Tensor of Quantitative Equational Theories

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Abstract

We develop a theory for the commutative combination of quantitative effects, their *tensor*, given as a combination of quantitative equational theories that imposes mutual commutation of the operations from each theory. As such, it extends the sum of two theories, which is just their unrestrained combination. Tensors of theories arise in several contexts; in particular, in the semantics of programming languages, the monad transformer for global state is given by a tensor.

We show that under certain assumptions on the quantitative theories the free monad that arises from the tensor of two theories is the categorical tensor of the free monads on the theories. As an application, we provide the first algebraic axiomatizations of labelled Markov processes and Markov decision processes. Apart from the intrinsic interest in the axiomatizations, it is pleasing they are obtained compositionally by means of the sum and tensor of simpler quantitative equational theories.

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1 Introduction

The theory of computational effects began with the work of Moggi [25, 26] seeking a unified category-theoretic account of the semantics of higher-order programming languages. He modelled computational effects (which he called notions of computation) by means of strong monads on a base category with cartesian closed structure. With Cenciarelli [5], he later extended the theory by allowing a compositional treatment of various semantic phenomena such as state, IO, exceptions, resumptions, etc, via the use of monad transformers. This work was followed up by the program of Plotkin and Power [27, 28] on an axiomatic understanding of computational effects as arising from operations and equations via the use of Lawvere theories (see also [14]). In a fundamental contribution [12] together with Hyland they developed a unified modular theory for algebraic effects that supports their combination by taking the *sum* and *tensor* of their Lawvere theories. This allowed them to recover in a more pleasing algebraic structural way many of the monad transformers considered by Moggi.

Quantitative equational theories, introduced by Mardare et al. [21], are a logical framework generalising the standard concept of equational logic to account for a concept of approximate equality. They are used to describe quantitative effects as monads on categories of metric spaces. Following the work of Hyland et al. [12], in [1] we developed a theory for the sum of quantitative equational theories, and proved that it corresponds to the categorical sum of quantitative effects as monads. As a major example, we axiomatised Markov processes with discounted probabilistic bisimilarity distance [7] as the sum of two theories: interpolative barycentric algebras (which axiomatises probability distributions with the Kantorovich metric [21]) and contractive operators (used to express the transition to the next state).

Whereas the sum of two monads is the simplest combination supporting both given effects with no interactions between them, the tensor additionally requires commutation of these effects over each other. Some of the most important monad transformers have an elegant abstract description using tensor. Specifically, Moggi's transformers for state, reader, and writer are examples of tensors [12].

In the present paper we extend the work initiated in [1], and develop the theory for the *tensor of quantitative equational theories.* The main contributions are:

1. we prove that the tensor of quantitative theories corresponds to the categorical tensor of their induced quantitative effects as strong monads;
2. we give quantitative axiomatisations to the *quantitative reader* and *writer monads*, from which we obtain analogues of Moggi's transformers at the level of theories using tensor;
3. we provide the first axiomatization of *labelled Markov processes* and *Markov decision processes* with their discounted bisimilarity metrics.

For the proof of (1) we introduce the concept of *pre-operation of a strong functor*, which we use to conveniently characterise the commutative bialgebras for the monads (which correspond to the Eilenberg-Moore algebras for their tensor). Crucially, this allows us to carry out the technical development directly at the level of quantitative equational theories without passing via a correspondence with metric-enriched Lawvere theories.

The axiomatisations in (3) are two major examples for our compositional theory quantitative effects. Specifically, we obtain the discounted bisimilarity metrics for labelled Markov processes and Markov decision processes with rewards by complementing the axiomatization for Markov processes presented in [1]. We model reactions to action labels by tensoring with the theory of quantitative reading computations (corresponding to Moggi's reader monad transformer); while rewards are recovered by tensoring with the theory of quantitative writing computations (corresponding to Moggi's writer monad transformer). We will illustrate our compositional approach by decomposing the proposed axiomatisations into their basic components and showing how to combine them step-by-step to get the desired result.

Further Related Work. In [12, 11] the tensor of (enriched) Lawvere theories is characterized as the colimit of certain commutative cocones, and the correspondence with the tensor of monads is obtained via the equivalence between Lawvere theories and monads. Since it is not hard to show that (basic) quantitative equational theories can be characterised as metric-enriched Lawvere theories, one may think to recover the correspondence with the tensor of monads via the equivalence with Lawvere theories. Alas, quantitative equational theories and Lawvere theories are not equivalent, as the latter allows generic operations with metric spaces as arities, while the framework of Mardare et al. [21] does not. An equivalence with *discrete* Lawvere theories [13] (where arities are just countable ordinals) does not hold either, because quantitative equations implicitly impose the existence operations with non-discrete arities which cannot be expressed in the framework of discrete Lawvere theories.

The above arguments required us to follow a different path, which lead us to the introduction of pre-operations for a strong functor F. Pre-operations are related to Plokin and Power's *algebraic operations* [29, 30] in the sense that their assignment to F-algebras are the appropriate version of algebraic operations for functors. Moreover, when considered over a strong monad T they correspond to generic effects of type $I \to Tv$ (*i.e.*, Kleisli maps of type $I \to v$, where I is the identity for the monoidal product). The reason why we consider pre-operations over functors, and not just monads, is to relate the operations of an algebraic monad with those of its signature. This was crucial in the technical development of Section 5.

Finally, we remark that quantitative equational theories are a natural kind of enriched equational theory expressive enough to recover many examples of interest in computer science (see [21, 1, 24]), but not corresponding to metric-enriched Lawvere theories. In this respect, it is nice that also this simpler subclass of enriched theories are closed under sum and tensor.

2 Preliminaries and Notation

An *extended metric space* is a pair (X, d) consisting of a set X equipped with a distance function $d\colon X \times X \to [0, \infty]$ allowed to have infinite values, satisfying: (i) $d(x, y) = 0$ iff $x = y$, (ii) $d(x, y) = d(y, x)$ and (iii) $d(x, z) \le d(x, y) + d(y, z)$.

A sequence (x_i) in X is *Cauchy* if $\forall \epsilon > 0, \exists N, \forall i, j \ge N, d(x_i, x_j) \le \epsilon$. If every Cauchy sequence converges, the extended metric space (X, d) is said to be *complete*. If a space is not complete it can be completed by a well-known construction called *Cauchy completion*. We write $\overline{(X, d)}$, or just $\overline{X}$, for the completion of (X, d).

We denote by **Met** the category of extended metric spaces with morphisms the non-expansive maps, *i.e.* the $f\colon (X, d_X) \to (Y, d_Y)$ such that $d_X(x, y) \ge d_Y(f(x), f(y))$. This category is both complete (*i.e.*, have all limits) and cocomplete (*i.e.*, have all colimits). We will consider also the full subcategory **CMet** of complete extended metric spaces.

The categorical properties of extended metric spaces are much nicer than usual metric spaces. In particular, we note that **Met** is a symmetric monoidal category, with monoidal product $(X, d_X) \,\square\, (Y, d_Y)$ being the extended metric space with underlying set $X \times Y$ and extended metric $d_{X \square Y}((x, y)(x', y')) = d_X(x, x') + d_Y(y, y')$ (*cf.* [19]). Note that this is not the cartesian product in **Met** (for which $+$ above would be replaced by max).

The monoidal product $\square$ introduced above defines a closed monoidal structure on **Met**, with internal hom $[(X, d_X), (Y, d_Y)]$ given by the set of non-expansive maps from X to Y with point-wise supremum extended metric $d_{[X,Y]}(f, g) = \sup_{x \in X} d(f(x), g(x))$.

Finally, we recall the basic definitions of strong functor (and monad), strong natural transformations, and fix the notation (for more details see e.g. [17, 18]). Let $\mathbf{V}$ be a symmetric monoidal closed category with monoidal product[1] $\square\colon \mathbf{V} \times \mathbf{V} \to \mathbf{V}$, unit object $I \in \mathbf{V}$, and internal hom-functor $[-,-]\colon \mathbf{V} \times \mathbf{V} \to \mathbf{V}$. We will denote the *counit* (or evaluation map) of the adjunction $(V \,\square\, -) \dashv [V, -]$ by $ev^V : V \,\square\, [V, -] \Rightarrow Id$ and the *unit* (or co-evaluatation map) by $\overline{ev}^V : Id \to [V, V \,\square\, -]$.

A functor $F\colon \mathbf{V} \to \mathbf{V}$ is *strong* with *monoidal strength* $t_{V,W} : V \,\square\, F(W) \to F(V \,\square\, W)$, if t is a natural transformation satisfying the coherence conditions $F\lambda \circ t = \lambda$ and $t \circ (id \,\square\, t) \circ \alpha = F\alpha \circ t$, w.r.t. the associator α and left unitor λ of $\mathbf{V}$. The dual strength $\hat{t}_{V,W} : F(W) \,\square\, V \to F(W \,\square\, V)$ is given by $\hat{t} = F(t) \circ t \circ s$, where $s\colon V \,\square\, W \to W \,\square\, V$ is the natural isomorphism of the symmetric monoidal category $\mathbf{V}$. A natural transformation $\theta\colon F \Rightarrow G$ is said *strong* if F, G are strong functors with strengths t, s, respectively, and $s \circ (id \,\square\, \theta) = \theta \circ t$, meaning that θ interacts well with the strengths.

A monad (T, η, μ) with unit $\eta\colon Id \Rightarrow T$ and multiplication $\mu\colon TT \Rightarrow T$, is *strong* if T is a strong functor with strength t such that $t \circ (id \,\square\, \eta) = \eta$ and $\mu \circ tt = t \circ (id \,\square\, \mu)$.

Note that strong functors (resp. monads) on a symmetric monoidal closed category $\mathbf{V}$ are equivalent to $\mathbf{V}$-enriched functors (resp. monads) on the self-enriched category $\mathbf{V}$ [17].

[1] We denote the monoidal product by $\square$ to avoid confusion with other tensorial operations we will deal with in this paper, e.g., the tensor of monads.

3 Quantitative Equational Theories

Quantitative equations were introduced in [21]. In this framework equalities $t \equiv_\varepsilon s$ are indexed by a positive rational number, to capture the idea that t is "within ε" of s. This informal notion is formalised in a manner analogous to traditional equational logic. In this section we review this formalism.

Let Σ be a signature of function symbols $f\colon n \in \Sigma$ of arity $n \in \mathbb{N}$. Let X be a countable set of variables, ranged over by $x, y, z, \ldots$. We write $\mathbb{T}(\Sigma, X)$ for the set of Σ-terms freely generated over X, ranged over by $t, s, u, \ldots$.

A *substitution of type* Σ is a function $\sigma\colon X \to \mathbb{T}(\Sigma, X)$, canonically extended to terms as $\sigma(f(t_1, \ldots, t_n)) = f(\sigma(t_1), \ldots, \sigma(t_n))$; we write $\mathcal{S}(\Sigma)$ for the set of substitutions of type Σ.

A *quantitative equation of type* Σ over X is an expression of the form $t \equiv_\varepsilon s$, for $t, s \in \mathbb{T}(\Sigma, X)$ and $\varepsilon \in \mathbb{Q}_{\geq 0}$. We use $\mathcal{V}(\Sigma, X)$ to denote the set of quantitative equations of type Σ over X, and its subsets will be ranged over by $\Gamma, \Theta, \ldots$. Let $\mathcal{E}(\Sigma, X)$ be the set of *conditional quantitative equations* on $\mathbb{T}(\Sigma, X)$, which are expressions of the form

$$\{t_1 \equiv_{\varepsilon_1} s_1, \ldots, t_n \equiv_{\varepsilon_n} s_n\} \vdash t \equiv_\varepsilon s\,,$$

for arbitrary $s_i, t_i, s, t \in \mathbb{T}(\Sigma, X)$ and $\varepsilon_i, \varepsilon \in \mathbb{Q}_{\geq 0}$.

A *quantitative equational theory of type* Σ *over* X is a set $\mathcal{U}$ of conditional quantitative equations on $\mathbb{T}(\Sigma, X)$ closed under the relation $\vdash$ as axiomatised below, for arbitrary $x, y, z, x_i, y_i \in X$, terms $s, t \in \mathbb{T}(\Sigma, X)$, rationals $\varepsilon, \varepsilon' \in \mathbb{Q}_{\geq 0}$, and $\Gamma, \Theta \subseteq \mathcal{V}(\Sigma, X)$,

$$
\begin{aligned}
&(\mathsf{Refl})\ \vdash x \equiv_0 x\,,\\
&(\mathsf{Symm})\ \{x \equiv_\varepsilon y\} \vdash y \equiv_\varepsilon x\,,\\
&(\mathsf{Triang})\ \{x \equiv_\varepsilon z, z \equiv_{\varepsilon'} y\} \vdash x \equiv_{\varepsilon+\varepsilon'} y\,,\\
&(\mathsf{Max})\ \{x \equiv_\varepsilon y\} \vdash x \equiv_{\varepsilon+\varepsilon'} y\,,\ \text{for all } \varepsilon' > 0\,,\\
&(\mathsf{Cont})\ \{x \equiv_{\varepsilon'} y \mid \varepsilon' > \varepsilon\} \vdash x \equiv_\varepsilon y\,,\\
&(f\text{-}\mathsf{NE})\ \{x_i \equiv_\varepsilon y_i \mid i{=}1..n\} \vdash f(x_1, \ldots, x_n) \equiv_\varepsilon f(y_1, \ldots, y_n)\,,\ \text{for } f\colon n \in \Sigma\,,\\
&(\mathsf{Subst})\ \text{If } \Gamma \vdash t \equiv_\varepsilon s, \text{ then } \{\sigma(t) \equiv_\varepsilon \sigma(s) \mid t \equiv_\varepsilon s \in \Gamma\} \vdash \sigma(t) \equiv_\varepsilon \sigma(s)\,,\ \text{for } \sigma \in \mathcal{S}(\Sigma)\,,\\
&(\mathsf{Ass})\ \text{If } t \equiv_\varepsilon s \in \Gamma, \text{ then } \Gamma \vdash t \equiv_\varepsilon s\,,\\
&(\mathsf{Cut})\ \text{If } \Gamma \vdash \Theta \text{ and } \Theta \vdash t \equiv_\varepsilon s, \text{ then } \Gamma \vdash t \equiv_\varepsilon s\,,
\end{aligned}
$$

where we write $\Gamma \vdash \Theta$ to mean that $\Gamma \vdash t \equiv_\varepsilon s$ holds for all $t \equiv_\varepsilon s \in \Theta$.

The rules (Subst), (Cut), (Ass) are the usual rules of equational logic. The axioms (Refl), (Symm), (Triang) correspond, respectively, to reflexivity, symmetry, and the triangle inequality; (Max) represents inclusion of neighborhoods of increasing diameter; (Cont) is the limiting property of a decreasing chain of neighborhoods with converging diameters; and (f-NE) expresses non-expansivness of $f \in \Sigma$.

A set A of conditional quantitative equations *axiomatises* a quantitative equational theory $\mathcal{U}$, if $\mathcal{U}$ is the smallest quantitative equational theory containing A.

The models of these theories, called *quantitative* Σ*-algebras*, are Σ-algebras in **Met**.

