Quantum alternation

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Quantum programming languages

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- Quantum $\lambda$-calculus: hard to give semantics.
- Measurement calculus: low-level, close to implementation.
- Selinger’s Quantum Programming Language: Quantum data and classical control.
- There are more.
Example

Simple program

```
input b: bit;
input p, q: qbit;

b := measure p;
if b then q := N(q) else p := N(p);
output p, q
```

$N$ is the NOT operation on a qubit.

bit and qbit separate datatypes.
The conditional is based on a classical boolean.
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Simple program

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q = |0⟩;
q := H(q);
if q then skip else p := N(p);
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Here $H$ is the one-qubit Hadamard gate. $q$ is in the state $\sqrt{2}(|0⟩ + |1⟩)$ just before the conditional. The if is producing a controlled not. Does this make sense?

Quantum alternation is problematic in general.
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Cones and positive elements

- A **cone** $C$ in a vector space $V$ is a *subset* of $V$ such that

  1. If $x, y \in C$ then $x + y \in C$,
  2. If $x \in C$ and $r \in \mathbb{R}^+$ then $r \cdot x \in C$,
  3. If $x \in C$ and $-x \in C$ then $x = 0$.

Given a cone we can define a notion of **positive** element by saying $x$ is positive if $x \in C$. We induce a partial order $\leq_C$ by $x \leq_C y$ if $y - x \in C$. 

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A **positive map** is a map from $\mathcal{B}(\mathcal{H})$ to itself such that it takes positive operators to positive operators.
Completely positive maps

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- A superoperator is a cp map that is also trace non-increasing.
Notation

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- We write $CP(M_n, M_k)$ for completely positive maps from $M_n$ to $M_k$. 
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- A ***-algebra** is an algebra equipped with a unary operation $*$ such that: (i) $a^{**} = a$, (ii) $(ab)^* = b^*a^*$ and (iii) $(\lambda a)^* = \overline{\lambda}a^*$, where $\lambda \in \mathbb{C}$. 

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The matrix algebras $M_n$ are all $C^*$-algebras with the $\ast$ being $\dagger$ (adjoint).
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If it is complete in the norm we have a Banach algebra.

A *-algebra is an algebra equipped with a unary operation \( * \) such that: (i) \( a^{**} = a \), (ii) \( (ab)^* = b^* a^* \) and (iii) \( (\lambda a)^* = \overline{\lambda} a^* \), where \( \lambda \in \mathbb{C} \).

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The bounded operators on a Hilbert space form a C*-algebra.
About $C^*$-algebras

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- Every commutative unital $C^*$-algebra is isomorphic to the set of continuous functions on a compact Hausdorff space (Gelfand duality).
Representations

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- However, abstract $C^*$-algebras can be represented in a concrete way as a subalgebra of $\mathcal{B}(\mathcal{H})$.
- A representation of a $C^*$-algebra $\mathcal{A}$ is a homomorphism $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ for some Hilbert space.
Three ways to understand CP maps

Let us consider maps on spaces of matrices. Suppose that $\phi$ is a CP map and $A$ is a matrix:

$$
\phi(A) = \sum_i K_i^\dagger AK_i
$$

where $K_i$ are matrices satisfying

$$
\sum_i K_i K_i^\dagger \leq I.
$$

This decomposition is not unique. If $\phi$ is $M_n \rightarrow M_k$ then $K_i$ are all $n \times k$ and there are fewer than $n \cdot k$ of them.

Choi: The action of $\phi \in \text{CP}(M_n, M_k)$ can be given explicitly as a matrix in $M_{nk}$ depending on the particular Kraus decomposition.

