Equational reasoning for probabilistic programming
Part II: Quantitative equational logic

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Approximate equations: $s \approx_\varepsilon t$, $s$ is within $\varepsilon$ of $t$.

Definitely not an equivalence relation;

it defines a *uniformity* (but we won’t stress this point of view).

Quantitative analogue of equational reasoning.

completeness results, universality of free algebras, Birkhoff-like variety theorem, monads ....
Moggi 1988: How to incorporate “effects” into denotational semantics?

(Strong) Monads!

Plotkin, Power (and then many others): view effects algebraically. Monads are given by operations and equations.

Categorically: equational presentations are Lawvere theories (but we won’t talk about them here either).

A monad of great interest: Lawvere (1964) The category of probabilistic mappings.

Probabilistic reasoning requires measure theory but,
measure theory works best on Polish spaces (topological space underlying separable complete metric spaces).
Metric ideas present in semantics from the start: Jaco de Bakker’s school.
Algebras will come with metric structure and quantitative equational theories will define monads on $\text{Met}$. 
Quantitative equations

- Signature $\Omega$, variables $X$ we get terms $\mathbb{T}X$.
- Quantitative equations: $\mathcal{V}(\mathbb{T}X)$:

  \[ s =_{\varepsilon} t, \quad s, t \in \mathbb{T}X, \quad \varepsilon \in \mathbb{Q} \cap [0, 1] \]

- A substitution $\sigma$ is a map $X \to \mathbb{T}X$; we write $\Sigma(X)$ for the set of substitutions.
- Any $\sigma$ extends to a map $\mathbb{T}X \to \mathbb{T}X$.
- Quantitative inferences: $\mathcal{E}(\mathbb{T}X) = \mathcal{P}_{\text{fin}}(\mathcal{V}(\mathbb{T}X)) \times \mathcal{V}(\mathbb{T}X)$

  \[ \{ s_1 =_{\varepsilon_1} t_1, \ldots, s_n =_{\varepsilon_n} t_n \} \vdash s =_{\varepsilon} t \]
Deducibility relations

(Refl) \( \emptyset \vdash t =_0 t \)

(Symm) \( \{ t =_\varepsilon s \} \vdash s =_\varepsilon t. \)

(Triang) \( \{ t =_\varepsilon s, s =_{\varepsilon'} u \} \vdash t =_{\varepsilon+\varepsilon'} u. \)

(Max) For \( e' > 0 \), \( \{ t =_\varepsilon s \} \vdash t =_{\varepsilon+\varepsilon'} s. \)

(Arch) For all \( \varepsilon \geq 0 \), \( \{ t =_{\varepsilon'} s \mid \varepsilon' > \varepsilon \} \vdash t =_\varepsilon s. \) Infinitary!

(NExp) For \( f : n \in \Omega \),
\[
\{ t_1 =_\varepsilon s_1, \ldots, t_n =_\varepsilon s_n \} \vdash f(t_1, \ldots t_i, \ldots t_n) =_{\varepsilon} f(s_1, \ldots s_i, \ldots s_n)
\]

(Subst) If \( \sigma \in \Sigma(X) \), \( \Gamma \vdash t =_\varepsilon s \) implies \( \sigma(\Gamma) \vdash \sigma(t) =_\varepsilon \sigma(s) \).

(Cut) If \( \Gamma \vdash \phi \) for all \( \phi \in \Gamma' \) and \( \Gamma' \vdash \psi \), then \( \Gamma \vdash \psi \).

(Assumpt) If \( \phi \in \Gamma \), then \( \Gamma \vdash \phi \).
Quantitative equational theories

- Given $S \subseteq \mathcal{E}(TX)$, $\vdash_S$: smallest deducibility relation containing $S$.
- Equational theory: $\mathcal{U} = \vdash_S \cap \mathcal{E}(TX)$. 
Quantitative algebras