▶ **Definition 1** (Quantitative Algebra)**.** *A* quantitative Σ-algebra *is a tuple* $\mathcal{A} = (A, \Sigma^{\mathcal{A}})$*, where* A *is an extended metric space and* $\Sigma^{\mathcal{A}} = \{f^{\mathcal{A}}\colon A^n \to A \mid f\colon n \in \Sigma\}$ *is a set of non-expansive* interpretations *(i.e., satisfying* $\max_i d_A(a_i, b_i) \geq d_A(f^{\mathcal{A}}(a_1, \ldots, a_n), f^{\mathcal{A}}(b_1, \ldots, b_n))$*).*

The morphisms between quantitative Σ-algebras are non-expansive Σ-homomorphisms. Quantitative Σ-algebras and their morphism form a category, denoted by $\mathbf{QA}(\Sigma)$.

$\mathcal{A} = (A, \Sigma^{\mathcal{A}})$ *satisfies* the conditional quantitative equation $\Gamma \vdash t \equiv_\varepsilon s$ in $\mathcal{E}(\Sigma, X)$, written $\Gamma \models_{\mathcal{A}} t \equiv_\varepsilon s$, if for any assignment $\iota \colon X \to A$, the following implication holds

$$\big(\forall t' \equiv_{\varepsilon'} s' \in \Gamma \,, d_A(\iota(t'), \iota(s')) \leq \varepsilon'\big) \Rightarrow d_A(\iota(t), \iota(s)) \leq \varepsilon \,,$$

where $\iota(t)$ is the homomorphic interpretation of t in $\mathcal{A}$.

A quantitative algebra $\mathcal{A}$ is said to *satisfy* (or be a *model* for) the quantitative theory $\mathcal{U}$, if $\Gamma \models_{\mathcal{A}} t \equiv_\varepsilon s$ whenever $\Gamma \vdash t \equiv_\varepsilon s \in \mathcal{U}$. We write $\mathbb{K}(\Sigma, \mathcal{U})$ for the collection of models of a theory $\mathcal{U}$ of type Σ.

Sometimes it is convenient to consider the quantitative Σ-algebras whose carrier is a complete extended metric space. This class of algebras forms a full subcategory of $\mathbf{QA}(\Sigma)$, written $\mathbf{CQA}(\Sigma)$. Similarly, we write $\mathbb{CK}(\Sigma, \mathcal{U})$ for the full subcategory of quantitative Σ-algebras in $\mathbf{CQA}(\Sigma)$ which are models of $\mathcal{U}$.

The following lifts the Cauchy completion of metric spaces to quantitative algebras.

▶ **Definition 2.** *(Algebra Completion) The* Cauchy completion *of a quantitative Σ-algebra $\mathcal{A} = (A, \Sigma^{\mathcal{A}})$, is the quantitative Σ-algebra $\overline{\mathcal{A}} = (\overline{A}, \Sigma^{\overline{\mathcal{A}}})$, where $\overline{A}$ is the Cauchy completion of A and $\Sigma^{\overline{\mathcal{A}}} = \{ f^{\overline{\mathcal{A}}} \colon \overline{A}^n \to \overline{A} \mid f \colon n \in \Sigma \}$ is such that for Cauchy sequences $(b^i_j)_j$ converging to $b^i \in \overline{A}$, for $1 \leq i \leq n$, $f^{\overline{\mathcal{A}}}(b^1, \dots, b^n) = \lim_j f^{\mathcal{A}}(b^1_j, \dots, b^n_j)$.*

The above extends to a functor $\mathbb{C} \colon \mathbf{QA}(\Sigma) \to \mathbf{CQA}(\Sigma)$ which is the left adjoint to the functor embedding $\mathbf{CQA}(\Sigma)$ into $\mathbf{QA}(\Sigma)$.

The completion of quantitative Σ-algebras extends also to a functor from $\mathbb{K}(\Sigma, \mathcal{U})$ to $\mathbb{CK}(\Sigma, \mathcal{U})$, whenever $\mathcal{U}$ can be axiomatised by a collection of *continuous schemata*, which are conditional quantitative equations of the form

$$\{x_i \equiv_{\varepsilon_i} y_i \mid i = 1..n\} \vdash t \equiv_\varepsilon s \,, \qquad \text{for all } \varepsilon \geq f(\varepsilon_1, \dots, \varepsilon_n),$$

where $f \colon \mathbb{R}^n_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a continuous real-valued function, and $x_i, y_i \in X$. We call such a theory *continuous.*

Free Monads on Quantitative Equational Theories

To every signature Σ, one can associate a *signature endofunctor* (also called Σ) on $\mathbf{Met}$ by:

$$\Sigma = \coprod_{f:n \in \Sigma} Id^n \,.$$

It is easy to see that, by couniversality of the coproduct, quantitative Σ-algebras correspond to Σ-algebras for the functor Σ in $\mathbf{Met}$, and the morphisms between them to non-expansive homomorphisms of Σ-algebras. Below we pass between the two points of view as convenient.

▶ **Theorem 3** (Free Algebra [21])**.** *The forgetful functor $\mathbb{K}(\Sigma, \mathcal{U}) \to \mathbf{Met}$ has a left adjoint.*

The left adjoint assigns to any $X \in \mathbf{Met}$ a *free quantitative Σ-algebra* $(T_X, \psi^{\mathcal{U}}_X)$ satisfying $\mathcal{U}$, from which one canonically obtains the monad $(T_{\mathcal{U}}, \eta^{\mathcal{U}}, \mu^{\mathcal{U}})$, with functor $T_{\mathcal{U}} \colon \mathbf{Met} \to \mathbf{Met}$ mapping $X \in \mathbf{Met}$ to the carrier T_X of the free algebra.

A similar free construction also holds for quantitative algebras in $\mathbf{CQA}(\Sigma)$ for continuous quantitative equational theories, implying that the forgetful functor from $\mathbb{CK}(\Sigma, \mathcal{U})$ to $\mathbf{CMet}$ has a left adjoint. In particular, $\mathbb{C}T_{\mathcal{U}}$ is the free monad on $\mathcal{U}$ in $\mathbf{CMet}$, provided that the quantitative equational theory is continuous.

Finally, let T-**Alg** denote the category of Eilenberg-Moore (EM) algebras for a monad T. In [1], it is shown that, whenever the quantitative theory $\mathcal{U}$ is *basic*, *i.e.*, it can be axiomatised by a set of conditional equations of the form

$$\{x_1 \equiv_{\varepsilon_1} y_1, \ldots, x_n \equiv_{\varepsilon_n} y_n\} \vdash t \equiv_\varepsilon s\,,$$

where $x_i, y_i \in X$ (*cf.* [22]), then EM $T_\mathcal{U}$-algebras are in 1-1 correspondence with the quantitative algebras satisfying $\mathcal{U}$:

▶ **Theorem 4.** *For any basic quantitative equational theory $\mathcal{U}$ of type Σ, $T_\mathcal{U}$-**Alg** $\cong \mathbb{K}(\Sigma, \mathcal{U})$.*

4 Tensor of Strong Monads

In this section we provide the definition of *tensor of strong monads* on a generic symmetric monoidal closed category $\mathbf{V}$. The presentation follows and generalises that of Manes [20], which considers only the case of monads on **Set**.

Let v be an object in $\mathbf{V}$. As $\mathbf{V}$ is self-enriched, it has all v-fold *powers* (or v-*powers*) X^v, of any object $X \in \mathbf{V}$, defined as $X^v = [v, X]$ [16]. Moreover, $(-)^v : \mathbf{V} \to \mathbf{V}$ is a strong functor with strength $\xi_{X,Y} : X \,\square\, Y^v \to (X \,\square\, Y)^v$ obtained by currying

$$v \,\square\, (X \,\square\, Y^v) \xrightarrow{\cong} X \,\square\, (v \,\square\, Y^v) \xrightarrow{X \square ev} X \,\square\, Y\,.$$

Let $F : \mathbf{V} \to \mathbf{V}$ be a strong functor with strength t. The v-*power* functor $(-)^v$ is be lifted to F-algebras by mapping (A, a) to $(A, a)^v = (A^v, a^v \circ \sigma_A)$, where $\sigma_A : FA^v \Rightarrow (FA)^v$ is the strong natural transformation obtained from t by currying $Tev^v_A \circ t_{v,A^v}$. Hence F-algebras are closed under powers of $\mathbf{V}$-objects.

▶ **Definition 5** (Pre-operation of a strong functor). *Let $F : \mathbf{V} \to \mathbf{V}$ be a strong functor and $v \in \mathbf{V}$. A v-ary* pre-operation *of F is a strong natural transformation of type $(-)^v \Rightarrow F$.*

We denote by $\mathcal{O}_F(v)$ the set of v-ary pre-operations of F. An assignment of $g \in \mathcal{O}_F(v)$ to an F-algebra (A, a) is the composite $a^g = a \circ g_A$. We call a^g an *operation* of (A, a).

▶ **Proposition 6.** *Let $(A, a), (B, b)$ be F-algebras of a strong endofunctor F on $\mathbf{V}$ and $f : A \to B$ a morphism in $\mathbf{V}$. Then, the following are equivalent:*
1. *f is a F-homomorphisms from (A, a) to (B, b);*
2. *For every $v \in \mathbf{V}$ and $g \in \mathcal{O}_F(v)$, $f \circ a^g = b^g \circ f^v$.*

The above proposition indicates that F-algebras are precisely characterised by their operations. In some situations, depending on the functor F, one gets the same characterisation with much fewer operations. We identify this property with the following definition.

▶ **Definition 7** (Density). *A set $\mathcal{D}$ of pre-operations of a strong functor $F : \mathbf{V} \to \mathbf{V}$ is* dense, *if for any F-algebras (A, a), (B, b) and $f : A \to B$ in $\mathbf{V}$, the following are equivalent:*
1. *f is a F-homomorphisms from (A, a) to (B, b);*
2. *For every v-ary pre-operation $g \in \mathcal{D}$, $f \circ a^g = b^g \circ f^v$.*

Let F, G be two strong endofunctors on $\mathbf{V}$. A $\langle F, G\rangle$-*bialgebra* is a triple (A, a, b) consisting of an object $A \in \mathbf{V}$ with both a F-algebra structure $a : FA \to A$ and a G-algebra structure $b : GA \to A$. A morphism of $\langle F, G\rangle$-bialgebras is an arrow that is simultaneously a F- and G-homomorphism. Denote by $\langle F, G\rangle$-**biAlg** the category of $\langle F, G\rangle$-bialgebras.

▶ **Proposition 8.** *Let (A, a, b) be a $\langle F, G\rangle$-bialgebra. The following statements are equivalent:*

1. *For all* $v \in \mathbf{V}$ *and* $g \in \mathcal{O}_F(v)$*,* a^g *is a* G*-homomorphism;*
2. *For all* $w \in \mathbf{V}$ *and* $h \in \mathcal{O}_G(w)$*,* b^h *is a* F*-homomorphism.*

Diagrammatically:

$$\begin{tikzcd} GA^v \arrow[r, "\bar{b}"] \arrow[d, "G(a^g)"'] \arrow[dr, phantom, "(1)"] & A^v \arrow[d, "a^g"] \\ GA \arrow[r, "b"] & A \end{tikzcd} \quad \text{iff} \quad \begin{tikzcd} FA^w \arrow[r, "\bar{a}"] \arrow[d, "F(b^h)"'] \arrow[dr, phantom, "(2)"] & A^w \arrow[d, "b^h"] \\ FA \arrow[r, "a"] & A \end{tikzcd}$$

where $(A, a)^w = (A^w, \bar{a})$ *and* $(A, b)^v = (A^v, \bar{b})$*.*

▶ **Definition 9** (Commutative bialgebra)**.** *A* $\langle F, G\rangle$*-bialgebra* (A, a, b) *is* commutative *if it satisfies either of the equivalent conditions of Proposition 8.*

In the case the functors F and G admit dense sets of pre-operations, commutativity for their bialgebras can be more conveniently expressed in the following way.

▶ **Proposition 10.** *Let* $\mathcal{D}$ *and* $\mathcal{E}$ *be dense sets of pre-operations for* F *and* G*, respectively. A* $\langle F, G\rangle$*-bialgebra* (A, a, b) *is* commutative *iff it satisfies either of the equivalent conditions:*
1. *For all* $g \in \mathcal{D}$*,* a^g *is a* G*-homomorphism;*
2. *For all* $h \in \mathcal{E}$*,* b^h *is a* F*-homomorphism.*

Let (T, η, μ) be a strong monad on $\mathbf{V}$. Note that, as T is a strong functor and the EM-algebras for T are closed under powers of $\mathbf{V}$-objects, all the results and definitions given in this section extends to EM-algebras for T.

Let (T, η, μ), (T', η', μ') be two strong monads on $\mathbf{V}$. A EM $\langle T, T'\rangle$*-bialgebra* is a triple (A, a, a') consisting of an object $A \in \mathbf{V}$ with both a EM T-algebra structure $a \colon TA \to A$ and a EM T'-algebra structure $a' \colon T'A \to A$. We say that a EM $\langle T, T'\rangle$-bialgebra (A, a, b) is *commutative* if it is so as a $\langle T, T'\rangle$-bialgebra for the functors T, T'. We denote by $\langle T, T'\rangle$-$\mathbf{biAlg}$ the category of EM $\langle T, T'\rangle$-bialgebras and by $(T \otimes T')$-$\mathbf{biAlg}$, the full subcategory of the commutative EM $\langle T, T'\rangle$-bialgebras.

▶ **Definition 11** (Tensor of monads)**.** *If the forgetful functor* $(T \otimes T')$-$\mathbf{biAlg} \to \mathbf{V}$ *has left adjoint, then the monad induced by the adjunction is the* tensor *of* T, T'*, denoted* $T \otimes T'$*.*

Note that the tensor of monads does not necessarily exist (see [4] for counterexamples). However, when it does $T \otimes T' \cong T' \otimes T$, as the categories of commutative biagebras $(T \otimes T')$-$\mathbf{biAlg}$ and $(T' \otimes T)$-$\mathbf{biAlg}$ are isomorphic.

5 Tensor of Quantitative Theories

In this section, we develop the theory for the *tensor* of quantitative equational theories. The main result is that the free monad on the tensor of two theories is the tensor of the monads on the theories. In the proof given, we use the fact that the quantitative theories are *basic*, as this allows us to exploit the correspondence between the algebras of a theory $\mathcal{U}$ and the EM-algebras of the monad $T_{\mathcal{U}}$ (Theorem 4).

Let Σ, Σ' be two disjoint signatures. Following Freyd [8] (and [12]), we define the tensor of two quantitative equational theories $\mathcal{U}, \mathcal{U}'$ of respective types Σ and Σ', written $\mathcal{U} \otimes \mathcal{U}'$, as the smallest quantitative theory containing $\mathcal{U}, \mathcal{U}'$ and the quantitative equations

$$\vdash f(g(x^1_1, \dots, x^1_m), \dots, g(x^n_1, \dots, x^n_m)) \equiv_0 g(f(x^1_1, \dots, x^n_1), \dots, f(x^1_m, \dots, x^n_m)) \,, \tag{1}$$

for all $f \colon n \in \Sigma$ and $g \colon m \in \Sigma'$, expressing that the operations of one theory commute with the operations of the other.