Stinespring: For any completely positive map $\theta: A \rightarrow B$ ($H$) there is a triple $(\pi, V, K)$ where $K$ is a Hilbert space, $\pi: A \rightarrow B (K)$ is a representation and $V: H \rightarrow K$ such that

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\theta(a) = V^\dagger \pi(a) V.
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- **Stinespring**: For any completely positive map $\theta : A \to B(\mathcal{H})$ there is a triple $(\pi, V, K)$ where $K$ is a Hilbert space, $\pi : A \to B(K)$ is a representation and $V : \mathcal{H} \to K$ such that
  \[ \theta(a) = V^\dagger \pi(a) V. \]
Any completely positive map can be realized as a “twisted” homomorphism.
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If $\theta \in CP(M_n, M_k)$ then the minimal Stinespring representation is in $M_m$ where $m \leq n^2 k$. 
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- Let $\mathcal{E} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be a superoperator.
- By Stinespring, there exists an ancilla $\mathcal{A}$ and an operator $V : \mathcal{K} \to \mathcal{H} \otimes \mathcal{A}$ such that

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Choose a basis $\{e_i\}_{i=1}^k$ for $\mathcal{A}$ and define $V_i : \mathcal{K} \to \mathcal{H}$ by

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Easy to check $\mathcal{E}(\rho) = \sum_{i=1}^k V_i^* \rho V_i.$
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Easy to check $\mathcal{E}(\rho) = \sum_{i=1}^k V_i^* \rho V_i$.

The $V_i$ give a Kraus representation for $\mathcal{E}$. 

---

**Stinespring to Kraus**

- Let $\mathcal{H}$ and $\mathcal{K}$ be two finite-dimensional Hilbert spaces and $B(\mathcal{H}), B(\mathcal{K})$ the Banach algebras of bounded linear operators.
- Let $\mathcal{E} : B(\mathcal{H}) \rightarrow B(\mathcal{K})$ be a superoperator.
- By Stinespring, there exists an ancilla $\mathcal{A}$ and an operator $V : \mathcal{K} \rightarrow \mathcal{H} \otimes \mathcal{A}$ such that
  $$\mathcal{E}(\rho) = V^* (\rho \otimes I_{\mathcal{A}}) V.$$
- Choose a basis $\{e_i\}_{i=1}^k$ for $\mathcal{A}$ and define $V_i : \mathcal{K} \rightarrow \mathcal{H}$ by
  $$\forall \psi \in \mathcal{K}, \quad V\psi = \sum_{i=1}^k (V_i \psi) \otimes e_i.$$
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Panangaden (McGill University)  
Quantum alternation  
Amsterdam, 8th May 2015  
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Löwner order on density matrices

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Löwner order on density matrices

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- \( A \sqsubseteq B \) if \( B - A \) is positive.
- Recall density matrices are defined to have trace \( \leq 1 \), so the zero matrix is the smallest element in this order.
- In this order, every increasing sequence has a least upper bound (lub). Such a structure is called a directed-complete partial order (dcpo).
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- Note it is not a lattice.
- Least upper bounds of increasing sequences co-incide with topological limits in the euclidean topology.
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- A function from a dcpo to another dcpo is called **Scott continuous** if it preserves lubs of increasing sequences.
Iteration

- Loop in the flowchart.
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- This sum can be proven to converge yielding a density matrix with trace $\leq 1$. 
Recursion

- Part of the program can call itself.
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- The meaning is given by a least upper bound of the increasing sequence.
- Because the density matrices form a dcpo we are sure that the lubs exist.
- Recursion can implement iteration but not the other way around.
What do we want?

Suppose we have a qubit $q$ and two superoperators $S, T : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$ then the quantum alternation $(q\text{Alt})(q; S, T)$ should be a superoperator from $\mathcal{B}(\mathcal{Q} \otimes \mathcal{H}) \rightarrow \mathcal{B}(\mathcal{Q} \otimes \mathcal{K})$. 
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We want the operation to be monotone so we can use this inside recursions.
Can we really do all this?

- No!
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- No!
- It is not possible to make it compositional and stick with superoperators.

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Can we really do all this?

