Causality and Order

Prakash Panangaden
Dedicated to Glynn Winskel
Event Structures

Glynn Winskel’s 1981 thesis: Events in Computation was a landmark.

I saw it in 1982 just as I was moving from Physics (relativity) to Computer Science.

It, and Lamport’s earlier paper, influenced me greatly.
Summary

- The Spacetime canvas
- Causality conditions
- Axioms for causal orders
- Differences between causal structure in spacetime and in computation
What is causality?

- The mathematical theory of alibis!
- I did not do it! I could not have done it, I was nowhere nearby!!
- How far do you have to be for this to be a cast iron alibi?
- Spacetime structure has this built in.
Spacetime

- Events
- Light cones
- World lines
- Spatial slices
The Spacetime Canvas

Events: primitive “point like” — they form a set.

Topological space: 4-dimensional manifold;
Hausdorff, locally compact, separable; these imply paracompact.

Smooth structure: now we can define tangent vectors.
Spacetime is now equipped with a “smoothly varying” collection of 4-dimensional tangent vector spaces attached to each point: the tangent bundle.

Causal structure: light cones at each point.
A pair of cones at each tangent space. The paths of light rays.

Affine connection: how to parallel transport vectors.
Now we can define geodesics. The paths of dust particles.
Finishing touches

Now we know the trajectories of dust and light rays.

The only remaining ingredient: the metric.

Usually the metric is given up front and all the other structures are derived from it.

But conceptually, physically and mathematically it comes last.
Causal structure:

Newtonian view

Thursday, 20 June, 13

A SLICE of SIMULTANEITY: "NOW"
Einstein-Minkowski
Causal Structure

Light cones delimit regions of causal influence.

Basic assumption: time-orientability, \( i.e. \) there is a consistent distinction between future- and past-pointing light cones.

A vector \textit{inside} the light-cone is \textbf{timelike}.

A curve whose tangent vector is everywhere future pointing and timelike is called a timelike curve.

A vector \textit{on the light cone} is \textbf{null}.

A vector \textit{outside the light cone} is \textbf{spacelike}.
Material particles travel on timelike curves and light travels on null curves.

A spacelike \textbf{slice} is a maximal 3-dimensional hypersurface where every two points are spacelike related.

A decomposition of spacetime into a family of disjoint spacelike slices is called a \textbf{foliation}.

In Newtonian spacetime there is a \textit{unique, canonical} foliation.

In relativistic spacetimes, there are many possible foliations.
Causal orders

Chronological order, $x \ll y$: there is a smooth timelike curve (with future-pointing tangent) from $x$ to $y$.
(Regarded as irreflexive.)

Causal order, $x \leq y$: there is a piecewise-smooth curve from $x$ to $y$ with a future-pointing tangent vector that is timelike or null.

Fundamental assumption: $\leq$ is a partial order.

$J^+(x) := \{y \mid x \leq y\}$, $J^-(x) := \{y \mid y \leq x\}$

$I^+(x) := \{y \mid x \ll y\}$, $I^-(x) := \{y \mid y \ll x\}$ “open sets”
Causality Assumptions

1. $x \ll y \Rightarrow y \ll x$  Chronology condition

2. $(x \leq y) \& (y \leq x) \Rightarrow x = y$  Causality condition

Usually, we think that this is enough to rule out causal anomalies.

But, in the continuum more complicated causal anomalies can occur.

There is a hierarchy of increasingly stringent causality conditions that are imposed.

Causality $\rightarrow \ldots \rightarrow$ Global hyperbolicity.
Stricter Assumptions

Future/Past distinguishing: \[ I^\pm(x) = I^\pm(y) \Rightarrow x = y. \]

There are 3 possible conditions here; they are all different.
This satisfies causality and past distinguishability but not future distinguishability.

The light cones tip over, become horizontal, and tip back up. A small vertical strip has been removed to break the closed null curve that would otherwise occur. Here $I^+(p) = I^+(q)$. 

Thursday, 20 June, 13
Stricter Assumptions

Future/Past distinguishing: \( I^\pm(x) = I^\pm(y) \Rightarrow x = y. \)

There are 3 possible conditions here; they are all different.

Strong causality at \( p \): Every neighbourhood of \( p \) contains a sub-neighbourhood such that no causal curve intersects it more than once.