5.1 Density of Symbolic Pre-operations

Towards our main result, we identify a dense set of pre-operations for the free monads on quantitative equational theories which, in turn, will gives us a simpler characterization for commutative bialgebras for these monads (*cf.* Proposition 10).

First notice that any signature functor $\Sigma = \coprod_{f:n\in\Sigma} Id^n$ in $\mathbf{Met}$ is strong, as it is the coproduct of the strong functors $Id^n \cong (-)^{\underline{n}}$, where $\underline{n} \in \mathbf{Met}$ denotes the set $\{1,\ldots,n\}$ equipped with the discrete extended metric assigning infinite distance to distinct elements. Moreover, the injections $in_f \colon (-)^{\underline{n}} \Rightarrow \Sigma$ are strong natural transformations, hence they are $\underline{n}$-ary pre-operations of Σ (*cf.* Definition 5).

▶ **Proposition 12.** *$\mathcal{S}_\Sigma = \{in_f \mid f\colon n \in \Sigma\}$ is a dense set of pre-operations of Σ.*

In the following, the pre-operations in $\mathcal{S}_\Sigma$ will be called *symbolic*, and to simplify the notation, for any $f\colon n\in\Sigma$ and Σ-algebra (A,a), we write a^f instead of a^{in_f}.

Let $\mathcal{U}$ be a quantitative equational theory of type Σ. Then, also the monad $T_\mathcal{U}$ is strong, with strength $\zeta_{X,Y}\colon X \,\square\, T_\mathcal{U} Y \to T_\mathcal{U}(X \,\square\, Y)$ obtained by uncurrying the unique map $h_{X,Y}$ that, by Theorem 3, makes the following diagram commute

$$\begin{tikzcd}
Y \arrow[r, "\eta^{\mathcal{U}}_Y"] \arrow[dr, bend right, "\beta_{X,Y}"'] & T_\mathcal{U} Y \arrow[d, dashed, "h_{X,Y}"'] & & \Sigma T_\mathcal{U} Y \arrow[ll, "\psi^{\mathcal{U}}_Y"'] \arrow[d, "\Sigma h_{X,Y}"] \\
& (T_\mathcal{U}(X \,\square\, Y))^X & & \Sigma(T_\mathcal{U}(X \,\square\, Y))^X \arrow[ll, "\psi^{\mathcal{U}}_{(T(X\square Y))^X}"']
\end{tikzcd}$$

where $\beta_{X,Y}$ is the currying of $\eta^{\mathcal{U}}_{X\square Y}\colon X \,\square\, Y \to T_\mathcal{U}(X \,\square\, Y)$.

Since a monad is strong iff both its unit and multiplication are strong natural transformations, both $\eta^\mathcal{U}$, $\mu^\mathcal{U}$ are strong. Moreover, also $\psi^\mathcal{U}\colon \Sigma T_\mathcal{U} \Rightarrow T_\mathcal{U}$ is strong.

Thus any pre-operation $g \in \mathcal{O}_\Sigma(v)$ can be tuned into an pre-operation of $T_\mathcal{U}$ as the composite

$$(-)^v \xrightarrow{g} \Sigma \xrightarrow{\Sigma\eta^{\mathcal{U}}} \Sigma T_\mathcal{U} \xrightarrow{\psi^{\mathcal{U}}} T_\mathcal{U}\,.$$

In particular, when the theory $\mathcal{U}$ is basic, by exploiting Theorem 4, the above transformation allows us to turn any dense set of pre-operations of Σ into a dense set of pre-operations of $T_\mathcal{U}$.

▶ **Proposition 13.** *Let $\mathcal{U}$ be a basic quantitative theory of type Σ and $\mathcal{D}$ a dense set of pre-operations of Σ. Then $\{\psi^\mathcal{U} \circ \Sigma\eta^\mathcal{U} \circ g \mid g \in \mathcal{D}\}$ is a dense set of pre-operations of $T_\mathcal{U}$.*

By combining Propositions 12 and 13, we have that $\mathcal{S}_{T_\mathcal{U}} = \{\psi^\mathcal{U} \circ \Sigma\eta^\mathcal{U} \circ in_f \mid f\colon n\in\Sigma\}$ is a dense set of pre-operations of $T_\mathcal{U}$. We call also these pre-operations *symbolic* and we simplify the notation by writing $a^{\langle f\rangle}$ instead of $a^{(\psi^\mathcal{U} \circ \Sigma\eta^\mathcal{U} \circ in_f)}$, for $f\colon n\in\Sigma$ and $(A,a) \in T_\mathcal{U}$-$\mathbf{Alg}$.

Thus, as an immediate consequence of Propositions 10 and 13, we obtain the following simpler characterization for commutative $\langle T_\mathcal{U}, T_{\mathcal{U}'}\rangle$-bialgebras.

▶ **Corollary 14.** *Let $\mathcal{U}$, $\mathcal{U}'$ be basic quantitative theories respectively of type Σ, Σ'. A $\langle T_\mathcal{U}, T_{\mathcal{U}'}\rangle$-bialgebra (A,a,b) is commutative iff it satisfies either of the equivalent conditions*

1. *For all $f\colon n\in\Sigma$, $a^{\langle f\rangle}$ is a $T_{\mathcal{U}'}$-homomorphism;*
2. *For all $g\colon n\in\Sigma'$, $b^{\langle g\rangle}$ is a $T_\mathcal{U}$-homomorphism.*

5.2 Tensor of Free Monads on Quantitative Theories

Let $\mathcal{U},\mathcal{U}'$ be basic quantitative theories respectively of type Σ, Σ'. We show that any model for $\mathcal{U} \otimes \mathcal{U}'$ is a $\langle \mathcal{U} \otimes \mathcal{U}' \rangle$*-bialgebra*: an extended metric space A with both a Σ-algebra structure $a\colon \Sigma A \to A$ satisfying $\mathcal{U}$ and a Σ'-algebra structure $b\colon \Sigma' A \to A$ satisfying $\mathcal{U}'$ and respecting the diagrammatic condition below, for all $f\colon n \in \Sigma$ and $g\colon m \in \Sigma'$

$$\begin{tikzcd} A^{\underline{n}} \arrow[r, "a^f"] & A & A^{\underline{m}} \arrow[l, "b^g"'] \\ (A^{\underline{m}})^{\underline{n}} \arrow[u, "(b^g)^{\underline{n}}"] \arrow[rr, "\chi", "\cong"'] & & (A^{\underline{n}})^{\underline{m}} \arrow[u, "(a^f)^{\underline{m}}"'] \end{tikzcd} \tag{2}$$

Formally, we denote by $(\mathcal{U}\otimes\mathcal{U}')$-**biAlg** the category of $\langle\mathcal{U}\otimes\mathcal{U}'\rangle$-bialgebras, with morphisms the non-expansive homomorphisms preserving both algebraic structures. Then, the following isomorphism of categories holds.

▶ **Proposition 15.** $\mathbb{K}(\Sigma+\Sigma', \mathcal{U}\otimes\mathcal{U}') \cong (\mathcal{U}\otimes\mathcal{U}')$-**biAlg**, *for* $\mathcal{U},\mathcal{U}'$ *basic quantitative theories.*

Moreover, by adapting the isomorphism of Theorem 4 and exploiting the density of symbolic pre-operations (*cf.* Corollary 14) the following is also true.

▶ **Proposition 16.** $(\mathcal{U}\otimes\mathcal{U}')$-**biAlg** $\cong (T_{\mathcal{U}}\otimes T_{\mathcal{U}'})$-**biAlg**, *for* $\mathcal{U},\mathcal{U}'$ *basic quantitative theories.*

By combining the above two propositions we get the main theorem of this section.

▶ **Theorem 17.** *Let* $\mathcal{U},\mathcal{U}'$ *be basic quantitative theories. Then, the monad* $T_{\mathcal{U}\otimes\mathcal{U}'}$ *in* **Met** *is the tensor of monads* $T_{\mathcal{U}} \otimes T_{\mathcal{U}'}$.

Proof. By Propositions 15 and 16 the forgetful functor from $(T_{\mathcal{U}} \otimes T_{\mathcal{U}'})$-**biAlg** to **Met** has a left adjoint and the monad generated by this adjunction is isomorphic to $T_{\mathcal{U}\otimes\mathcal{U}'}$. Thus, by definition of tensor of monads, $T_{\mathcal{U}\otimes\mathcal{U}'} \cong T_{\mathcal{U}} \otimes T_{\mathcal{U}'}$. ◀

The above results do not require any specific property of **Met**, apart that its morphisms are non-expansive maps. Thus, when the quantitative equational theories are continuous, we can reformulate an alternative version of Theorem 17 which is valid in **CMet**.

▶ **Theorem 18.** *Let* $\mathcal{U},\mathcal{U}'$ *be continuous quantitative theories. Then,* $\mathbb{C}T_{\mathcal{U}\otimes\mathcal{U}'}$ *in* **CMet** *is the tensor of monads* $\mathbb{C}T_{\mathcal{U}} \otimes \mathbb{C}T_{\mathcal{U}'}$.

6 Quantitative Reader Algebras

Let E be a finite set or input values and fix an enumeration $E = \{e_1, \dots, e_n\}$ for it. The *quantitative reader algebras* of type E are the algebras for the signature

$$\Sigma_{\mathcal{R}_E} = \{\mathsf{r}\colon |E|\}$$

having only one operator r of arity equal to the number of the input values in E, and satisfying the following axioms

$$(\mathsf{Idem}) \vdash x \equiv_0 \mathsf{r}(x, \dots, x)\,,$$
$$(\mathsf{Diag}) \vdash \mathsf{r}(x_{1,1}, \dots, x_{n,n}) \equiv_0 \mathsf{r}(\mathsf{r}(x_{1,1}, \dots, x_{1,n}), \dots, \mathsf{r}(x_{n,1}, \dots, x_{n,n}))\,.$$

The quantitative theory induced by the axioms above, written $\mathcal{R}_E$, is called *quantitative theory of reading computations.*

Intuitively, the term $\mathsf{r}(t_1, \ldots, t_n)$ can be interpreted as the computation that proceeds as t_i after reading the value e_i from its input. The axiom (Idem) says that if we ignore the value of the input the reading of it is not observable; (Diag) says that the resulting computation after reading the input is the same no matter how many times we read it.

▶ Remark 19. For the binary case ($|E| = 2$) we can think of r as an *if-then-else* statement $b?(S, T)$ checking for the value of a fixed global Boolean variable b and proceeding as S when $b = \text{true}$, and as T otherwise. In this case, (Idem) and (Diag) express the standard program equivalences $S \equiv b?(S, S)$ and $b?(S, T) \equiv b?(b?(S, T), b?(S, T))$.

In the following, when the set E is clear from the context, we use $\mathcal{R}$ in place of $\mathcal{R}_E$.

On Metric Spaces

Let E be a finite set. We denote by $\underline{E}$ the extended metric space on E equipped with the indiscrete metric that assigns infinite distance to any pair of distinct elements.

Consider the $\underline{E}$-power functor $(-)^{\underline{E}} \colon \mathbf{Met} \to \mathbf{Met}$, assigning to each $X \in \mathbf{Met}$ the metric space $[E, X]$ of (necessarily non-expansive) maps from $\underline{E}$ to X.

This functor has a monad structure, with unit $\kappa \colon Id \Rightarrow (-)^{\underline{E}}$ and multiplication $\Delta \colon ((-)^{\underline{E}})^{\underline{E}} \Rightarrow (-)^{\underline{E}}$, respectively given as follows, for $x \in X$, $e \in E$, and $f \in E \to X^{\underline{E}}$

$$\kappa_X(x)(e) = x\,, \qquad\qquad \Delta_X(f)(e) = f(e)(e)\,.$$

This is also known as *reader monad* (also called *environment monad* or *function monad*).

▶ Remark 20. The reader monad is always well defined in a cartesian closed category. Fix an object E. The reader monad $(-)^E$ has unit and multiplication respectively given by

$$X \cong X^1 \xrightarrow{X^!} X^E \qquad \text{and} \qquad (X^E)^E \cong X^{E\times E} \xrightarrow{X^\delta} X^E\,,$$

where $! \colon E \to 1$ is the unique map to the terminal object and $\delta \colon E \to E \times E$ the diagonal map $\delta = \langle id, id\rangle$. However, this definition does not generalise to arbitrary monoidal closed categories, and $\mathbf{Met}$ is such a counterexample. The specific problem with $\mathbf{Met}$ is that $\delta \colon E \to E \,\Box\, E$ is not well-defined for arbitrary $E \in \mathbf{Met}$, as non-expansivness requires that

$$d_E(e, e') \geq d_{E\Box E}(\delta(e), \delta(e')) = d_E(e, e') + d_E(e, e')\,,$$

which holds only when E has the discrete metric. This is the reason why in our treatment we restrict the set of input values to have discrete metric.

The reader monad $(-)^{\underline{E}}$ is isomorphic to the free monad $T_{\mathcal{R}}$. In other words, the quantitative theory $\mathcal{R}$ of reading computations axiomatises the reader monad.

▶ **Theorem 21.** *The monads $T_{\mathcal{R}}$ and $(-)^{\underline{E}}$ in $\mathbf{Met}$ are isomorphic.*

Let T be a strong monad with strength t. The natural transformation $\lambda_X : TX^{\underline{E}} \Rightarrow (TX)^{\underline{E}}$ obtained from the strength t by currying $Tev^{\underline{E}}_X \circ t_{\underline{E},X^{\underline{E}}}$, is a distributive law of monads. Distributive laws induce a notion of monad composition [2], so Moggi's reader monad transformer $T \mapsto (T-)^{\underline{E}}$ is also available in $\mathbf{Met}$. The following says that we can recover this monad transformer as the operation of tensoring with the reader monad.

▶ **Theorem 22** (Tensoring with Reader Monad)**.** *Let T be a strong monad. Then, $T \otimes (-)^{\underline{E}}$ exists and is given as the monad composition $(T-)^{\underline{E}}$.*

By using the above result in combination with Theorem 17, we obtain an analogous transformer at the level of quantitative equational theories as follows.

▶ **Corollary 23.** *Let $\mathcal{U}$ be a basic quantitative equational theory. Then, $(T_{\mathcal{U}}-)^{\underline{E}}$ is the free monad on the theory $\mathcal{U} \otimes \mathcal{R}$ in $\mathbf{Met}$.*

On Complete Metric Spaces

The category **CMet** has finite products. Since, we assumed the set of input values E to be finite, the functor $(-)^{\underline{E}}$ is isomorphic to the finite product $(-)^n$, for $n = |E|$. Therefore the power functor $(-)^{\underline{E}}$, preserves Cauchy completeness and can be restricted to an endofunctor on **CMet**. Thus also the reader monad restricts to **CMet**.

Because $\mathcal{R}$ is a continuous quantitative theory, the free monad on $\mathcal{R}$ in **CMet** is $\mathbb{C}T_{\mathcal{R}}$. Thus, by restricting Theorem 21 to quantitative algebras over **CMet**, we obtain:

▶ **Theorem 24.** *The monads $\mathbb{C}T_{\mathcal{R}}$ and $(-)^{\underline{E}}$ in* **CMet** *are isomorphic.*

In virtue of the above characterisation, by instantiating Theorem 22 in the category of complete extended metric spaces, in combination with Theorems 17 we obtain the following variant of the quantitative reader theory transformer on continuous quantitative theories.