- No!
- It is not possible to make it compositional and stick with superoperators.
- Can we define it in a monotone way?
- I am *almost* sure this is impossible.
Basic scheme

- $\mathcal{H}$ Hilbert space with orthonormal basis $\{e_i\}_{i=1}^n$, $\mathcal{K}$ another Hilbert space.
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$$

- If $\mathcal{H}$ is a qubit then we have $\left( |0\rangle \langle 0| \otimes U_0 \right) + \left( |1\rangle \langle 1| \otimes U_1 \right)$.
- Action: $\left( \sum_i e_i \otimes \psi_i \right) \mapsto \left( \sum_i e_i \otimes U_i \psi_i \right)$. 
Examples I

Syntax: if $q$ then $U_0$ else $U_1$. 
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- Toffoli gate uses nested if:
  **if** $q_0$ **then** skip **else** **if** $q_1$ **then** skip **else** $q_2^* = N$.
- Very useful for describing algorithms especially if there are only unitaries.
Examples II: Deutsch’s algorithm

Given a function $f : \{0, 1\} \rightarrow \{0, 1\}$ we can determine if $f$ is a constant function or not, $f(0) = f(1)$ or not using only one computation of $f$. 
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```plaintext
new qbit x, y
x* = H
y* = N; H
if x then y* = U_0 else y* = U_1
x* = H
```
Example III: Quantum Fourier transform

for $i = 1$ to $n$ do
    $q_i := H$

for $k = 2$ to $n - i + 1$ do
    if $q_{k+i-1}$ then skip else $q_i := R_k$

Here $R_k$ is the phase shift gate defined by $R_k = \Pi_0 + e^{i\theta} \Pi_1$ with $\theta = \frac{2\pi}{2^k}$. 
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- Simple and intuitive, but
- can we extend it to quantum operations that are not unitaries?
What is a Kraus form?

- A superoperator describes the most general physical transformation of a system.
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- A particular Kraus form comes from a particular choice of basis of the environment, as we saw.
- A basis corresponds to a particular choice of measurement. Thus the particular Kraus representation is dictated by how the experimenter chooses to describe the environment.
Semantics in terms of Kraus forms

Our position: Do not try to give semantics in terms of superoperators, give the semantics in terms of the Kraus forms.
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- We give compositional semantics but in terms of specific choices of Kraus operators, we do not try to give compositional superoperator semantics.
Given unitary operators $U$, $V$ on $\mathcal{H}$ and a qubit $q$ (space $Q$) we define

$$|0\rangle\langle0| \otimes U + |1\rangle\langle1| \otimes V = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$$

as the quantum alternation of $U$ and $V$. 
Alternation of Kraus forms

- Given superoperators $\mathcal{E}, \mathcal{F}$ with Kraus forms

$$
\mathcal{E} \rho = \sum_{i=1}^{m} \mathcal{E}^* i \rho \mathcal{E}^i \\
\mathcal{F} \rho = \sum_{j=1}^{n} \mathcal{F}^* j \rho \mathcal{F}^j
$$

We define a family of operators $K_{i,j}$ by

$$
K_{i,j} = |0\rangle\langle 0| \otimes (1/\sqrt{m} \mathcal{E}^i 0 0 1/\sqrt{n} \mathcal{F}^j)
$$

This defines a superoperator $S(\rho) = \sum_{i,j} K^*_{i,j} \rho K_{i,j}$. 
Given superoperators $\mathcal{E}, \mathcal{F}$ with Kraus forms

$\mathcal{E} \rho = \sum_{i=1}^{m} E_i^* \rho E_i$ and $\mathcal{F} \rho = \sum_{j=1}^{n} F_j^* \rho F_j$, 

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Alternation of Kraus forms

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$$E \rho = \sum_{i=1}^{m} E_i^{*} \rho E_i \quad \text{and} \quad F \rho = \sum_{j=1}^{n} F_j^{*} \rho F_j,$$

- we define a family of operators $K_{i,j}$ by

$$K_{i,j} = |0 \otimes \frac{1}{\sqrt{n}} E_i \rangle + |1 \otimes \frac{1}{\sqrt{m}} F_j \rangle = \begin{pmatrix} \frac{1}{\sqrt{n}} E_i & 0 \\ 0 & \frac{1}{\sqrt{m}} F_j \end{pmatrix}.$$
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  \[
  \mathcal{E}\rho = \sum_{i=1}^{m} E_i^* \rho E_i \quad \text{and} \quad \mathcal{F}\rho = \sum_{j=1}^{n} F_j^* \rho F_j,
  \]