- $\Omega$: signature; $A = (A, d)$, $A$ an $\Omega$-algebra and $(A, d)$ a metric space.
- All functions in $\Omega$ are nonexpansive.
- Morphisms are $\Omega$-algebra homomorphisms that are nonexpansive.
- $\mathbb{T}X$ is an $\Omega$-algebra. $\sigma : \mathbb{T}X \to A$, $\Omega$-homomorphism.
- $(A, d)$ satisfies $\{ s_i = \varepsilon_i \ i = 1, \ldots, n \} \vdash s =_\varepsilon t$ if

$$\forall \sigma, \ d(\sigma(s_i), \sigma(t_i)) \leq \varepsilon_i, \ i = 1, \ldots, n$$

implies

$$d(\sigma(s), \sigma(t)) \leq \varepsilon.$$ 

- We write $\{ s_i =_\varepsilon t_i \ i = 1, \ldots, n \} \models_A s =_\varepsilon t$.
- We write $\mathbb{K}(\mathcal{U}, \Omega)$ for the algebras satisfying $\mathcal{U}$. 
A metric on $T_X$

$$d^U(s, t) = \inf \{ \varepsilon \mid \emptyset \vdash s =^\varepsilon t \in U \}$$

- Why not use the following?

$$d^U(s, t) = \inf \{ \varepsilon \mid \forall V \in \mathcal{P}_f(\mathcal{V}(X)), V \vdash s =^\varepsilon t \in U \}$$

- They are the same!
- The (pseudo)metric can take on infinite values.
- The kernel is a congruence for $\Omega$.
- If we take the quotient we get an (extended) metric space.
- The resulting algebra is in $\mathbb{K}(\Omega, U)$.
- We can do this for any set $M$ of generators and produce a “free” quantitative algebra.
∀A ∈ K(U, Ω), Γ ⊨_A φ if and only if [Γ ⊨ φ] ∈ U.

- Analogue of the usual completeness theorem for equational logic.
- Right to left is by definition.
- Left to right is by a model construction.
- The proof needs to deal with quantitative aspects and uses the archimedean property.
Starting from a **metric space** \((M, d)\) we can define \(TM\) by adding constants for each \(m \in M\)

and axioms \(\emptyset \vdash m =_e n\) for every rational \(e\) such that \(d(m, n) \leq e\).

Call this extended signature \(\Omega_M\) and the extended theory \(U_M\).

Any algebra in \(\mathbb{K}(U_M, U_M)\) can be viewed as an algebra in \(\mathbb{K}(\Omega, U)\) by forgetting about the interpretation of the constants from \(M\).

Given any \(\alpha : M \to A\) non-expansive we can turn \(A = (A, d)\) into an algebra in \(\mathbb{K}(\Omega_M, U_M)\) by interpreting each \(m \in M\) as \(\alpha(m) \in A\).
$\mathcal{U}_M$ is consistent if and only if the map $\eta_M$ is an isometry.

We have a monad on $\textbf{Met}$. 

$\mathcal{U}(\Omega, \mathcal{U})$
Birkhoff Variety Theorems

- Three kinds of equations: (a) unconditional equations
- (b) basic equations: assumptions of the form $x =_e y$, $x, y$ variables.
- (c) Horn clauses, assumptions may involve terms.

Usual variety theorem says: a class of algebras is equationally definable if and only if it is closed under products, homomorphic images and subalgebras.

We have to consider a new kind of closure property.
Reflexive homomorphisms

- A \( c \)-reflexive homomorphism \( f \) between QA’s \( A, B \), where \( c \) is a cardinal number, is a homomorphism with the property that for any subset \( B' \subset B \) with \(|B'| < c\), there is a subset \( A' \subset A \) with \( f(A') = B' \) and \( f \) restricted to \( A' \) is an isometry.

- If \( \mathcal{U} \) is an unconditional theory then \( \mathbb{K}(\Omega, \mathcal{U}) \) is closed under homomorphic images.

- If \( \mathcal{U} \) is a basic equational theory with every conditional equation having only finitely many assumptions then \( \mathbb{K}(\Omega, \mathcal{U}) \) is closed under \( \aleph_0 \)-reflexive homomorphisms.

- If \( \mathcal{U} \) is a basic equational theory then \( \mathbb{K}(\Omega, \mathcal{U}) \) is closed under \( \aleph_1 \)-reflexive homomorphisms.

- A \( c \)-variety is a class of algebras closed under products, subalgebras and \( c \)-reflexive homomorphisms.

- A \( c \)-equational class is a class of algebras defined by \( c \)-basic conditional equations.
The main theorem

\( \mathcal{K} \) is a \( c \)-variety if and only if it is a \( c \)-basic equational class.

- \( \mathcal{K} \) is an unconditional equational class iff it is a variety.
- \( \mathcal{K} \) is a finitary-basic equational class iff it is an \( \aleph_0 \)-variety.
- \( \mathcal{K} \) is a basic equational class iff it is an \( \aleph_1 \)-variety.