▶ **Corollary 25.** *Let $\mathcal{U}$ be a continuous quantitative theory. Then, $(\mathbb{C}T_{\mathcal{U}}-)^{\underline{E}}$ is the free monad on the theory $\mathcal{U} \otimes \mathcal{R}$ in* **CMet**.

7 Quantitative Writer Algebras

Fix an extended metric space $\Lambda \in$ **Met** of *output values* having monoid structure $(\Lambda, *, 0)$ with non-expansive multiplication operation $*\colon \Lambda \times \Lambda \to \Lambda$.

The *quantitative writer algebras* of type Λ are the algebras for the signature

$$\Sigma_{\mathcal{W}_\Lambda} = \{\mathsf{w}_\alpha \colon 1 \mid \alpha \in \Lambda\}$$

having a unary operator w_α, for each output value $\alpha \in \Lambda$, and satisfying the following axioms

$$\begin{aligned}
&(\mathsf{Zero}) \ \vdash x \equiv_0 \mathsf{w}_0(x)\,,\\
&(\mathsf{Mult}) \ \vdash \mathsf{w}_\alpha(\mathsf{w}_{\alpha'}(x)) \equiv_0 \mathsf{w}_{\alpha * \alpha'}(x)\,,\\
&(\mathsf{Diff}) \ \{x \equiv_\varepsilon x'\} \vdash \mathsf{w}_\alpha(x) \equiv_\delta \mathsf{w}_{\alpha'}(x')\,, \text{ for } \delta \geq d_\Lambda(\alpha,\alpha') + \varepsilon\,.
\end{aligned}$$

The quantitative theory induced by the axioms above, written $\mathcal{W}_\Lambda$, is called *quantitative theory of writing computations*.

The term $\mathsf{w}_\alpha(t)$ represents the computation that proceeds as t after writing α on the output tape. The axiom (Zero) says that writing the identity element 0 is not observable on the tape; (Mult) says that consecutive writing operations are stored in the tape in the order of execution; (Diff) compares two computations w.r.t. the distance of their output values.

In the following, when the metric space Λ of output values is clear from the context, we use $\mathcal{W}$ in place of $\mathcal{W}_\Lambda$.

On Metric Spaces

Let $(\Lambda \,\square\, -)\colon$ **Met** $\to$ **Met** be the functor assigning to each extended metric space X the space $(\Lambda \,\square\, X)$. By exploiting the monoid structure of Λ, the functor $(\Lambda \,\square\, -)$ can be given a monad structure with unit $\tau \colon Id \Rightarrow (\Lambda \,\square\, -)$ and multiplication $\varsigma \colon (\Lambda \,\square\, (\Lambda \,\square\, -)) \Rightarrow (\Lambda \,\square\, -)$, respectively given as follows, for arbitrary $x \in X$ and $\alpha, \alpha' \in \Lambda$

$$\tau_X(x) = (0, x)\,, \qquad \varsigma_X((\alpha, (\alpha', x))) = (\alpha * \alpha', x)\,.$$

This monad is also known as *writer monad* (also called *complexity monad*). Note that, the non-expansiveness of the maps above crucially depends on the assumption that the multiplication $*$ in Λ is non-expansive.

The next theorem says that the writer monad $(\Lambda \,\square\, -)$ has a quantitative equational presentation in terms of the theory $\mathcal{W}$ of writing computations.

▶ **Theorem 26.** *The monads $T_{\mathcal{W}}$ and $(\Lambda \,\square\, -)$ in* **Met** *are isomorphic.*

Let T be a strong monad with strength t. There is a canonical distributive law of the monad $(\Lambda \,\square\, -)$ over T, obtained using the strength $t_{\Lambda,-} \colon (\Lambda \,\square\, T-) \Rightarrow T(\Lambda \,\square\, -)$ of T. So $T(\Lambda \,\square\, -)$ acquires a canonical monad structure [2], and we obtain Moggi's writer monad transformer $T \mapsto T(\Lambda \,\square\, -)$ in **Met**.

In [12], Hyland et al. observed that Moggi's writer monad transformer can be equivalently recovered as the operation of tensoring with the writer monad.

▶ **Theorem 27** (Tensoring with Writer Monad [12])**.** *Let T be a strong monad. Then, the monad composition $T(\Lambda \,\square\, -)$ is given as $T \otimes (\Lambda \,\square\, -)$.*

By combining the above with Theorems 17 and 26, we get an analogous transformer at the level of quantitative equational theories as follows:

▶ **Corollary 28.** *Let $\mathcal{U}$ be a basic quantitative theory. Then, $T_{\mathcal{U}}(\Lambda \,\square\, -)$ is the free monad on the theory $\mathcal{U} \otimes \mathcal{W}$ in* **Met***.*

On Complete Metric Spaces

If we assume the monoid $(\Lambda, *, 0)$ to be over a complete extended metric space Λ, the writer monad $(\Lambda \,\square\, -)$ is well defined also in **CMet**.

Since $\mathcal{W}$ is axiomatised by a continuous schema of quantitative conditional equations the free monad on $\mathcal{W}$ in **CMet** is given by $\mathbb{C}T_{\mathcal{W}}$. Thus, by restricting the use of Theorem 26 to quantitative algebras over complete extended metric spaces, we obtain:

▶ **Theorem 29.** *The monads $\mathbb{C}T_{\mathcal{W}}$ and $(\Lambda \,\square\, -)$ in* **CMet** *are isomorphic.*

Thus, by similar arguments as before, we obtain the following variant of Corollary 28.

▶ **Corollary 30.** *Let $\mathcal{U}$ be a continuous quantitative theory. Then, $\mathbb{C}T_{\mathcal{U}}(\Lambda \,\square\, -)$ is the free monad on the theory $\mathcal{U} \otimes \mathcal{W}$ in* **CMet***.*

8 The Algebras of Labeled Markov Processes

In this section we show how to obtain a quantitative equational axiomatization of labelled Markov processes with discounted bisimilarity metric as the combination, via sum and tensor, of the following simpler quantitative equational theories:

(a) *The quantitative theory of interpolative barycentric algebras $\mathcal{B}$* from [21] (see also Appendix B) over the signature $\Sigma_{\mathcal{B}} = \{+_e \colon 2 \mid e \in [0,1]\}$ extends M. H. Stone's theory of barycentric algebras [32] (a.k.a. abstract convex algebras) with the following axiom

$$\text{(IB)} \quad \{x \equiv_\varepsilon y, x' \equiv_{\varepsilon'} y'\} \vdash x +_e x' \equiv_\delta y +_e y', \ \text{ for } \delta \geq e\varepsilon + (1-e)\varepsilon'$$

expressing that the distance between convex combinations is obtained as the convex interpolation of the distance of their sub-terms. This theory will be used to axiomatise probability distributions with Kantorovich metric [15] (see also Appendix A).

(b) *The pointed quantitative theory*, defined as the free quantitative theory $\mathcal{U}_{\mathbf{0}}$ (*i.e.*, the one imposing no additional axioms) for a signature $\Sigma_{\mathbf{0}} = \{\mathbf{0} \colon 0\}$ consisting of a single constant $\mathbf{0}$ symbol. This will be used to axiomatise termination.

(c) *The quantitative theory $\mathcal{R}_A$ of reading computations* (*cf.* Section 6) will be used to axiomatise the reaction to the choice of a label from a set A of action labels.

(d) *The quantitative theory of contractive operators* discussed in [1], is the theory obtained by imposing a Lipschitz contractive axiom for each operator in the signature. In our case, we consider a signature $\Sigma_\diamond = \{\diamond\colon 1\}$ with only one unary operator and the contractive theory $\mathcal{U}_\diamond$ generated from the axiom

$$(\diamond\text{-Lip}) \quad \{x =_\varepsilon y\} \vdash \diamond(x) \equiv_\delta \diamond(y)\,, \text{ for } \delta \geq c\varepsilon\,,$$

where $c \in (0,1)$ is a fixed *contractive factor* for the operator $\diamond$. This theory will be used to axiomatise the transition to a next state with discount factor c.

Formally, we define the quantitative theory $\mathcal{U}_{\mathbf{LMP}}$ of labelled Markov processes as the following combination of quantitative theories, with signature $\Sigma_{\mathbf{LMP}}$ given by the disjoint union of those from its component theories:

$$\Sigma_{\mathbf{LMP}} = \Sigma_{\mathcal{B}} + \Sigma_{\mathbf{0}} + \Sigma_{\mathcal{R}_A} + \Sigma_\diamond\,, \qquad \mathcal{U}_{\mathbf{LMP}} = ((\mathcal{B} + \mathcal{U}_{\mathbf{0}}) \otimes \mathcal{R}_A) + \mathcal{U}_\diamond\,.$$

Following [33, Section 6], we regard A-labelled Markov processes over extended metric spaces as $(\Delta(1 + -))^{\underline{A}}$-coalgebras in **Met**, where Δ is the *Kantorovich functor* assigning to each $X \in \mathbf{Met}$ the space of Radon probability measures with finite moment over X equipped with Kantorovich metric. In [33] it is shown that the *probabilistic bisimilarity distance* on a labelled Markov processes (X, τ) is equal to the (pseudo)metric

$$\mathbf{d}_{(X,\tau)}(x, x') = d_Z(h(x), h(x'))\,,$$

where $h\colon X \to Z$ is the unique homomorphism to the final coalgebra (Z, ω).

Similarly to [1], we slightly extend the type of the coalgebras to encompass the case when the probabilistic bisimilarity distance is discounted by a factor $0 < c < 1$. Explicitly, we consider coalgebras for the functor $(\Delta(1 + c \cdot -))^{\underline{A}}$, where $(c \cdot -)$ is the *rescaling functor*, mapping a metric space (X, d_X) to $(X, c \cdot d_X)$. This will not change the essence of the results from [33] that are used in this section to characterise the probabilistic bisimilarity metric.

In the reminder of the section we prove that the theory $\mathcal{U}_{\mathbf{LMP}}$ axiomatizes (the monad of) A-labelled Markov processes with c-discounted bisimilarity metric.

On Metric Spaces

We characterise the monad $T_{\mathcal{U}_{\mathbf{LMP}}}$ in steps. First, note that $T_{\mathcal{U}_{\mathbf{0}}} \cong 1^* = (1 + -)$ is the *maybe monad*, *i.e.*, freely generated monad on the constant terminal object functor 1. As the monad $(1 + -)$ is isomorphic to $(1F)^*$, for any functor F, by [1, Theorems 4.4 and 5.2], and [12, Theorem 4], we obtain the following isomorphism of monads in **Met**:

$$T_{\mathcal{B}+\mathcal{U}_{\mathbf{0}}} \cong T_{\mathcal{B}} + T_{\mathcal{U}_{\mathbf{0}}} \cong \Pi(1 + -)\,,$$

where $\Pi(1 + -)$ is the *finite sub-distribution monad* with functor assigning to $X \in \mathbf{Met}$ the space of finitely supported Borel sub-probability measures with Kantorovich metric. Thus, $\mathcal{B} + \mathcal{U}_{\mathbf{0}}$ axiomatizes finitely supported sub-probability distributions with Kantorovich metric.

From the above, Theorem 17 and Corollary 23, we further get the monad isomorphism

$$T_{(\mathcal{B}+\mathcal{U}_{\mathbf{0}})\otimes\mathcal{R}_A} \cong \Pi(1 + -) \otimes (-)^{\underline{A}} \cong (\Pi(1 + -))^{\underline{A}}\,,$$

saying that tensoring with the theory $\mathcal{R}_A$ of reading computations corresponds to axiomatically adding the capability of reacting to the choice of an action label.

By [1, Theorem 6.3], $T_{\mathcal{U}_\diamond}$ is isomorphic to the free monad over the rescaling functor $(c\cdot -)$. Hence, by [1, Theorem 4.4] and [12, Corollary 2] we get the following last isomorphism

$$T_{\mathcal{U}_{\mathbf{LMP}}} = T_{((\mathcal{B}+\mathcal{U}_{\mathbf{0}})\otimes\mathcal{R}_A)+\mathcal{U}_\diamond} \cong \mu y.(\Pi(1+c\cdot y + -))^{\underline{A}}.$$

Explicitly, this means that, the free monad on $\mathcal{U}_{\mathbf{LMP}}$ assigns to an arbitrary metric space $X \in \mathbf{Met}$ the *initial solution* of the following functorial equation in $\mathbf{Met}$

$$LMP_X \cong (\Pi(1+c\cdot LMP_X + X))^{\underline{A}}.$$

In particular, when $X = 0$ is the empty metric space (*i.e.*, the initial object in $\mathbf{Met}$) the above corresponds to the isomorphism on the initial $(\Pi(1+c\cdot -))^{\underline{A}}$-algebra. The isomorphism gives us also a $(\Pi(1+c\cdot -))^{\underline{A}}$-coalgebra structure on LMP_0, which can be converted into a labeled Markov process (LMP_0, τ_0) via a post-composition with the inclusion $\Pi(-) \hookrightarrow \Delta(-)$.

The key aspect is that the metric of LMP_0 is exactly the bisimilarity metric.

▶ **Lemma 31.** *d_{LMP_0} is the c-discounted probabilistic bisimilarity metric on (LMP_0, τ_0).*

▶ Remark 32. For a less abstract description of (LMP_0, τ_0), notice that the elements of LMP_0 are (equivalence classes of) ground terms over the signature $\Sigma_{\mathbf{LMP}}$, which one can interpret as pointed (or rooted) acyclic labelled Markov processes quotiented by bisimilarity.

On Complete Metric Spaces

Since all the quantitative theories considered are continuous, we can replicate the same steps also while interpreting the theory $\mathcal{U}_{\mathbf{LMP}}$ over complete metric spaces, obtaining the monad

$$\mathbb{C}T_{\mathcal{U}_{\mathbf{LMP}}} \cong \mu y.\Delta(1+c\cdot y + -)^{\underline{A}}.$$

By following similar arguments to [1, Section 8.3], one can prove that the the functorial equation $LMP_X \cong \Delta(1+c\cdot LMP_X + X)^{\underline{A}}$ has a unique solution. Thus by applying the monad above on $X = 0$ we recover the carrier of the final $(\Delta(1+c\cdot -))^{\underline{A}}$-coalgebra, equipped with c-discounted probabilistic bisimilarity metric.

▶ Remark 33. While by interpreting the theory $\mathcal{U}_{\mathbf{LMP}}$ over $\mathbf{Met}$ we can only characterise Markov processes that are acyclic, by doing it over $\mathbf{CMet}$ we obtain an algebraic representation of all bisimilarity classes as the elements of the final coalgebra. Thus, among others, we also recover Markov processes with cyclic structures as the limit of all their finite unfoldings.

9 The Algebras of Markov Decision Processes with Rewards

As a last example, we provide a quantitative axiomatization of Markov decision processes with rewards equipped with discounted bisimilarity metric. As the construction is similar to Section 8, we avoid repeating the details of each step of the monad characterization.

