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Scaling Behavior of Semiclassical Gravity

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Abstract

Using the idea of metric scaling we examine the scaling behavior of the stress tensor of a scalar quantum field in curved space-time. The renormalization of the stress tensor results in a departure from naive scaling. We view the process of renormalizing the stress tensor as being equivalent to renormalizing the coupling constants in the Lagrangian for gravity (with terms quadratic in the curvature included). Thus the scaling of the stress tensor is interpreted as a nonnaive scaling of these coupling constants. In particular, we find that the cosmological constant and the gravitational constant approach UV fixed points. The constants associated with the terms which are quadratic in the curvature logarithmically diverge. This suggests that quantum gravity is asymptotically scale invariant.

§(1): *Introduction*

In this paper we discuss the scaling behavior of the stress tensor for a quantum field on a curved background space-time. We first develop a renormalization group equation for the stress tensor and then use the idea of metric scaling [1] to obtain scaling equations.

We shall regard the process of stress-tensor renormalization not as the process of renormalizing operator insertion [2, 3] but as the process of renormalizing the coupling constants in the action functional for semiclassical relativity (see, for example, Parker [4] or Bunch [5]). This view emphasizes the role of the stress tensor as being the source for the geometry in classical relativity.

In order to renormalize the stress tensor of a quantum field theory on a curved background space-time, it is necessary to subtract singular terms that are proportional to geometric quantities. This subtraction can be viewed as renormalizing the coefficients in the Lagrangian for the geometry, although the geometry itself is not being quantized. It is not necessary to add further terms to the

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matter Lagrangian, so that stress tensor renormalization does not affect mass, coupling constant, or wave function renormalization.

There are three interesting new points that arise when renormalizing a stress tensor on a curved space-time that do not exist in the usual flat space-time renormalization program. First, even the free-field stress tensor must be renormalized. This is because in a generic space-time, there is no well-defined notion of normal ordering and the corresponding renormalization must be carried out in detail. Second, the renormalization of the stress tensor is an additive prescription, i.e., divergences in the matter stress tensor are added onto the gravity Lagrangian. Unlike the renormalization of mass, coupling constant, and wave function that arise when renormalizing the Green's functions, the renormalization of the gravity Lagrangian cannot be viewed as multiplicative (i.e., as the bare coupling constants being the renormalized constant times some singular terms). Finally, in the general curved space-time the counterterms needed to renormalize the matter stress tensor add to the geometry Lagrangian not only term proportional to R (and hence appear as a renormalization of Newton's constant) but also terms proportional to $1, R^2, R_{\mu\nu}R^{\mu\nu}$, and $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$. This result is reminiscent of the Green's function renormalization in the $\lambda\phi^4$ theory: There, the flat space-time theory suggested a minimally coupled theory in a curved space-time (i.e., a Lagrangian with covariant derivatives but with no additional couplings to the geometry). However, in renormalizing the theory, it is necessary to add terms proportional to $R\phi^2$ to the Lagrangian [6–9]. Thus the renormalization forces one to consider a more sophisticated Lagrangian for the matter. Likewise, in renormalizing the stress tensor, it is found that the renormalized geometry Lagrangian contains not only the expected Einstein term κR but additional non-Einstein terms.

These three points have quite unexpected consequences for the scaling properties of the stress tensor, and hence for the back-reaction problem at high energies. A consequence of the nontrivial renormalization of the free field implies a nontrivial scaling for its stress tensor. This result is to be contrasted with the case of the Green's function for the free field: It has trivial scaling. Next, because the stress tensor renormalization is additive instead of multiplicative, the renormalization group (RG) equations for the coefficients in the gravity Lagrangian are nonhomogeneous. This fact is in contrast to the RG equations for mass and coupling constants. There, for example, if the theory does not have a mass in it, the theory, when scaled to high energies, still has no effective mass. The case of the stress tensor is just the opposite: If a theory of matter and geometry has no cosmological constant when examined at low energies, nonetheless, the high-energy behavior of the theory may have an effective cosmological constant. Similar results hold for the other non-Einstein terms.

The fact that the stress tensor (and hence the renormalized gravity Lagrangian) have these nontrivial scaling properties has drastic effects on the asymptotic behavior of quantum fields in curved space-time. We have shown elsewhere [1] that curvature may have important effects on a field at high energies, particularly those that are expected in the early universe. The results of this paper indicate that in the early universe not only can the back-reaction not be neglected but that it may be a dominant process. This is seen from our finding that the coupling constants associated with terms which are quadratic in the curvature get very large [10]. These terms arise in the process of renormalizing the stress tensor, i.e., in incorporating the effects of the matter on the geometry and are not in the original gravity action.

