$ for the set of characters on A.

Just as we moved from ultrafilters to boolean algebra homomorphisms in Stone duality, we have

Proposition

For a commutative unital Banach algebra (CUBA) $\tau \mapsto \ker(\tau)$ is a bijection between $\Omega(A)$ and the set of maximal ideals.

Proposition

For a CUBA A, $\forall a \in A,\ \sigma(a) = \{\tau(a) | \tau \in \Omega(A)\}$.

Topologizing $\Omega(A)$

A general strategy for defining topologies

Choose a set of functions $\mathcal{F}$ from a set X to a topological space Y. Define the *weakest* (fewest open sets) topology that makes every function in $\mathcal{F}$ continuous: $\sigma(X, \mathcal{F})$.

The Gelfand topology

Weakest topology that makes all the maps

$$\Omega(A) \longrightarrow \mathbb{C} : \text{eval}_a(\tau) = \tau(a)$$

continuous.

Compactness

Theorem

A is a CUBA if and only if $\Omega(A)$ is a compact Hausdorff space.

If A is not unital then $\Omega(A)$ is locally compact. Adding a unit to A is the same as the "one-point compactification" of $\Omega(A)$.

Functoriality

$\Omega(\cdot)$ as a functor

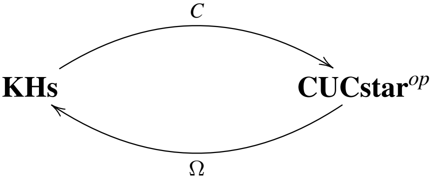
If $h : A \longrightarrow B$ is a Banach algebra map then $\Omega(h) : \Omega(B) \longrightarrow \Omega(A)$ is $f \mapsto h \circ f$.

$C(\cdot)$ as a functor

If $f : X \longrightarrow Y$ is a continuous function between compact Hausdorff spaces then $C(f) : C(Y) \longrightarrow C(X)$ given by $C(f)(g) = g \circ f$ is a Banach algebra map, i.e. a norm-decreasing homomorphism.

But wouldn't we like it to be an isometry?

Finally! Gelfand duality



Theorem

For a commutative unital C^*-algebra A the map $a \mapsto \text{eval}_a : A \to C(\Omega(A))$ is an isometric $*$-isomorphism.

This is even true for non-unital algebras if one uses C_0.

Corollary

For two commutative C^* algebras A and B, $\Omega(A)$ and $\Omega(B)$ are homeomorphic iff A and B are *isometrically* $*$-isomorphic.