Let $(\mathbb{R}, +, 0)$ be the standard monoid structure on the reals. We define the quantitative theory $\mathcal{U}_{\mathbf{MDP}}$ of Markov decision processes with real-valued rewards as follows

$$\Sigma_{\mathbf{MDP}} = \Sigma_{\mathcal{B}} + \Sigma_{\mathcal{W}_{\mathbb{R}}} + \Sigma_{\mathcal{R}_A} + \Sigma_\diamond\,, \qquad \mathcal{U}_{\mathbf{MDP}} = ((\mathcal{B}\otimes\mathcal{U}_{\mathcal{W}_{\mathbb{R}}})\otimes\mathcal{R}_A) + \mathcal{U}_\diamond\,,$$

where $\mathcal{W}_{\mathbb{R}}$ is the theory of writing computations and the other theories are as in Section 8.

For convenience, we regard Markov decision processes over metric spaces as the coalgebras for the functor $(\Delta(\mathbb{R}\,\square\, c\cdot -))^{\underline{A}}$ on $\mathbf{Met}$, where the endofunctor $(\mathbb{R}\,\square\, -)$ is used to encode the metric differences at each decision step for the real-valued reward available for two states. Via this coalgebraic representation, the c-discounted *probabilistic bisimilarity distance* on this structures can be defined as in [33] (following the same definition of Section 8).

▶ Remark 34. In [31] a Markov decision process is defined as a tuple $(S, p(\cdot|s,a), r(s,a))$ with a Markov kernel $p\colon S \times A \to \Delta(S)$ and randomised reward function $r\colon S \times A \to \Delta(\mathbb{R})$. Our coalgebraic representation is the natural generalisation over metric spaces, where the randomness of the Markov kernel and reward function is combined as a probability measure on $(\mathbb{R} \,\Box\, c \cdot S)$, by regarding $\mathbb{R}$ and S as extended metric spaces (for each $a \in A$).

On Metric Spaces and Complete Metric Spaces

Similarly to what we have done in Section 8 for labelled Markov processes, we relate Markov decision processes and their c-discounted probabilistic bisimilarity pseudometric with the free monads on the theory $\mathcal{U}_{\mathbf{MDP}}$ in **Met** and **CMet**.

The only step that changes in the characterisation of $T_{\mathcal{U}_{\mathbf{MDP}}}$, regards the combination of theories $\mathcal{B} \otimes \mathcal{U}_{\mathcal{W}_{\mathbb{R}}}$, which is dealt using Corollary 28. Thus, similarly to Section 8 we get

$$T_{\mathcal{U}_{\mathbf{MDP}}} = T_{((\mathcal{B}+\otimes\mathcal{U}_{\mathcal{W}_{\mathbb{R}}})\otimes\mathcal{R}_A)+\mathcal{U}_\diamond} \cong \mu y.\Pi((\mathbb{R} \,\Box\, y) + -)^{\underline{A}}\,.$$

The metric on the initial solution for the functorial fixed point definition corresponds to the c-discounted probabilistic bisimilarity (pseudo)metric on its coalgebra structure.

Similar considerations apply also when interpreting the theories in the category **CMet** of complete metric spaces, as the argument follows without issues because $\mathbb{R}$ a complete metric space. Thus we obtain the following characterisation for the monad:

$$\mathbb{C}T_{\mathcal{U}_{\mathbf{LMP}}} \cong \mu y.\Delta((\mathbb{R} \,\Box\, y) + -)^{\underline{A}}\,.$$

Again, the metric on the solution for the above functorial fixed point definition corresponds to the c-discounted probabilistic bisimilarity metric. Moreover, as the fixed point solution is unique, $\mathbb{C}T_{\mathcal{U}_{\mathbf{LMP}}}0$ is an algebraic characterization of the final $(\Delta(\mathbb{R} \,\Box\, c \cdot -))^{\underline{A}}$-coalgebra.

10 Conclusions

We studied the commutative combination of quantitative effects as the tensor of their quantitative equational theories. The key result in this regard is Theorem 17, asserting that the tensor of two quantitative theories corresponds to the categorical tensor of their free monads. In addition to this general result, we show how to extend to the quantitative algebraic setting Moggi's notions of reader and writer monad transformers.

We illustrate the applicability of our theoretical development with two examples: labeled Markov processes and Markov decision processes. Apart from the intrinsic interest in their quantitative equational axiomatisations, what is particularly pleasant is the systematic compositional way with which one can obtain quantitative axiomatisations of different variants of Markov processes by just combining theories as new basic ingredients.

An example that escapes our compositional treatment via sum and tensor is the combination of probabilities and non-determinism as illustrated in [24]. A possible future work in this direction is to extend the combination of theories with another operator: the distributive tensor (see [13, Section 6]). Following a similar intuition by Cheng [6], we claim that these correspond in a suitable way to Garner's weak distributive law [9]. Our claim seems promising in the light of the work [10, 3] which consider equational axiomatisations combining probabilities and non-determinism.

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A Kantorovich Metric on Extended Metric Spaces

We assume the reader is familiar with the notions of σ-algebras, measurable functions, and Borel probability measures. Next we review some facts about metrics between extended spaces of probability distributions from [23].

Let M be an extended metric space. A Borel probability measure μ over M is *Radon* if for any Borel set $E \subseteq M$, $\mu(E)$ is the supremum of $\mu(K)$ over all compact subsets K of E. Examples of Radon probability measures are finitely supported probability measures on any metric space and generic Borel probability measures over complete separable metric spaces.

A Radon probability measure μ over M has *finite moment* (of order 1) if, for some (equivalently all) m_0, the integral

$$\int d_M(m_0, -) \,\mathrm{d}\mu$$

is finite. By restricting our attention to Radom measures of finite moment the following is a well-defined metric [23].

The *Kantorovich metric* (or W_1 metric) between Radon probability measures μ, ν of finite moment over M is defined as:

$$\mathcal{K}(d_M)(\mu,\nu) = \min\left\{ \int d_M \,\mathrm{d}\omega \mid \omega \in \mathcal{C}(\mu,\nu) \right\}.$$

where $\mathcal{C}(\mu,\nu)$ is the set of *couplings* for a pair of Radon measures (μ,ν), that is, a Radon probability measures ω on the product space $M \times M$, such that, for all Borel sets $E \subseteq M$

$$\omega(E \times M) = \mu(E) \qquad \text{and} \qquad \omega(M \times E) = \nu(E)\,.$$

We write $\Delta(M)$ for the set of Radon probability measures with finite moment over M, equipped with the Kantorovich metric and $\Pi(M)$ for the subspace of $\Delta(M)$ of the finitely supported Borel probability measures over M.

▶ **Theorem 35** (Theorem 2.7 [23]). *Let M be a complete extended metric space. Then, $\Delta(M)$ is a complete metric space generated as the Cauchy completion of $\Pi(M)$, i.e., $\overline{\Pi(M)} \cong \Delta(M)$.*

B Interpolative Barycentric Algebras

In this section we recall interpolative barycentric algebras from [21], which are the quantitative algebras for the signature

$$\Sigma_{\mathcal{B}} = \{+_e \colon 2 \mid e \in [0,1]\}$$

having a binary operator $+_e$, for each $e \in [0,1]$ (a.k.a. *barycentric signature*), and satisfying the following axioms

$$
\begin{aligned}
&(\mathsf{B1}) \vdash x +_1 y \equiv_0 x\,,\\
&(\mathsf{B2}) \vdash x +_e x \equiv_0 x\,,\\
&(\mathsf{SC}) \vdash x +_e y \equiv_0 y +_{\bar{e}} x\,,\\
&(\mathsf{SA}) \vdash (x +_e y) +_{e'} z \equiv_0 x +_{ee'} (y +_{\frac{e'-ee'}{1-ee'}} z)\,,\ \text{ for } e, e' \in [0,1)\,,\\
&(\mathsf{IB}) \{x \equiv_\varepsilon y, x' \equiv_{\varepsilon'} y'\} \vdash x +_e x' \equiv_\delta y +_e y',\ \text{ for } \delta \geq e\varepsilon + (1-e)\varepsilon'\,.
\end{aligned}
$$

The quantitative theory induced by the axioms above, written $\mathcal{B}$, is called *interpolative barycentric quantitative equational theory.* The axioms (B1), (B2), (SC), (SA) are those of *barycentric algebras* (a.k.a. *abstract convex sets*) due to M. H. Stone [32] where (SC) stands for *skew commutativity* and (SA) for *skew associativity*; (IB) is the *interpolative barycentric axiom* introduced in [21].

On Metric Spaces

Let $\Pi \colon \mathbf{Met} \to \mathbf{Met}$ be the functor assigning to each $X \in \mathbf{Met}$ the metric space $\Pi(X)$ of finitely supported Borel probability measures with Kantorovich metric and acting on morphisms $f \colon X \to Y$ as $\Pi(f)(\mu) = \mu \circ f^{-1}$, for $\mu \in \Pi(X)$.

This functor has a monad structure, with unit $\delta \colon Id \Rightarrow \Pi$ and multiplication $m \colon \Pi^2 \Rightarrow \Pi$, given as follows, for $x \in X$, $\Phi \in \Pi^2(X)$, and Borel subset $E \subseteq X$

$$\delta_X(x) = \delta_x\,, \qquad\qquad m_X(\Phi)(E) = \int v_E \, \mathrm{d}\Phi\,,$$

where δ_x is the Dirac distribution at x, and $v_E \colon \Pi(X) \to [0,1]$ is the evaluation function, taking $\mu \in \Pi(X)$ to $\mu(E) \in [0,1]$. This monad is also known as the *finite distribution monad.*

For any $X \in \mathbf{Met}$, one can define a quantitative $\Sigma_{\mathcal{B}}$-algebra $(\Pi(X), \phi_X)$ as follows, for arbitrary $\mu, \nu \in \Pi X$

$$\phi_X \colon \Sigma_{\mathcal{B}} \Pi X \to \Pi X \qquad\qquad \phi_X(in_{+_e}(\mu,\nu)) = e\mu + (1-e)\nu\,.$$

This quantitative algebra satisfies the interpolative barycentric theory $\mathcal{B}$ (*cf.* [21, Theorem 10.4]) and is isomorphic to the free quantitative $\mathcal{B}$-algebra (*cf.* [21, Theorem 10.5]).

Thus, as shown in [21], Π is isomorphic to the free monad $T_{\mathcal{B}}$ on the theory $\mathcal{B}$ of interpolative barycentric algebras.

▶ **Theorem 36.** *The monads $T_{\mathcal{B}}$ and Π in $\mathbf{Met}$ are isomorphic.*

On Complete Metric Spaces

Define the functor $\Delta\colon \mathbf{CMet} \to \mathbf{CMet}$ assigning to each $X \in \mathbf{CMet}$ the complete metric space $\Delta(X)$ of Radon probability measures with finite moment and equipped with Kantorovich metric; acting on morphisms $f\colon X \to Y$ as $\Delta(f)(\mu) = \mu \circ f^{-1}$, for $\mu \in \Delta(X)$. This functor has a monad structure, defined similarly to the one for Π. It is known as the *Kantorovich monad*.

By exploiting Lemma 35, one can verify that $\Delta(X)$ with its canonical barycentric algebra structure is the free interpolative barycentric algebra in **CMet** (Theorem 3.7 [23]). As the canonical monad structure on $\mathbb{C}\Pi$ is isomorphic to the one on Δ in **CMet**, by Theorem 36, we obtain the following.

▶ **Theorem 37.** *The monads $\mathbb{C}T_{\mathcal{B}}$ and Δ in* **CMet** *are isomorphic.*

Note that, since $\mathcal{B}$ is axiomatised by a continuous schema of quantitative equations, the free monad on $\mathcal{B}$ in **CMet** is given by $\mathbb{C}T_{\mathcal{B}}$. In other words, [23, Theorem 3.7] provides an algebraic characterisation of the Kantorovich monad.

C Omitted Proofs

Proof. (of Proposition 6) (1) $\Rightarrow$ (2) follows by definition of a^g, b^g and naturality of g. As for (2) $\Rightarrow$ (1), note that since $\mathbf{V}$ is a symmetric monoidal closed category, we have a 1-1 correspondence between strong and $\mathbf{V}$-enriched endofunctors on $\mathbf{V}$, and also between strong and $\mathbf{V}$-enriched natural transformations [17]. Therefore, by (the weak form of) the enriched Yoneda lemma (*cf.* [16]), there exists a natural bijection between strong natural transformations $g \in \mathcal{O}_F(A)$ and the (generalised) elements of FA, *i.e.*, morphisms of the form $I \to FA$, obtained via the composition

$$I \xrightarrow{id_A} A^A \xrightarrow{g_A} FA\,.$$

Thus, for any $e\colon I \to FA$, there exists $\hat{e} \in \mathcal{O}_F(A)$ such that $\hat{e}_A \circ id_A = e$. Therefore, by naturality of $\hat{e}$, definition of $a^{\hat{e}}$, $b^{\hat{e}}$, and (2), the following diagram commute

$$\begin{tikzcd}
I \arrow[r, "e"] \arrow[dr, "id_A"'] \arrow[dd, "e"'] & FA \arrow[rr, "a"] & & A \arrow[dd, "f"] \\
& A^A \arrow[u, "\hat{e}_A"'] \arrow[r, "f^A"] \arrow[dl, "\hat{e}_A"'] & B^A \arrow[d, "\hat{e}_B"] & \\
FA \arrow[rr, "Ff"] & & FB \arrow[r, "b"] & B
\end{tikzcd}$$

implying that $f^I \circ a^I = b^I \circ (Ff^I)$. Then (1) follows by the naturality of the isomorphism $V \xrightarrow{\cong} V^I$ (obtained by currying $\lambda\colon I \,\Box\, V \xrightarrow{\cong} V$) and the commutativity of the diagram

$$\begin{tikzcd}
FA \arrow[r, "\cong"] \arrow[d, "Ff"'] \arrow[rrr, bend left, "a"] & (FA)^I \arrow[r, "a^I"] \arrow[d, "(Ff)^I"'] & A^I \arrow[d, "f^I"] & A \arrow[l, "\cong"'] \arrow[d, "f"] \\
FB \arrow[r, "\cong"] \arrow[rrr, bend right, "b"'] & (FB)^I \arrow[r, "b^I"] & B^I & B \arrow[l, "\cong"']
\end{tikzcd}$$

◀

▶ **Proposition 38.** *Let (A, a) be a F-algebra of a strong endofunctor F on $\mathbf{V}$. Then, for any $v, w \in \mathbf{V}$ and $g \in \mathcal{O}_F(v)$ the following commute*

$$\begin{tikzcd}
(A^v)^w \arrow[rr, "\chi", "\cong"'] \arrow[dr, "(a^g)^w"'] & & (A^w)^v \arrow[dl, "\bar{a}^g"] \\
& A^w &
\end{tikzcd}$$

where $(A, a)^w = (A^w, \bar{a})$ and χ is the canonical isomorphism.

