Semantics of Probabilistic Languages

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Outline

1. Introduction
2. Semantics of a language with while loops
3. Partially additive categories
4. Back to semantics
Syntax

Kozen’s Language

\[ S ::= x_i := f(\vec{x}) | S_1; S_2 | \text{if } B \text{ then } S_1 \text{ else } S_2 | \text{while } B \text{ do } S. \]

- There are a fixed set of variables \( \vec{x} \) taking values in a measurable space \((X, \Sigma_X)\).
- \( f \) is a measurable function.
- \( B \) is a measurable subset.
Outline of the semantics

- State transformer semantics: distribution (measure) transformer semantics.
- Meaning of statements: Markov kernels \( i.e. \) SRel morphisms.
- The only subtle part: how to give fixed-point semantics to the while loop?
Partially additive categories

Partially additive monoids

- Back to **SRel** structure.
- Can we “add” **SRel** morphisms?
- Not always, the sum may exceed 1, but we can define *summable families* which may even be countably infinite.
- The homsets of **SRel** form *partially additive monoids*.

A partially additive monoid is a pair \((M, \sum)\) where \(M\) is a nonempty set and \(\sum\) is a partial function which maps some countable subsets of \(M\) to \(M\). We say that \(\{x_i \mid i \in I\}\) is summable if \(\sum_{i \in I} x_i\) is defined.
Partially additive categories

Axioms for partially-additive monoids

1. The sums can be rearranged at will.

2. **Partition-Associativity:** Suppose that \( \{x_i | i \in I\} \) is a countable family and \( \{I_j | j \in J\} \) is a countable partition of \( I \). Then \( \{x_i | i \in I\} \) is summable iff for every \( j \in J \) \( \{x_i | i \in I_j\} \) is summable and \( \{\sum_{i \in I_j} x_i | j \in J\} \) is summable. In this case we require

\[
\sum_{i \in I} x_i = \sum_{j \in J} \sum_{i \in I_j} x_i.
\]

3. **Unary-sum:** A singleton family is always summable.

4. **Limit:** If \( \{x_i | i \in I\} \) is countable and every finite subfamily is summable then the whole family is summable.
Zero morphisms

The sum of the empty family exists, call it 0. It is the identity for $\sum$. 
Let $C$ be a category. A **partially additive structure** on $C$ is a partially additive monoid structure on the homsets of $C$ such that if $\{f_i : X \to Y | i \in I\}$ is summable, then $\forall W, Z, g : W \to X, h : Y \to Z$, we have that $\{h \circ f_i | i \in I\}$ and $\{f_i \circ g | i \in I\}$ are summable and, furthermore, the equations

$$h \circ \sum_{i \in I} f_i = \sum_{i \in I} h \circ f_i, \quad (\sum_{i \in I} f_i) \circ g = \sum_{i \in I} f_i \circ g$$

hold.

A category has **zero morphisms** if there is a distinguished morphism in every homset – we write $0_{XY}$ for the distinguished member of $\text{hom}(X, Y)$ – such that $\forall W, X, Y, Z, f : W \to X, g : Z \to Y$ we have $g \circ 0_{WZ} = 0_{XY} \circ f$.

If a category has a partially additive structure it has zero morphisms.
**SRel** has partially additive structure

- A family \( \{ h_i : X \to Y | i \in I \} \) in **SRel** is summable if
  \[
  \forall x \in X. \sum h_i(x, Y) \leq 1.
  \]

  We define the sum by the evident pointwise formula.

- Partition associativity follows immediately from the fact that we are dealing with absolute convergence since all the values are nonnegative.

- The unary sum axiom is immediate.

- The limit axiom follows from the fact that the finite partial sums are bounded by 1.

- Countable additivity follows from the fact that each \( h_i \) is countably additive and the sums in question can be rearranged since we have only nonnegative terms.

- The verification of the two distributivity equations is by the monotone convergence theorem.
Partially additive categories

Quasi-projections

Let $C$ be a category with countable coproducts and zero morphisms and let $\{X_i | i \in I\}$ be a countable family of objects of $C$.