Imposing strong causality on a compact set means that no causal curve can get imprisoned in this set.

It is possible to have a spacetime satisfying the previous conditions but violating strong causality.
Figure 38. A space–time satisfying the causality, future and past distinguishing conditions, but not satisfying the strong causality condition at \( p \). Two strips have been removed from a cylinder; light cones are at \( \pm 45^\circ \).

Picture taken from: *The Large-Scale Structure of Spacetime* by Hawking and Ellis.
Consequences of Strong Causality

Define \( \ll x, y \gg = I^+(x) \cap I^-(y) \).

If \( \ll x, y \gg \cap \ll p, q \gg \neq \emptyset \) then there are \( u, v \) such that \( \ll u, v \gg \subseteq \ll x, y \gg \cap \ll p, q \gg \).

Use these sets as the base for a topology: the Alexandrov topology.

Penrose discovered a strong link between strong causality and the manifold topology.
Penrose's Theorem

**Theorem** (Penrose) The following are equivalent:

1. $M$ is strongly causal,
2. The Alexandrov topology agrees with the manifold topology,
3. The Alexandrov topology is Hausdorff.
Even stricter assumptions

Stable causality: opening up the light cones “a bit” does not create closed causal curves.

This implies the presence of a global time function.

Causal continuity: the volume of the past and future sets should vary continuously. (Picture on next slide)

Causal simplicity: $J^+(x)$ and $J^-(x)$ are closed.

This means there are no holes in space-time.

Global hyperbolicity: $J^+(x) \cap J^-(y)$ is compact.
CAUSAL CONTINUITY

VOLUME OF PAST & FUTURE SETS, $I^\pm(x)$ SHOULD VARY CONTINUOUSLY WITH $x$.

$q \in I^-(p')$ BUT $q \notin I^-(p)$.

$K \subset M - I^-(p)$
$K \not\subset M - I^-(p')$
Topology from Causality

Hawking and King had shown how to reconstruct the topology from $\ll$ and $\leq$.

Malament had shown how to reconstruct the topology from the class of continuous timelike curves.

For causally simple space times Keye Martin and I showed that the $\ll$ order is the way-below relation of the causal order.

Thus one can reconstruct $I^\pm$ sets from $\leq$.

Which allows one to reconstruct the topology from the order.
Kronheimer-Penrose Axioms

A causal Space is a set $X$ equipped with two partial orders: $\leq$, $\ll$ satisfying:

- $\leq$ is a partial order,
- $\ll$ is transitive and irreflexive,
- $\ll \subseteq \leq$,
- if $x \leq y$ and $y \ll z$ then $x \ll z$,
- if $x \ll y$ and $y \leq z$ then $x \ll z$. 
Kronheimer and Penrose use *three* relations: $\leq$, $\ll$ and $\rightarrow$, the last is called the **horismos**.

**Definition:** $x \rightarrow y$ if $x \leq y$ but not $x \ll y$.

Is there an abstract definition?

A reflexive binary relation $R$ is called **horismotic** if whenever $(x_i)_{1 \leq i \leq n}$ is a finite sequence with $x_iRx_{i+1}$ for $1 \leq i \leq n$, then for any $1 \leq j \leq k \leq n$:

(i) $x_1Rx_n$ implies that $x_jRx_k$ and

(ii) $x_nRx_1$ implies $x_j = x_k$. 

The horismos is horismotic

The \( \rightarrow \) relation of a causal space is horismotic.

Let \( x \leq y \leq z \) in a causal space.
If \( x \rightarrow z \) then \( x \rightarrow y \rightarrow z \).
Why does that matter?

Imagine a mirror at $x_2$ which bounces a light ray from $x_1$ to $x'_3$.

The path from $x_1$ to $x'_3$ can be deformed into a timelike path by smoothing the corner.

Anything on the path from $x_1$ to $x_3$ has to be related to $x_1$.

It defines the horismos as the boundary of the causal future without mentioning maximum speeds.
Conclusions

What we do not have in any of the computer science models of causal structure that I have seen: the horismos!

Interesting interplay between causal order and topology.

Recently Martin and I have incorporated the metric using his theory of measurements.

Where does differential structure fit in?

Quantum space times? Perhaps event structure will give us a clue.
Glynn in 1987