Proof. By the universality of the counit $ev\colon (w \,\Box\, -) \Rightarrow Id$ of the adjunction $(w \,\Box\, -) \dashv (-)^w$ it suffices to show that the following two diagrams commute:

$$\begin{tikzcd}
& w \,\Box\, (A^v)^w \arrow[dl, "a^g \circ ev"'] \arrow[d, "w\Box(a^g)^w"] \\
A & w \,\Box\, A^w \arrow[l, "ev"]
\end{tikzcd}
\qquad
\begin{tikzcd}
w \,\Box\, (A^v)^w \arrow[r, "w\Box\chi"] \arrow[d, "a^g \circ ev"'] & w \,\Box\, (A^w)^v \arrow[d, "w\Box\bar{a}^g"] \\
A & w \,\Box\, A^w \arrow[l, "ev"]
\end{tikzcd}$$

The diagram to the left commutes by naturality of the counit ev; the one to the right commutes as follows, where ξ and t are respectively the strengths of $(-)^v$ and F

$$\begin{tikzcd}
& w \,\Box\, (A^w)^v \arrow[d, "\xi"] \arrow[dddr, bend left, "w\Box g"] & \\
w \,\Box\, (A^v)^w \arrow[ur, "w\Box\chi"] \arrow[d, "ev"'] & (w \,\Box\, A^w)^v \arrow[dl, "ev^v"'] \arrow[d, "g"] & \\
A^v \arrow[d, "g"'] & F(w \,\Box\, A^w) \arrow[dl, "Fev"'] & w \,\Box\, FA^w \arrow[l, "t"'] \arrow[d, "w\Box\sigma"] \\
FA \arrow[d, "a"'] & & w \,\Box\, (FA)^w \arrow[ll, "ev"'] \arrow[d, "w\Box a^w"] \\
A & & w \,\Box\, A^w \arrow[ll, "ev"']
\end{tikzcd}$$

by naturality of the counit ev; definition of ξ and χ; definition of the law $\sigma\colon F(-)^w \Rightarrow (F-)^w$; definition of a^g, $\bar{a}^g$; by $\bar{a} = a^w \circ \sigma$; and because g is strong. ◀

Proof. (of Proposition 8) (1) $\Rightarrow$ (2) By Proposition 6, we prove (2) by showing that for all $v \in \mathbf{V}$ and $g \in \mathcal{O}_F(v)$, $b^h \circ \bar{a}^g = a^g \circ (b^h)^v$. This is shown by the diagram below

$$\begin{tikzcd}
(A^w)^v \arrow[rr, "\bar{a}^g"] \arrow[dr, bend left, "\chi"] \arrow[dd, "(b^h)^v"'] & & A^w \arrow[dd, "b^h"] \\
& (A^v)^w \arrow[ul, bend left, "\chi^{-1}"] \arrow[ur, "(a^g)^w"'] \arrow[dl, "\bar{b}^h"'] & \\
A^v \arrow[rr, "a^g"] & & A
\end{tikzcd}$$

which commutes by Proposition 38, (1), definition of a^g, and naturality of g. The implication (2) $\Rightarrow$ (1) is similar. ◀

Proof. (of Proposition 10) The equivalence of the statements (1), (2) follows as in Proposition 8, by using the density of $\mathcal{D}$ and $\mathcal{E}$ in lieu of Proposition 6.

Assume (A, a, b) is a commutative $\langle F, G' \rangle$-bialgebra. Then, (1) follows trivially because, $\mathcal{D}$ is a subset of pre-operations of F. For the converse implication, assume (1) and let $h \in \mathcal{O}_G(w)$ for some $w \in \mathbf{V}$. We want to show that

$$\begin{tikzcd} FA^w \arrow[r, "\bar{a}"] \arrow[d, "F(b^h)"'] & A^w \arrow[d, "b^h"] \\ \Sigma A \arrow[r, "a"] & A \end{tikzcd}$$

commutes, where $(A, a)^w = (A^w, \bar{a})$. By density of $\mathcal{D}$, it suffices to show that for all v-ary pre-operation $g \in \mathcal{D}$, $b^h \circ \bar{a}^g = a^g \circ (b^h)^v$. This follows by

$$\begin{tikzcd} (A^w)^v \arrow[rr, "\bar{a}^g"] \arrow[dd, "(b^h)^v"'] \arrow[dr, bend left, "\chi"] & & A^w \arrow[dd, "b^h"] \\ & (A^v)^w \arrow[ul, bend left, "\chi^{-1}"] \arrow[ur, "(a^g)^w"'] \arrow[dl, "\bar{b}^h"] & \\ A^v \arrow[rr, "a^g"] & & A \end{tikzcd}$$

which commutes by Proposition 38, (1), definition of a^g, and naturality of g. ◀

Proof. (of Proposition 12) Let $(A, a), (B, b)$ be Σ-algebras in **Met** and $h \colon A \to B$ a non-expansive map. We want to prove the equivalence of

1. f is a Σ-homomorphisms from (A, a) to (B, b);
2. For every $f \colon n \in \Sigma$, $h \circ a^f = b^f \circ h^v$.

(1) $\Rightarrow$ (2) follows by definition of a^f, b^f and naturality of $in_f \colon (-)^{\underline{n}} \Rightarrow \Sigma$. The implication (2) $\Rightarrow$ (1) follows by the universality of the coproduct, as $\Sigma = \coprod_{f:n \in \Sigma} Id^n$. ◀

▶ **Proposition 39.** *$T_{\mathcal{U}}$ is a strong monad with strength ζ.*

Proof. Naturality of ζ follows by definition and naturality of $\eta^{\mathcal{U}}$ and $\psi^{\mathcal{U}}$. The coherence conditions of a monoidal strength follow by universality of the evaluation and co-evaluation maps of the closed structure of **Met**, Theorem 3 and definition of $(T_{\mathcal{U}}, \eta^{\mathcal{U}}, \mu^{\mathcal{U}})$. ◀

Proof. (of Proposition 13) $(A, a), (B, b)$ be $T_{\mathcal{U}}$-algebras and $h \colon A \to B$ a non-expansive map. We want to prove the equivalence of

1. h is a $T_{\mathcal{U}}$-homomorphism from (A, a) to (B, b);
2. For every v-ary pre-operation $g \in \mathcal{D}$, $h \circ a^{(\psi^{\mathcal{U}} \circ \Sigma \eta^{\mathcal{U}} \circ g)} = b^{(\psi^{\mathcal{U}} \circ \Sigma \eta^{\mathcal{U}} \circ g)} \circ h^v$.

(1) $\Rightarrow$ (2) follows by definition of $a^{(\psi^{\mathcal{U}} \circ \Sigma \eta^{\mathcal{U}} \circ g)}$, $b^{(\psi^{\mathcal{U}} \circ \Sigma \eta^{\mathcal{U}} \circ g)}$ and naturality of $\psi^{\mathcal{U}} \circ \Sigma \eta^{\mathcal{U}} \circ g$. For the converse implication, recall that the isomorphism of categories from Theorem 4, maps a $T_{\mathcal{U}}$-algebra (A, a) to the Σ-algebra $(A, a \circ \psi^{\mathcal{U}}_A \circ \Sigma \eta^{\mathcal{U}}_A)$ and morphisms essentially to themselves. Thus (2) $\Rightarrow$ (1) follows by density of $\mathcal{D}$ and definition of $a^{(\psi^{\mathcal{U}} \circ \Sigma \eta^{\mathcal{U}} \circ g)}$, $b^{(\psi^{\mathcal{U}} \circ \Sigma \eta^{\mathcal{U}} \circ g)}$. ◀

Proof. (of Proposition 15) The isomorphism is given by the pair of functors

$$\begin{tikzcd} \mathbb{K}(\Sigma + \Sigma', \mathcal{U} \otimes \mathcal{U}') \arrow[r, bend left, "H"] & \mathbb{K}((\Sigma, \mathcal{U}) \otimes (\Sigma', \mathcal{U}')) \arrow[l, bend left, "K"] \end{tikzcd}$$

defined, for a $(\Sigma + \Sigma')$-algebra (A, a) satisfying $\mathcal{U} \otimes \mathcal{U}'$ and a $\langle \mathcal{U} \otimes \mathcal{U}' \rangle$-bialgebra (B, b, b'), respectively as

$$H(A, a) = (A, a \circ in_l, a \circ in_r)\,, \qquad K(B, b, b') = (B, [b, b'])\,,$$

where $[b, b']\colon \Sigma B + \Sigma' B \to B$ is the unique map induced by b and b' by universality of the coproduct. Both functors are identity on morphisms; it is easy to see that a homomorphism in one sense is also a homomorphism in the other.

The pair of functors above is the restriction of the isomorphic pair of functors used in the proof of [1, Proposition 4.1]. Thus, to show H and K are well defined we are just left to deal with checking that the restriction conditions on the subcategories are preserved both ways.

As for H, we prove that whenever $\mathcal{A} = (A, a)$ satisfies the quantitative equation in (1), then $(A, a \circ in_l, a \circ in_r)$ satisfies the commutativity of the diagram in (2). This follows as, for all $f\colon n \in \Sigma$ and $g\colon m \in \Sigma'$, by definition of algebraic interpretation $(-)^{\mathcal{A}}$, we have

$$f^{\mathcal{A}} = a \circ in_l \circ in_f = (a \circ in_l)^f \,,$$
$$g^{\mathcal{A}} = a \circ in_r \circ in_g = (a \circ in_r)^g \,.$$

Thus, the satisfiability (1) coincides with the commutativity of the diagram in (2).

For K we need to show that whenever (B, b, b') satisfies the commutativity of the diagram in (2), then $\mathcal{A} = (A, [b, b'])$ satisfies (1). This follows as, for all $f\colon n \in \Sigma$ and $g\colon m \in \Sigma'$, by definition of algebraic interpretation $(-)^{\mathcal{A}}$, we have

$$f^{\mathcal{A}} = [b, b'] \circ in_l \circ in_f = (b)^f \,,$$
$$g^{\mathcal{A}} = [b, b'] \circ in_r \circ in_g = (b')^g \,.$$

Thus, the commutativity of the diagram in (2) coincides with the satisfiability of (1). ◀

Proof. (of Proposition 16) Recall the isomorphism of categories from Theorem 4

$$\begin{tikzcd} T_{\mathcal{U}}\text{-}\mathbf{Alg} \arrow[r, bend left, "H"] & \mathbb{K}(\Sigma, \mathcal{U}) \arrow[l, bend left, "K"] \end{tikzcd}$$

mapping morphisms essentially to themselves and on objects acting as follows: for $(A, a) \in T_{\mathcal{U}}\text{-}\mathbf{Alg}$ and $(B, b) \in \mathbb{K}(\Sigma, \mathcal{U})$,

$$H(A, a) = (A, a \circ \psi^{\mathcal{U}}_A \circ \Sigma\eta^{\mathcal{U}}_A) \,, \qquad K(B, b) = (B, b_\flat) \,,$$

where $\beta_\flat\colon T_{\mathcal{U}} B \to B$ is the unique map that, by Theorem 3, satisfies the equations $b_\flat \circ \eta^{\mathcal{U}}_B = id_B$ and $b_\flat \circ \psi^{\mathcal{U}}_B = b \circ \Sigma b_\flat$. (for the details on the proof *cf.* [1, Theorem 4.2]).

Next we show that the obvious point-wise extension of the above functors on the categories of bialgebras $(\mathcal{U} \otimes \mathcal{U}')$-**biAlg** and $(T_{\mathcal{U}} \otimes T_{\mathcal{U}'})$-**biAlg** is an isomorphism of categories.

Clearly, since H and K are inverse with each other, so are their point-wise extensions. The only thing we are left to prove is that they are well defined; in particular that the respective commutative conditions are preserved.

Let $(A, a, b) \in (T_{\mathcal{U}} \otimes T_{\mathcal{U}'})$-**biAlg**. We need to show that condition (2) is satisfied by $(A, a \circ \psi^{\mathcal{U}}_A \circ \Sigma\eta^{\mathcal{U}}_A, b \circ \psi^{\mathcal{U}}_A \circ \Sigma\eta^{\mathcal{U}}_A)$. Let $(A, b)^{\underline{n}} = (A^{\underline{n}}, \bar{b})$. By Corollary 14 and Propositions 12, 13, we have that the bottom square diagram below commutes for all $f\colon n \in \Sigma$ and all $g\colon m \in \Sigma'$, while the top commute by Proposition 38:

$$\begin{tikzcd} (A^{\underline{m}})^{\underline{n}} \arrow[d, "\chi"'] \arrow[dr, bend left, "(b^{\langle g \rangle})^{\underline{n}}"] & \\ (A^{\underline{n}})^{\underline{m}} \arrow[r, "\bar{b}^{\langle g \rangle}"] \arrow[d, "(a^{\langle f \rangle})^{\underline{m}}"'] & A^{\underline{n}} \arrow[d, "a^{\langle f \rangle}"] \\ A^{\underline{m}} \arrow[r, "b^{\langle g \rangle}"] & A \end{tikzcd}$$

Since $a^{\langle f\rangle} = (a \circ \psi^{\mathcal{U}}_A \circ \Sigma\eta^{\mathcal{U}}_A)^f$ and $b^{\langle g\rangle} = (b \circ \psi^{\mathcal{U}}_A \circ \Sigma\eta^{\mathcal{U}}_A)^g$, the above diagram proves that condition (2) holds.

Let $(A, a, b) \in (\mathcal{U} \otimes \mathcal{U}')$-**biAlg**. We need to show that $(A, a_\flat, b_\flat)$ is a $\langle T_{\mathcal{U}}, T_{\mathcal{U}'}\rangle$-bialgebra. By Corollary 14, it is sufficient to prove that the following diagram commutes for all $g \colon m \in \Sigma'$,

$$\begin{tikzcd}
T_{\mathcal{U}} A^{\underline{m}} \arrow[r, "\overline{a_\flat}"] \arrow[d, "T_{\mathcal{U}}(b_\flat^{\langle g\rangle})"'] & A^{\underline{m}} \arrow[d, "b_\flat^{\langle g\rangle}"] \\
T_{\mathcal{U}} A \arrow[r, "a_\flat"] & A
\end{tikzcd} \tag{3}$$

where $(A, a_\flat)^{\underline{m}} = (A^{\underline{m}}, \overline{a_\flat})$.

Toward proving (3), first notice that the diagram below commutes for all $f \colon n \in \Sigma$ and $g \colon m \in \Sigma'$

$$\begin{tikzcd}
(A^{\underline{n}})^{\underline{m}} \arrow[rr, "\bar{a}^f"] \arrow[dr, bend left, "\chi^{-1}"] \arrow[dd, "(b^g)^{\underline{n}}"'] & & A^{\underline{m}} \arrow[dd, "b^g"] \\
& (A^{\underline{m}})^{\underline{n}} \arrow[ul, bend left, "\chi"] \arrow[ur, "(a^f)^{\underline{m}}"'] \arrow[dl, phantom, "(2)"] & \\
A^{\underline{n}} \arrow[rr, "a^f"] & & A
\end{tikzcd} \tag{4}$$

for $(A, a)^{\underline{m}} = (A^{\underline{m}}, \bar{a})$ and $(A, b)^{\underline{n}} = (A^{\underline{n}}, \bar{b})$. Indeed, the bottom commutes because (A, a, b) satisfies (2), and the top triangle does by Proposition 38.