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  \[
  K_{i,j} = |0 \otimes \frac{1}{\sqrt{n}}E_i + 1 \otimes \frac{1}{\sqrt{m}}F_j\rangle = \begin{pmatrix}
  \frac{1}{\sqrt{n}}E_i & 0 \\
  0 & \frac{1}{\sqrt{m}}F_j
  \end{pmatrix}.
  \]

- This defines a superoperator
  
  \[
  S(\rho) = \sum_{i,j} K_{i,j}^* \rho K_{i,j}.
  \]
What Stinespring says

If one looks at the Stinespring dilation corresponding to the above construction we see that the ancilla spaces (environments) of the two Kraus forms are tensored together.
Kraus semantics

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- The meaning of a construct will be given by a set of Kraus operators.
- Sequential composition

\[
[P; Q] = [Q] \circ [P] = \{E_i \circ F_j \mid E_i \in [P], F_j \in [Q]\}.
\]
Kraus semantics

- We think of a superoperator as being given by a specific Kraus form.
- We write the composition of Kraus forms as $S \bullet T$ where $S$ and $T$ are specific Kraus forms for the superoperators.
- We interpret commands in the quantum programming language as specific Kraus forms. So we can think of a superoperator as a set of Kraus operators.
- The meaning of a construct will be given by a set of Kraus operators.
- Sequential composition

\[
[P; Q] = [Q] \circ [P] = \{E_i \circ F_j \mid E_i \in [P], F_j \in [Q]\}.
\]

- Applying a unitary

\[
[\star = U] = \{U\}.
\]
More semantics

- Measure $q$, this has type $\tau \rightarrow \tau \oplus \tau$

  \[
  \llbracket \text{measure } q \rrbracket = \{ \text{in}_0 \circ \Pi_0, \text{in}_1 \circ \Pi_1 \}.\]
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- We do not give semantics for loops and conditionals.
Quantum alternation cannot be compositional

- More precisely: If the semantics is based on superoperators it cannot be compositional.

Consider $P \equiv e^{i\theta}I$ and $I$, as superoperators these are identical. But if $q$ then $I$ else $P$ is definitely not the same as if $q$ then $I$ else $I$; the latter is clearly the same as $I$ and the first is the controlled-phase gate.

This example arose from discussions with Mingsheng Ying and Yuan Feng at UTS Sydney based on an example due to Nengkun Yu.

One can think of quantum alternation as an algorithmic notation, it is not clear what it means physically.
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Quantum control even with just unitary operators, is not monotone with respect to the Löwner order.
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- Let $U, V$ be one-qubit unitaries and $\lambda, \mu \in [0, 1]$.
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Quantum control: semantics

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- By explicit calculation we can show that $R' \not\preceq R$. 
Is there any way to choose a canonical Kraus form?

Yes, mathematically there is, but does it mean anything physically?

There is an operator-algebra version of the Radon-Nikodym theorem due to Belavkin and Arveson (BARN).

One can show that every CP map is uniformly dominated by the tracial map from $\mathcal{M}_n$ to $\mathcal{M}_k$:

$$\text{trmap}(C) = \frac{1}{n} \text{tr}(C) I_k.$$  

The BARN then gives a Kraus decomposition.

One can give a denotational semantics based on these "canonical" Kraus forms but there is little reason to think that this has physical significance.
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- However, they did not give complete rules. For example, one cannot nest quantum conditionals.
Ying-Yu-Feng 2014

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- Defined a superoperator semantics and noted lack of compositionality.
- Implicit in their superoperator semantics is our Kraus semantics.
- Perhaps one should view the superoperator semantics as an *abstract interpretation* of the Kraus semantics.
- Did not note non-monotonicity but had a different approach to recursion based on Fock space [Ying 2015].
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Perhaps quantum alternation and recursion is not allowed in nature!
Thank you!