Causality, Order, Information and Topology

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1 Introduction
Outline

1. Introduction

2. Causal Structure
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2. Causal Structure
3. Domain Theory
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Causal structure - mathematically modelled as a partial order - can be taken to be the fundamental structure of spacetime.
Overview

- Causal structure - mathematically modelled as a partial order - can be taken to be the fundamental structure of spacetime.
- The topology can be derived from this.
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Spacetime carries a natural domain structure.
Scott’s vision: computability should be continuity in some topology.
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Data types are domains (ordered topological spaces) and computable functions are continuous.
The causal order alone determines the topology of globally hyperbolic spacetimes. [CMP Nov’06]
Summary of Results

- The causal order alone determines the topology of globally hyperbolic spacetimes. [CMP Nov’06]
- A (globally hyperbolic) spacetime can be given domain structure: approximate points. [CMP Nov’06]
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The space of causal curves in the Vietoridis topology is compact (cf. Sorkin-Woolgar) [GRG ’06]

The geometry can be captured by a Martin “measurement.” [AMS Symposia in Pure and Applied Math 2012]
The layers of spacetime structure

- Set of events: no structure
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- Topology: 4 dimensional real manifold, Hausdorff, paracompact,...
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- Parallel transport: affine structure.
- Lorentzian metric: gives a length scale.
The causal structure of spacetime

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- At every point a pair of "cones" is defined in the tangent space: future and past light cone. A vector on the cone is called null or lightlike and one inside the cone is called timelike.
- We assume that spacetime is time-orientable: there is a global notion of future and past.
- A timelike curve from $x$ to $y$ has a tangent vector that is everywhere timelike: we write $x \preceq y$. (We avoid $x \ll y$ for now.) A causal curve has a tangent that, at every point, is either timelike or null: we write $x \leq y$. 
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Other axioms describe the interaction of $<$ and $\preceq$.

The $\leq$ and $\ll$ orders satisfy all the axioms of a causal space.
$I^+(x) := \{ y \in M | x \preceq y \}$; similarly $I^-$

Chronology: $x \preceq y \Rightarrow y \not\preceq x$.

Causality: $x \leq y$ and $y \leq x$ implies $x = y$. 

$I^+$ and $I^-$ are always open sets in the manifold topology; $J^+$ and $J^-$ are not always closed sets.
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- $J^+(x) := \{ y \in M | x \preceq y \}$; similarly $J^-$.
- $I^\pm$ are always open sets in the manifold topology; $J^\pm$ are not always closed sets.
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Causality Conditions

\[ I^{\pm}(p) = I^{\pm}(q) \implies p = q. \]
Causality Conditions

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- Strong causality at \( p \): Every neighbourhood \( \mathcal{O} \) of \( p \) contains a neighbourhood \( \mathcal{U} \subset \mathcal{O} \) such that no causal curve can enter \( \mathcal{U} \), leave it and then re-enter it.
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- In such a spacetime a future directed causal curve cannot get trapped in a compact set.
- Stable causality: perturbations of the metric do not cause violations of causality.
- Causal simplicity: for all \( x \in M \), \( J^\pm(x) \) are closed.
Global Hyperbolicity

- Spacetime has good initial data surfaces for global solutions to hyperbolic partial differential equations (wave equations). [Leray]
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- Spacetime has good initial data surfaces for global solutions to hyperbolic partial differential equations (wave equations). [Leray]
- Global hyperbolicity: $M$ is strongly causal and for each $p, q$ in $M$, $[p, q] := J^+(p) \cap J^-(q)$ is compact.
The Alexandrov Topology

Define

\[ \langle x, y \rangle := I^+(x) \cap I^-(y). \]

The sets of the form \( \langle x, y \rangle \) form a base for a topology on \( M \) called the Alexandrov topology.