Going back to proving (3), by Theorem 3, it suffices to show that both $b_\flat^{\langle g\rangle} \circ \overline{a_\flat}$ and $a_\flat \circ T_{\mathcal{U}}(b_\flat^{\langle g\rangle})$ are the (unique) homomorphic extension of a along $b_\flat^g$. This is shown by the following diagrams

$$\begin{tikzcd}
A^{\underline{m}} \arrow[r, "\eta^{\mathcal{U}}"] \arrow[d, "b_\flat^{\langle g\rangle}"'] & T_{\mathcal{U}} A^{\underline{m}} \arrow[d, "T_{\mathcal{U}} b_\flat^{\langle g\rangle}"] \\
A \arrow[r, "\eta^{\mathcal{U}}"] \arrow[dr, "id"'] & T_{\mathcal{U}} A \arrow[d, "a_\flat"] \\
& A
\end{tikzcd}
\qquad
\begin{tikzcd}
A^{\underline{m}} \arrow[r, "\eta^{\mathcal{U}}"] \arrow[dr, "id"'] \arrow[ddr, bend right, "b_\flat^{\langle g\rangle}"'] & T_{\mathcal{U}} A^{\underline{m}} \arrow[d, "\overline{a_\flat}"] \\
& A^{\underline{m}} \arrow[d, "b_\flat^{\langle g\rangle}"] \\
& A
\end{tikzcd}$$

$$\begin{tikzcd}
T_{\mathcal{U}} A^{\underline{m}} \arrow[d, "T_{\mathcal{U}} b_\flat^{\langle g\rangle}"'] & \Sigma T_{\mathcal{U}} A^{\underline{m}} \arrow[l, "\psi^{\mathcal{U}}"'] \arrow[d, "\Sigma T_{\mathcal{U}} b_\flat^{\langle g\rangle}"] \\
T_{\mathcal{U}} A \arrow[d, "a_\flat"'] & \Sigma T_{\mathcal{U}} A \arrow[l, "\psi^{\mathcal{U}}"'] \arrow[d, "\Sigma a_\flat"] \\
A & \Sigma A \arrow[l, "a"']
\end{tikzcd}$$

$$\begin{tikzcd}
& T_{\mathcal{U}} A^{\underline{m}} \arrow[d, "\overline{a_\flat}"'] & \Sigma T_{\mathcal{U}} A^{\underline{m}} \arrow[l, "\psi^{\mathcal{U}}"'] \arrow[d, "\Sigma\overline{a_\flat}"] & \\
\Sigma' T_{\mathcal{U}'} A \arrow[d, "id"] & A^{\underline{m}} \arrow[d, "in_g"] & \Sigma A^{\underline{m}} \arrow[l, "\bar{a}"'] \arrow[d, "\Sigma in_g"'] & \Sigma\Sigma' T_{\mathcal{U}'} A \arrow[d, "id"] \\
\Sigma' T_{\mathcal{U}'} A \arrow[d, "\psi^{\mathcal{U}'}"] & \Sigma' A \arrow[l, "\Sigma' b_\flat"] \arrow[ul, "\Sigma' \eta^{\mathcal{U}'}"'] \arrow[d, "b"] & \Sigma\Sigma' A \arrow[r, "\Sigma\Sigma' b_\flat"'] \arrow[ur, "\Sigma\Sigma' \eta^{\mathcal{U}'}"] \arrow[d, "\Sigma b"] & \Sigma\Sigma' T_{\mathcal{U}'} A \arrow[d, "\psi^{\mathcal{U}'}"] \\
T_{\mathcal{U}'} A \arrow[r, "b_\flat"] & A & \Sigma A \arrow[l, "a"'] & \Sigma T_{\mathcal{U}'} \arrow[l, "\Sigma b_\flat"']
\end{tikzcd}$$

that commute by definitions of $a_\flat$, $b_\flat$, $b_\flat^{\langle g\rangle}$, b^g; by naturality of $\eta^{\mathcal{U}}$, $\psi^{\mathcal{U}}$; since $(A, a_\flat)^{\underline{m}}$ is a EM $T_{\mathcal{U}}$-algebra; because by Theorem 4 $(A, a_\flat)^{\underline{m}} = (A, a)^{\underline{m}}$; and since from Propositions 12, 13 and (4) we have that b^g is a Σ-homomorphism. ◀

Proof. (of Theorem 18) The tensor $\mathcal{U} \otimes \mathcal{U}'$ of continuous theories is also continuous, so that, by [1, Theorem 3.4], the free monad on it in **CMet** is $\mathbb{C}T_{\mathcal{U}\otimes\mathcal{U}'}$. Moreover, by exploiting the universal property of [1, Theorem 3.4], we can refactor the proofs of Propositions 15 and 16 to obtain the isomorphism $\mathbb{CK}(\Sigma + \Sigma', \mathcal{U} \otimes \mathcal{U}') \cong (\mathbb{C}T_{\mathcal{U}} \otimes \mathbb{C}T_{\mathcal{U}'})$-**biAlg**. Thus, by definition of tensor of monads, $\mathbb{C}T_{\mathcal{U}\otimes\mathcal{U}'} \cong \mathbb{C}T_{\mathcal{U}} \otimes \mathbb{C}T_{\mathcal{U}'}$. ◀

C.1 Quantitative Reader Algebras

For any $X \in \mathbf{Met}$, we define the quantitative $\Sigma_{\mathcal{R}}$-algebra $(X^{\underline{E}}, \rho_X)$ as follows, for arbitrary maps $f_1, \ldots, f_n \colon \underline{E} \to X$

$$\rho_X \colon \Sigma_{\mathcal{R}} X^{\underline{E}} \to X^{\underline{E}} \qquad\qquad \rho_X(in_{\mathsf{r}}(f_1, \ldots, f_n))(e_i) = f_i(e_i)\,.$$

This quantitative algebra satisfies the quantitative theory $\mathcal{R}$ of reading computations.

▶ **Proposition 40.** $(X^{\underline{E}}, \rho_X) \in \mathbb{K}(\Sigma_{\mathcal{R}}, \mathcal{R})$.

Proof. Let $\mathsf{r}^\rho = \rho_X \circ in_{\mathsf{r}}$ denote the interpretation of the operator symbol $\mathsf{r} \colon n \in \Sigma_{\mathcal{R}}$ in the algebra $(X^{\underline{E}}, \rho_X)$. Soundness for the axiom of non-expansiveness (r-NE) follows by the fact that ρ_X is a well defined map in **Met** as shown below

$$\begin{aligned}
&d_{X^{\underline{E}}}(\mathsf{r}^\rho(f_1, \ldots, f_n), \mathsf{r}^\rho(g_1, \ldots, g_n)) \\
&\qquad = \sup_{e_i} d_X(\mathsf{r}^\rho(f_1, \ldots, f_n)(e_i), \mathsf{r}^\rho(g_1, \ldots, g_n)(e_i)) \\
&\qquad = \sup_{e_i} d_X(f_i(e_i), g_i(e_i)) \\
&\qquad \leq \max_j \Big(\sup_{e_i \in E} d_X(f_j(e_i), g_j(e_i)) \Big) \\
&\qquad \leq \max_j d_{X^{\underline{E}}}(f_j, g_j)\,.
\end{aligned}$$

We are left to show that the algebra $(X^{\underline{E}}, \rho_X)$ satisfies the axioms (Idem) and (Diag). Soundness for (Idem) follows by definition of ρ as, for all $e_i \in E$

$$\mathsf{r}^\rho(f, \ldots, f)(e_i) = f(e_i)\,.$$

Soundness for (Diag) also follows by definition, as

$$\begin{aligned}
&\mathsf{r}^\rho(\mathsf{r}^\rho(f_{1,1}, \ldots, f_{1,n})), \ldots, \mathsf{r}^\rho(f_{n,1}, \ldots, f_{n,n}))(e_i) \\
&\quad = \mathsf{r}^\rho(f_{i,1}, \ldots, f_{i,n})(e_i) \\
&\quad = f_{i,i}(e_i) \\
&\quad = \mathsf{r}^\rho(f_{1,1}, \ldots, f_{n,n})(e_i)\,.
\end{aligned}$$

◀

Moreover, it is universal in the following sense:

▶ **Theorem 41.** *For any $\Sigma_{\mathcal{R}}$-algebra (A, a) satisfying $\mathcal{R}$ and non-expansive map $\beta \colon X \to A$, there exists a unique homomorphism $h \colon X^{\underline{E}} \to A$ of quantitative $\Sigma_{\mathcal{R}}$-algebras making the diagram below commute*

$$\begin{tikzcd}
X \arrow[r, "\kappa_X"] \arrow[dr, "\beta"'] & X^{\underline{E}} \arrow[d, dashed, "h"] & \Sigma_{\mathcal{R}} X^{\underline{E}} \arrow[l, "\rho_X"'] \arrow[d, "\Sigma_{\mathcal{R}} h"] \\
& A & \Sigma_{\mathcal{R}} A \arrow[l, "a"]
\end{tikzcd}$$

Proof. Let (A, a) be a quantitative $\Sigma_{\mathcal{R}}$-algebra satisfying $\mathcal{R}$ and $\beta\colon X \to A$ a non-expansive map. We define $h\colon X^{\underline{E}} \to A$ as follows, for arbitrary $f\colon \underline{E} \to X$

$$h(f) = a(in_{\mathsf{r}}(\beta(f(e_1)), \dots, \beta(f(e_n)))) .$$

As it is defined as the composition of non-expansive maps, then also h is non-expansive. Next we prove the commutativity of the diagram, that is, $h \circ \kappa_X = \beta$ and $h \circ \rho_X = a \circ \Sigma_{\mathcal{R}} h$.

Let $\mathsf{r}^\rho = \rho_X \circ in_{\mathsf{r}}$ and $\mathsf{r}^a = a \circ in_{\mathsf{r}}$ denote the interpretations of $\mathsf{r}\colon n \in \Sigma_{\mathcal{R}}$ in the algebras $(X^{\underline{E}}, \rho_X)$ and (A, a), respectively. Let $x \in X$. Then

$$\begin{aligned}
&(h \circ \kappa_X)(x) \\
&\quad = \mathsf{r}^a(\beta(\kappa_X(x)(e_1)), \dots, \beta(\kappa_X(x)(e_n))) && \text{(def. } h\text{)} \\
&\quad = \mathsf{r}^a(\beta(x), \dots, \beta(x)) && \text{(def. } \kappa\text{)} \\
&\quad = \beta(x) . && \text{(Idem)}
\end{aligned}$$

Let $f_1, \dots, f_n\colon \underline{E} \to X$. Then

$$\begin{aligned}
&(h \circ \rho_X)(in_{\mathsf{r}}(f_1, \dots, f_n)) \\
&\quad = \mathsf{r}^a(\beta(f_1(e_1)), \dots, \beta(f_n(e_n))) && \text{(def. } h \text{ and } \rho\text{)} \\
&\quad = \mathsf{r}^a\Big(\mathsf{r}^a(\beta(f_1(e_1)), \dots, \beta(f_1(e_n))), \dots \\
&\qquad\qquad \dots, \mathsf{r}^a(\beta(f_n(e_1)), \dots, \beta(f_n(e_1)))\Big) && \text{(Diag)} \\
&\quad = \mathsf{r}^a(h(f_1), \dots, h(f_n)) && \text{(def. } h\text{)} \\
&\quad = (a \circ \Sigma_{\mathcal{R}} h)(in_{\mathsf{r}}(f_1, \dots, f_n)) . && \text{(def. } \mathsf{r}^a \text{ and } \Sigma_{\mathcal{R}}\text{)}
\end{aligned}$$

Hence h is a $\Sigma_{\mathcal{R}}$-homomorphism.

It remains to prove the uniqueness of such a homomorphism. Assume there exists $g\colon X^{\underline{E}} \to A$ such that $g \circ \kappa_X = \beta$ and $g \circ \rho_X = a \circ \Sigma_{\mathcal{R}} g$. Next we prove $h = g$. Notice first that for any $f\colon X^{\underline{E}} \to X$, $f = \mathsf{r}^\rho(\kappa_X(f(e_1)), \dots, \kappa_X(f(e_n)))$, as for all $e_i \in E$, the following holds:

$$\begin{aligned}
f(e_i) &= \kappa_X(f(e_i))(e_i) && \text{(def. } \kappa\text{)} \\
&= \mathsf{r}^\rho(\kappa_X(f(e_1)), \dots, \kappa_X(f(e_n)))(e_i) . && \text{(def. } \rho\text{)}
\end{aligned}$$

From the above we have that, for all $f\colon X^{\underline{E}} \to X$,

$$\begin{aligned}
h(f) &= h(\mathsf{r}^\rho(\kappa_X(f(e_1)), \dots, \kappa_X(f(e_n)))) \\
&= \mathsf{r}^a((h \circ \kappa)(f(e_1)), \dots, (h \circ \kappa)(f(e_1))) && (h \text{ homo}) \\
&= \mathsf{r}^a(\beta(f(e_1)), \dots, \beta(f(e_1))) && (h \circ \kappa = \beta) \\
&= \mathsf{r}^a((g \circ \kappa)(f(e_1)), \dots, (g \circ \kappa)(f(e_1))) && (g \circ \kappa = \beta) \\
&= g(\mathsf{r}^\rho(\kappa_X(f(e_1)), \dots, \kappa_X(f(e_n)))) && (g \text{ homo}) \\
&= g(f)
\end{aligned}$$

Therefore, $g = h$. ◀

Proof. (of Theorem 21) By Theorem 41, the functors $(-)^{\underline{E}}$ and $T_{\mathcal{R}}$ are isomorphic and the units of the two monads coincide (up-to iso). We are left to prove that also the multiplications coincide (up-to iso). By Theorem 41, this follows by showing that the following diagram commutes

$$\begin{tikzcd}
X^{\underline{E}} \arrow[r, "\kappa_{X^{\underline{E}}}"] \arrow[dr, "id"'] & (X^{\underline{E}})^{\underline{E}} \arrow[d, dashed, "\Delta_X"] & \Sigma_{\mathcal{R}}(X^{\underline{E}})^{\underline{E}} \arrow[l, "\rho_{X^{\underline{E}}}"'] \arrow[d, "\Sigma_{\mathcal{R}}\Delta_X"] \\
& X^{\underline{E}} & \Sigma_{\mathcal{R}} X^{\underline{E}} \arrow[l, "\rho_X"]
\end{tikzcd}$$

$\Delta_X \circ \kappa_X = id$ holds since $(-)^{\underline{E}}$ is a monad. Finally, the right square diagram commutes as shown below

$$\begin{aligned}
&(\Delta_X \circ \rho_{X^{\underline{E}}})(in_{\mathsf{r}}(F_1, \dots, F_n))(e_i) \\
&\quad = \rho_{X^{\underline{E}}}(in_{\mathsf{r}}(F_1, \dots, F_n))(e_i)(e_i) && (\text{def. } \Delta) \\
&\quad = F_i(e_i)(e_i) && (\text{def. } \rho) \\
&\quad = \Delta_X(F_i)(e_i) && (\text{def. } \Delta_X) \\
&\quad = \rho_X(in_{\mathsf{r}}(\Delta_X(F_1), \dots, \Delta_X(F_n)))(e_i) && (\text{def. } \rho) \\
&\quad = (\rho_X \circ \Sigma_{\mathcal{R}}\Delta_X)(in_{\mathsf{r}}(F_1, \dots, F_n))(e_i) && (\text{def. } \Sigma_{\mathcal{R}})
\end{aligned}$$

for arbitrary $F_1, \dots, F_n \colon \underline{E} \to X^{\underline{E}}$. ◀

Proof. (of Theorem 22) Recall that the composite monad $(T-)^{\underline{E}}$ is the monad that arises from the adjunction with the forgetful functor λ-**biAlg** $\to$ **Met**, where λ-**biAlg** denotes the full subcategory of EM $\langle T, (-)^{\underline{E}} \rangle$-bialgebras (A, a, b) satisfying the commutativity of the diagram

$$\begin{tikzcd}
TA \arrow[r, "a"] & A & A^{\underline{E}} \arrow[l, "b"'] \\
T(A^{\underline{E}}) \arrow[u, "Tb"] \arrow[rr, "\lambda"] & & (TA)^{\underline{E}} \arrow[u, "(a)^{\underline{E}}"']
\end{tikzcd} \tag{5}$$

The bialgebras satisfying (5) are called, λ-bialgebras for the law $\lambda \colon T(-^{\underline{E}}) \Rightarrow (T-)^{\underline{E}}$ (see e.g., [2]). We show that the category of λ-bialgebras is identical to the category of commutative $\langle T \otimes (-)^{\underline{E}} \rangle$-bialgebras, that is, that the commutativity of the diagram above corresponds to either one of the equivalent conditions from Proposition 8.