For any $J \subset I$ we define the **quasi-projection** $PR_J : \bigsqcup_{i \in I} X_i \rightarrow \bigsqcup_{j \in J} X_j$ by

$$PR_J \circ \iota_i = \begin{cases} \iota_i & i \in J \\ 0 & i \notin J \end{cases}$$
We write $I \cdot X$ for the coproduct of $|I|$ copies of $X$. We define the **diagonal-injection** $\triangle$ by couniversality:

\[
\coprod (X_i | i \in I) \xrightarrow{\triangle} I \cdot \coprod (X_i | i \in I)
\]

We have a morphism $\sigma$ from $I \cdot X$ to $X$ given by:

\[
I \cdot X \xrightarrow{\sigma} X
\]
These maps in $\mathbf{SRel}$

\[ PR_J \left( (x, k), \bigsqcup_{j \in J} \right) = \begin{cases} 
\delta(x, A_k) & k \in J \\
0 & k \notin J 
\end{cases}. \]

The $\Delta$ morphism in $\mathbf{SRel}$ is

\[ \Delta \left( (x, k), \bigsqcup_{i \in I} (\bigsqcup_{j \in I} A^i_j) \right) = \delta(x, A^k_k). \]

The analogous map in $\mathbf{Set}$ is $\Delta((x, k)) = ((x, k), k)$.

Finally

\[ \sigma((x, k), A) = \delta(x, A) \]

in $\mathbf{SRel}$ while in $\mathbf{Set}$ we have $\sigma((x, k)) = x$. 
A **partially additive category**, $\mathcal{C}$, is a category with countable coproducts and a partially additive structure satisfying the following two axioms.

1. **Compatible sum axiom**: If $\{f_i | i \in I\}$ is a countable set of morphisms in $\mathcal{C}(X, Y)$ and there is a morphism $f : X \rightarrow I \cdot Y$ with $PR_i \circ f = f_i$ then $\{f_i | i \in I\}$ is summable.

2. **Untying axiom**: If $f, g : X \rightarrow Y$ are summable then $\iota_1 \circ f$ and $\iota_2 \circ g$ are summable as morphisms from $X$ to $Y + Y$. 
**SRel is a PAC**

The category **SRel** is a partially additive category.

All verifications are routine.
Iteration in a PAC

Arbib-Manes

Given \( f : X \to X + Y \) in a partially additive category, we can find a unique \( f_1 : X \to X \) and \( f_2 : X \to Y \) such that \( f = \iota_1 \circ f_1 + \iota_2 \circ f_2 \). Furthermore there is a morphism \( \dagger f = df \sum_{n=0}^{\infty} f_2 \circ f_1^n : X \to Y \). The morphism \( \dagger f \) is called the iterate of \( f \).

- First claim is trivial.
- The second is about the summability of a specific family.
- Can prove easily by induction that the finite subfamilies are summable.
- The limit axiom then guarantees that the whole family is summable.
Statements are \( \mathbf{SRel} \) morphisms of type \( (X^n, \Sigma^n) \to (X^n, \Sigma^n) \).

**Assignment:** \( x := f(\vec{x}) \)

\[
[x_i := f(\vec{x})](\vec{x}, \vec{A}) = \delta(x_1, A_1) \cdots \delta(x_{i-1}, A_{i-1}) \delta(f(\vec{x}), A_i) \delta(x_{i+1}, A_{i+1}) \cdots
\]

**Sequential Composition:** \( S_1; S_2 \)

\[
[S_1; S_2] = [S_2] \circ [S_1]
\]

where the composition on the right hand side is the composition in \( \mathbf{SRel} \).

**Conditionals:** \( \text{if } B \text{ then } S_1 \text{ else } S_2 \)

\[
[\text{if } B \text{ then } S_1 \text{ else } S_2](\vec{x}, \vec{A}) = \delta(\vec{x}, B)[S_1](\vec{x}, \vec{A}) + \delta(\vec{x}, B^c)[S_2](\vec{x}, \vec{A})
\]
While Loops: \texttt{while B do S}

\[
[\texttt{while B do S}] = h^*
\]

where we are using the \(*\) in \texttt{SRel} and the morphism

\[
h : (X^n, \Sigma^n) \rightarrow (X^n, \Sigma^n) + (X^n, \Sigma^n)
\]

is given by

\[
h(\vec{x}, \vec{A}_1 \cup \vec{A}_2) = \delta(\vec{x}, \texttt{B})[S](\vec{x}, \vec{A}_1) + \delta(\vec{x}, \texttt{B}^c)\delta(\vec{x}, \vec{A}_2).
\]
We can construct a category of probabilistic predicate transformers: $\text{SPT}$. Objects are measurable spaces.

Given $(X, \Sigma_X)$ we can construct the (Banach) space of bounded measurable functions on $X$ (the “predicates”) $\mathcal{F}(X)$.

A morphism $X \rightarrow Y$ in $\text{SPT}$ is a bounded (continuous) linear map from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$.

$\text{SPT} \simeq \text{SRel}^{op}$.

This gives us the structure needed for a $\text{wp}$ semantics.