Theorem (Penrose): TFAE:

1. \( (M, g) \) is strongly causal.
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Theorem (Penrose): TFAE:

1. \((M, g)\) is strongly causal.
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3. The Alexandrov topology is Hausdorff.
Scott’s “domain” theory

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- For “directed set” think “chain.”
- Computable functions are viewed as continuous with respect to a suitable topology: the Scott topology.
- Ideal (infinite) elements are limits of their (finite) approximations.
Examples of domains

- The integers with no relation between them and a special element \( \bot \) below all the integers: a flat domain.
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- Compact non-empty intervals of real numbers ordered by *reverse* inclusion (with $\mathbb{R}$ thrown in).
Examples of domains

- The integers with no relation between them and a special element \( \bot \) below all the integers: a flat domain.
- Sequences of elements from \( \{a, b\} \) ordered by prefix: the domain of streams.
- Compact non-empty intervals of real numbers ordered by reverse inclusion (with \( \mathbb{R} \) thrown in).
- \( X \) a locally compact space with \( K(X) \) the collection of compact subsets ordered by reverse inclusion.
In addition to $\leq$ there is an additional, (often) irreflexive, transitive relation written $\ll$: $x \ll y$ means that $x$ has a “finite” piece of information about $y$ or $x$ is a “finite approximation” to $y$. If $x \ll x$ we say that $x$ is finite.

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The Way-below relation

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The relation $x \ll y$ - pronounced $x$ is “way below” $y$ - is directly defined from $\leq$.

Official definition of $x \ll y$: If $X \subset D$ is directed and $y \leq (\bigvee X)$ then there exists $u \in X$ such that $x \leq u$. If a limit gets past $y$ then some finite stage of the limiting process already got past $x$. 
A continuous domain $D$ has a basis of elements $B \subset D$ such that for every $x$ in $D$ the set $x \downarrow := \{ u \in B | u \ll x \}$ is directed and $\bigvee (x \downarrow) = x$. 

A continuous function between domains is order monotone and preserves lubs (sups) of directed sets.

Why are directed sets so important? They are collecting consistent pieces of information. Surely the words "continuous function" should have something to do with topology?
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Domain theory continued

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The dream

Find a topology so that Turing computability is precisely continuity.
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- Find a topology so that Turing computability is precisely continuity.
- Scott’s topology comes close.
- All computable functions are Scott continuous but one still needs some recursion theoretic machinery to pin down exactly what computable means.
the open sets of $D$ are upwards closed and if $\mathcal{O}$ is open, then if $X \subset D$ is directed and $\bigvee X \in \mathcal{O}$ it must be the case that some $x \in X$ is in $\mathcal{O}$. 
the open sets of $D$ are upwards closed and if $O$ is open, then if $X \subset D$ is directed and $\bigsqcup X \in O$ it must be the case that some $x \in X$ is in $O$.

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The effectively checkable properties.

This topology is $T_0$ but not $T_1$. 
basis of the form

\[ \mathcal{O} \setminus \bigcup_i (x_i \uparrow) \].
Topologies of Domains 2: The Lawson topology

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- Says something about negative information.
- This topology is metrizable.
- It has the same Borel algebra as the Scott topology.
Topologies of Domains 3: The interval topology

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Topologies of Domains 3: The interval topology

- Basis sets of the form \([x, y] := \{u | x \ll u \ll y\}\).
- The domain theoretic analogue of the Alexandrov topology.
- Caveat: the “Alexandrov topology” means something else in the theory of topological lattices.
The role of way below in spacetime structure

**Theorem:** Let \((M, g)\) be a spacetime with Lorentzian signature. Define \(x \ll y\) as the way-below relation of the causal order. If \((M, g)\) is globally hyperbolic then \(x \ll y\) iff \(y \in I^+(x)\).
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Theorem: Let $(M, g)$ be a spacetime with Lorentzian signature. Define $x \ll y$ as the way-below relation of the causal order. If $(M, g)$ is globally hyperbolic then $x \ll y$ iff $y \in I^+(x)$.

One can recover $I$ from $J$ without knowing what smooth or timelike means.

Intuition: any way of approaching $y$ must involve getting into the timelike future of $x$. 

We can stop being coy about notational clashes: henceforth $\ll$ is way-below and the timelike order.
The definition of continuous domain - or poset - is biased towards approximation from below. If we symmetrize the definitions we get bicontinuity (details in the paper).
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Theorem: If \((M, g)\) is globally hyperbolic then \((M, \leq)\) is a bicontinuous poset. In this case the interval topology is the manifold topology.
An “abstract” version of globally hyperbolic

We define a globally hyperbolic poset \((X, \leq)\) to be bicontinuous and,
An “abstract” version of globally hyperbolic

We define a globally hyperbolic poset \((X, \leq)\) to be

1. bicontinuous and,
2. all segments \([a, b] := \{x : a \leq x \leq b\}\) are compact in the interval topology on \(X\).
An Important Example of a Domain: $\mathbb{I} \mathbb{R}$