One direction is easy, as if we assume (A, a, b) to be a commutative $\langle T \otimes (-)^{\underline{E}} \rangle$-bialgebra, then (5) is just the instantiation of (2) from Proposition 8 for $h = id \in \mathcal{O}_{(-)^{\underline{E}}}(\underline{E})$ as, by definition of lifting, $(A, a)^{\underline{E}} = (A^{\underline{E}}, (a)^{\underline{E}} \circ \lambda_A)$.

For the converse direction, assume (5) holds and let $g \in \mathcal{O}_T(v)$, for some $v \in$ **Met**. Then, asking that a^g is a $(-)^{\underline{E}}$-homomorphism (*i.e.*, condition (1) from Proposition 8) corresponds to the commutativity of the following diagram, as $(A, b)^v = (A^v, b^v \circ \sigma_A)$ and $(A, a)^{\underline{E}} = (A^{\underline{E}}, (a)^{\underline{E}} \circ \lambda_A)$:

$$\begin{tikzcd}
(A^v)^{\underline{E}} \arrow[r, "\sigma"] \arrow[d, "g^{\underline{E}}"'] & (A^{\underline{E}})^v \arrow[r, "b^v"] \arrow[d, "g"'] & A^v \arrow[d, "g"'] \\
(TA)^{\underline{E}} \arrow[dr, bend right, "a^{\underline{E}}"'] & T(A^{\underline{E}}) \arrow[r, "Tb"] \arrow[d, "(a)^{\underline{E}} \circ \lambda"'] & TA \arrow[d, "a"] \\
& A^{\underline{E}} \arrow[r, "b"] & A
\end{tikzcd}$$

The bottom-left square is (5), so commutes by hypothesis; the top-right square commutes by naturality of g; and finally, the left square commutes by Proposition 38 as, by definitions of the strengths of $(-)^v$ and $(-)^{\underline{E}}$, $\sigma\colon (A^v)^{\underline{E}} \Rightarrow (A^{\underline{E}})^v$ coincides with the canonical isomorphism (denoted as χ in Proposition 38).

Therefore, as the two categories of bialgebras coincide, by definition of tensor of monads, $T \otimes (-)^{\underline{E}} = (T-)^{\underline{E}}$. ◀

C.2 Quantitative Writer Algebras

For any $X \in \mathbf{Met}$, we define the quantitative $\Sigma_{\mathcal{W}}$-algebra $(\Lambda \square X, \omega_X)$ as follows, for arbitrary $\alpha, \alpha' \in \Lambda$ and $x \in X$

$$\omega_X \colon \Sigma_{\mathcal{W}}(\Lambda \square X) \to \Lambda \square X\,, \qquad \omega_X(in_{\mathsf{w}_\alpha}(\alpha', x)) = (\alpha * \alpha', x)\,.$$

This quantitative algebra satisfies the quantitative theory $\mathcal{W}$ of writing computations.

▶ **Proposition 42.** $((\Lambda \square X), \omega_X) \in \mathbb{K}(\Sigma_{\mathcal{W}}, \mathcal{W})$.

Proof. Let $\mathsf{w}^\omega_\alpha = \omega_X \circ in_{\mathsf{w}_\alpha}$ denote the interpretation of the operation $\mathsf{w}_\alpha \colon 1 \in \Sigma_{\mathcal{W}}$ in the algebra $(\Lambda \square X, \omega_X)$. Proving the soundness for (w_α-NE), for each $\alpha \in \Lambda$, is equivalent to show that the map ω is well-defined in $\mathbf{Met}$. This follows as

$$\begin{aligned}
& d_{(\Lambda \square X)}(\mathsf{w}^\omega_\alpha(\beta, x), \mathsf{w}^\omega_\alpha(\beta', x')) \\
&= d_{(\Lambda \square X)}((\alpha * \beta, x), (\alpha * \beta', x')) && \text{(def. } \omega) \\
&= d_\Lambda(\alpha * \beta, \alpha * \beta') + d_X(x, x') && \text{(def. } \square) \\
&\leq \max\left\{ d_\Lambda(\alpha, \alpha), d_\Lambda(\beta, \beta') \right\} + d_X(x, x') && (* \text{ non-exp}) \\
&= d_\Lambda(\beta, \beta') + d_X(x, x') && \text{(metric)} \\
&= d_{(\Lambda \square X)}((\beta, x), (\beta', x'))\,. && \text{(def. } \square)
\end{aligned}$$

We are missing to prove that the algebra $((\Lambda \square X), \omega_X)$ satisfies the axioms (Zero), (Mult), and (Diff). The first one holds trivially as $(\alpha, x) = (0 * \alpha, x)$ because 0 is the identity element of the monoid Λ of output values. The soundness of (Mult) follows by definition of ω as

$$\mathsf{w}^\omega_\alpha(\mathsf{w}^\omega_{\alpha'}(\beta, x)) = \mathsf{w}^\omega_\alpha((\alpha' * \beta, x))\,.$$

Finally, soundness for (Diff) follows by

$$\begin{aligned}
& d_{(\Lambda \square X)}(\mathsf{w}^\omega_\alpha(\beta, x), \mathsf{w}^\omega_{\alpha'}(\beta', x')) \\
&= d_\Lambda(\alpha * \beta, \alpha' * \beta') + d_X(x, x') && \text{(def. } \omega \text{ \& } \square) \\
&= d_\Lambda(\alpha * \beta, \alpha * \beta') + d_\Lambda(\alpha * \beta', \alpha' * \beta') + d_X(x, x') && \text{(triang. ineq.)} \\
&\leq d_\Lambda(\beta, \beta') + d_\Lambda(\alpha, \alpha') + d_X(x, x') && (* \text{ non-exp}) \\
&\geq d_\Lambda(\alpha, \alpha') + d_{(\Lambda \square X)}((\beta, x), (\beta', x'))\,, && \text{(def. } \square)
\end{aligned}$$

which concludes our proof. ◀

Moreover, the next result says that this algebra is actually the free quantitative $\Sigma_{\mathcal{W}}$-algebra on X in $\mathbb{K}(\Sigma_{\mathcal{W}}, \mathcal{W})$.

▶ **Theorem 43.** *For any $\Sigma_{\mathcal{W}}$-algebra (A, a) satisfying $\mathcal{W}$ and non-expansive map $\beta\colon X \to A$, there exists a unique homomorphism $h\colon X^{\underline{E}} \to A$ of quantitative $\Sigma_{\mathcal{W}}$-algebras making the diagram below commute*

$$\begin{tikzcd} X \arrow[r, "\tau_X"] \arrow[dr, "\beta"'] & \Lambda \,\Box\, X \arrow[d, dashed, "h"] & \Sigma_{\mathcal{W}}(\Lambda \,\Box\, X) \arrow[l, "\omega_X"'] \arrow[d, "\Sigma_{\mathcal{W}} h"] \\ & A & \Sigma_{\mathcal{W}} A \arrow[l, "a"] \end{tikzcd}$$

Proof. Let (A, a) be a $\Sigma_{\mathcal{W}}$-algebra satisfying $\mathcal{W}$ and $\beta\colon X \to A$ a non-expansive map, We define the map $h\colon \Lambda \,\Box\, X \to A$ as follows, for arbitrary $\alpha \in \Lambda$ and $x \in X$

$$h((\alpha, x)) = a(in_{\mathsf{w}_\alpha}(\beta(x)))\,.$$

Non-expansiveness of h follows by the fact that (A, a) satisfies the axiom (Diff) as shown below, where $\mathsf{w}^a_\alpha = a \circ in_{\mathsf{w}_\alpha}$ denotes the interpretation of $\mathsf{w}_\alpha\colon 1 \in \Sigma_{\mathcal{W}}$ in (A, a),

$$\begin{aligned} & d_A(h((\alpha, x)), h((\alpha, x))) \\ &\quad = d_A(\mathsf{w}^a_\alpha(\beta(x)), \mathsf{w}^a_{\alpha'}(\beta(x'))) && \text{(def. } h\text{)} \\ &\quad \le d_\Lambda(\alpha, \alpha') + d_A(\beta(x), \beta(x')) && \text{(Diff)} \\ &\quad \le d_\Lambda(\alpha, \alpha') + d_X(x, x') && (\beta \text{ non-exp}) \\ &\quad = d_{\Lambda \Box X}((\alpha, x), (\alpha', x'))\,. && \text{(def. } \Box\text{)} \end{aligned}$$

Next we prove $h \circ \tau_X = \beta$ and $h \circ \omega_X = a \circ \Sigma_{\mathcal{W}} h$.

Let $x \in X$. Then,

$$\begin{aligned} (h \circ \tau_X)(x) &= h((0, x)) && \text{(def. } \tau\text{)} \\ &= \mathsf{w}^a_0(\beta(x)) && \text{(def. } h\text{)} \\ &= \beta(x)\,. && \text{(Zero)} \end{aligned}$$

Let $x \in X$ and $\alpha, \alpha' \in \Lambda$. Then,

$$\begin{aligned} & (h \circ \omega_X)(in_{\mathsf{w}_\alpha}(\alpha', x)) \\ &\quad = \mathsf{w}^a_{\alpha * \alpha'}(\beta(x)) && \text{(def. } h \text{ and } \omega\text{)} \\ &\quad = \mathsf{w}^a_\alpha(\mathsf{w}^a_{\alpha'}(\beta(x))) && \text{(Mult)} \\ &\quad = \mathsf{w}^a_\alpha(h(\alpha', x)) && \text{(def. } h\text{)} \\ &\quad = (a \circ \Sigma_{\mathcal{W}} h)(in_{\mathsf{w}_\alpha}(\alpha', x))\,. && \text{(def. } \mathsf{w}^a_\alpha \text{ and } \Sigma_{\mathcal{W}}\text{)} \end{aligned}$$

Thus, h is a $\Sigma_{\mathcal{W}}$-homomorphism.

It remains to prove uniqueness of h. Notice first that, for any $\alpha \in \Lambda$ and $x \in X$, $(\alpha, x) = \mathsf{w}^\omega_\alpha(\tau(x))$, where $\mathsf{w}^\omega_\alpha = \omega_X \circ in_{\mathsf{w}_\alpha}$ denotes the interpretation of $\mathsf{w}_\alpha\colon 1 \in \Sigma_{\mathcal{W}}$ in $(\Lambda \,\Box\, X, \omega_X)$. Indeed, the following holds

$$\begin{aligned} (\alpha, x) &= (\alpha * 0, x) && \text{(0 identity)} \\ &= \mathsf{w}^\omega_\alpha(0, x) && \text{(def. } \omega\text{)} \\ &= \mathsf{w}^\omega_\alpha(\tau(x))\,. && \text{(def. } \tau\text{)} \end{aligned}$$

Assume there exists $g\colon \Lambda \Box X \to A$ such that $g \circ \tau_X = \beta$ and $g \circ \omega_X = a \circ \Sigma_{\mathcal{W}} g$. Then, the following holds:

$$\begin{aligned}
h((\alpha, x)) &= h(\mathsf{w}^{\omega}_{\alpha}(\tau(x))) \\
&= \mathsf{w}^{a}_{\alpha}(h(\tau(x))) && (h \text{ homo}) \\
&= \mathsf{w}^{a}_{\alpha}(\beta(x)) && (h \circ \tau = \beta) \\
&= \mathsf{w}^{a}_{\alpha}(g(\tau(x))) && (g \circ \tau = \beta) \\
&= g(\mathsf{w}^{\omega}_{\alpha}(\tau(x))) && (g \text{ homo}) \\
&= g((\alpha, x))
\end{aligned}$$

Therefore, $h = g$. ◀

Proof. (of Theorem 26) By Theorem 41, the functors $(\Lambda \Box -)$ and $T_{\mathcal{W}}$ are isomorphic and the units of the two monads coincide (up-to iso). We are left to prove that also the multiplications coincide (up-to iso). By Theorem 41, this follows by showing that the following diagram commutes

$$\begin{tikzcd}
(\Lambda \Box X) \arrow[r, "\tau_{\Lambda \Box X}"] \arrow[dr, "id"'] & (\Lambda \Box (\Lambda \Box X)) \arrow[d, dashed, "\varsigma_X"] & \Sigma_{\mathcal{W}}(\Lambda \Box (\Lambda \Box X)) \arrow[l, "\omega_{\Lambda \Box X}"'] \arrow[d, "\Sigma_{\mathcal{W}} \varsigma_X"] \\
& \Lambda \Box X & \Sigma_{\mathcal{W}}(\Lambda \Box X) \arrow[l, "\omega_X"']
\end{tikzcd}$$

$\varsigma_X \circ \tau_X = id$ holds since $(\Lambda \Box -)$ is a monad. The right square diagram commutes as shown below

$$\begin{aligned}
&(\varsigma_X \circ \omega_{\Lambda \Box X})(in_{\mathsf{w}_\alpha}(\alpha', (\alpha'', x))) \\
&\quad = \varsigma_X((\alpha * \alpha', (\alpha'', x))) && (\text{def. } \omega) \\
&\quad = (\alpha * \alpha' * \alpha'', x) && (\text{def. } \varsigma) \\
&\quad = \omega_X(in_{\mathsf{w}_\alpha}(\alpha' * \alpha'', x)) && (\text{def. } \omega_X) \\
&\quad = \omega_X(in_{\mathsf{w}_\alpha}(\varsigma_X(\alpha', (\alpha'', x)))) && (\text{def. } \varsigma) \\
&\quad = (\omega_X \circ \Sigma_{\mathcal{W}} \varsigma_X)(in_{\mathsf{w}_\alpha}(\alpha', (\alpha'', x))) && (\text{def. } \Sigma_{\mathcal{W}})
\end{aligned}$$

for arbitrary $x \in X$ and $\alpha, \alpha', \alpha'' \in \Lambda$. ◀

Proof. (of Lemma 31) Similar to [1, Lemma 8.4]. ◀