- The collection of compact intervals of the real line
  \[ \mathbb{I} \mathbb{R} = \{ [a, b] : a, b \in \mathbb{R} \land a \leq b \} \]
  ordered under reverse inclusion
  \[ [a, b] \sqsubseteq [c, d] \iff [c, d] \subseteq [a, b] \]
  is an $\omega$-continuous dcpo.
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- For directed $S \subseteq \mathbb{I}\mathbb{R}$, $\bigsqcup S = \bigcap S$. 
Interval Domains

\[ \mathbb{I}\mathbb{R} \text{ continued}. \]

- \( I \ll J \iff J \subseteq \text{int}(I) \), and

\[ \{ [p, q] : p, q \in \mathbb{Q} \land p \leq q \} \] is a countable basis for \( \mathbb{I}\mathbb{R} \).
$I \ll J \iff J \subseteq \text{int}(I)$, and

$\{[p, q] : p, q \in \mathbb{Q} \& p \leq q\}$ is a countable basis for $\mathbb{IR}$. 
The domain $\mathbb{IR}$ is called the interval domain.
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The “classical” structure lives on top - ideal points,

there is now a substrate of “approximate” elements.
The closed segments of a globally hyperbolic poset $X$

$\mathcal{I}X := \{[a, b] : a \leq b \& a, b \in X\}$

directed by reverse inclusion form a continuous domain with
The closed segments of a globally hyperbolic poset $X$

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$$[a, b] \ll [c, d] \equiv a \ll c \land d \ll b.$$
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$X$ has a countable basis iff $IX$ is $\omega$-continuous.
\[
\max(IX) \simeq X
\]

where the set of maximal elements has the relative Scott topology from IX.
If we have a countable dense subset $C$ of $\mathcal{M}$, a globally hyperbolic spacetime, then we can view the induced causal order on $C$ as defining a discrete poset. An ideal completion construction in domain theory, applied to a poset constructed from $C$ yields a domain $\mathbf{IC}$ with

$$\text{max}(\mathbf{IC}) \cong \mathcal{M}$$

where the set of maximal elements have the Scott topology. Thus from a countable subset of the manifold we can reconstruct the whole manifold.
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These two categories are equivalent.

Thus globally hyperbolic spacetimes are domains - not just posets - but

not with the causal order but, rather, with the order coming from the notion of intervals; i.e. from notions of approximation.
The domain consists of intervals \([x, y] = J^+(x) \cap J^-(y)\).
Spacetime as a domain

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The domain consists of intervals $[x, y] = J^+(x) \cap J^-(y)$. For globally hyperbolic spacetimes these are all compact. The order is inclusion. The maximal elements are the usual points $x = J^+(x) \cap J^-(x)$. The other elements are “approximate points.”
Other layers of structure

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Keye Martin defined a concept called a “measurement.” This is designed to capture quantitative notions on domains.

Metric notions can be related to these measurements.
Keye’s measurements

A measurement on $D$ is a function $\mu : D \rightarrow (\infty, 0]$ (reverse ordered) that is Scott continuous and satisfies some extra conditions.
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Keye’s measurements

- A *measurement* on $D$ is a function $\mu : D \to (\infty, 0]$ (reverse ordered) that is Scott continuous and satisfies some extra conditions.
- We write $\ker(\mu)$ for $\{x | \mu(x) = 0\}$ and $\mu_\epsilon(x) = \{y | y \subseteq x \text{ and } |\mu(x) - \mu(y)| \leq \epsilon\}$.
- For any Scott open set $U$ and any $x \in \ker(\mu)$

\[ x \in U \Rightarrow (\exists \epsilon > 0) x \in \mu_\epsilon \subseteq U. \]
Idea: $\mu(x)$ measures the “uncertainty” in $x$. 

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Maximal elements have zero uncertainty.
On a suitable domain of probability distributions Shannon entropy is a measurement.
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Unfortunately not! If $a$ and $b$ are null related then you get a nontrivial interval with zero volume.

However, any globally hyperbolic spacetime (in fact any stably causal one) has a \textit{global time function}. The difference in the global time function does give a measurement.

Knowing the global time function effectively gives the rest of the metric.
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Channel capacity as geometry?

In spacetime I want to be able to view a path from $p$ to $q$ as a “channel” and to measure its capacity. If we transmit messages from $p$ to $q$ the Hawking radiation produces noise and thus limits the capacity. The same thing happens in the Rindler wedge and we can use this to help with encrypting. Details in Bradler, Hayden, P. CMP 2012.

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