The scale parameter α will be interpreted as specifying a minimum "size" of the quantum fluctuations to be explicitly incorporated. Thus our view of the renormalization group is similar to the viewpoint originally espoused in statistical mechanics, through the formalism closely resembles traditional field theoretic formulations of the renormalization group [11].

§(2): *Scaling in Curved Space-Times*

In Minkowski space-time one defines a scale transformation as changing the coordinates by a constant factor:

$$x \to \alpha^{-1} x \tag{1}$$

while keeping the metric $\eta_{\mu\nu}$ fixed. The Cartesian coordinates all have dimensions of length and the transformation (1) is interpreted as changing the length scale of the problem. This transformation, however, relies on the natural vector-space structure of Minkowski space-time: First, linearity of the coordinate system is needed to give (1) a Lorentz-invariant meaning. Second, Cartesian coordinates (each of unit length) are needed to interpret (1) as a scale transformation.

In general curved space-time there is not a preferred set of Cartesian coordinate systems adapted to the metric structure (as there is in Minkowski space-time) in terms of which one can define a change of the length scale. What can be done, however, is to change the length scale by changing the metric tensor $g_{\mu\nu}$ itself. Thus, for any space-time, we say that the coordinates (Cartesian or otherwise) are mere manifold labels. All information about length resides in the metric tensor. In particular, then, all coordinates are dimensionless and each component of the metric tensor $g_{\mu\nu}$ has units of (length).[2] A scale change is then given by [1]

$$\begin{aligned} x^\mu &\to x^\mu \\ g_{\mu\nu} &\to \alpha^{-2} g_{\mu\nu} \end{aligned} \tag{2}$$

The advantage of this approach is that the coordinates have no metrical meaning; the metric does, and hence, this notion of scaling generalizes to curved space-times. A minor disadvantage is that tensor objects sometimes have different units than when coordinates carry the length units. We give a table indicating these differences. (We take units where $\hbar = c = 1$. We use mass units M.) See Table I.

§(3): *The RG Equation for $\langle T_{\mu\nu}\rangle$ in the $\lambda\phi^4$ Theory*

The expectation value of the stress-tensor of the matter field is given by the functional derivative of the generating function with respect to the background metric $g^{\mu\nu}$ [2-4]:

$$\langle T_{\mu\nu}\rangle = \frac{2}{\sqrt{g}}\,\frac{\delta}{\delta g^{\mu\nu}}\int [d\phi]\,\exp\left[i\int \sqrt{g}\;d^n x\;\pounds_B^{(\text{matter})}\right] \tag{3}$$

The matter Lagrangian is the bare one; i.e., it contains the counterterms due to mass and charge renormalization. For example, for the $\lambda\phi^4$ theory

$$\pounds_B^{(\text{matter})} = -\left(\frac{1}{2}\,g^{\mu\nu}\partial_\mu\phi_B\partial_\nu\phi_B + \frac{1}{2}\,m_B\phi_B^2 + \frac{1}{2}\,\xi_B\phi_B^2 + \frac{\lambda_B}{24}\,\phi_B^4\right) \tag{4}$$

However, even after mass, coupling constant, and wave function renormalization, the stress tensor (3) will still be divergent. These divergences, however, will be proportional to purely geometrical quantities, involving the metric, the curvature, and quadratic combinations of the curvature. As such, they can be removed by adding to $\pounds^{(\text{matter})}$ in the generating function the geometry Lagrangian $\pounds^{(\text{geometry})}$ and then renormalizing the coupling constants in $\pounds^{(\text{geometry})}$. The required $\pounds^{(\text{geometry})}$ is not just the usual Einstein Lagrangian κR but includes also a cosmological term Λ and terms quadratic in the curvature. As a consequence, the semiclassical field equations that give the effect of the matter on the geometry are a generalization of Einstein's equations, involving terms of second order in the curvature. Because these geometric constants have been renormalized, the consistency of the semiclassical field equations

$$\langle T_\mu{}^\nu\rangle = G_\mu{}^\nu \tag{5}$$

(in which the left-hand side has been calculated neglecting the effects of the matter on the geometry) is again an important question. We will study the effective coupling of the matter back to the geometry in the limits of high energies using the techniques of the renormalization group. Because the stress tensor is additively renormalized (instead of multiplicatively renormalized, as are the Green's

Table I. Units of Basic Quantities

		When $[x] = M^{-1}$	When $[g_{\mu\nu}] = M^{-2}$
Coordinate variables	x^μ	M^{-1}	1
	$\partial/\partial x^\mu$	M^1	1
	$d^n x$	M^n	1
Geometrical quantities and coupling constants	$g_{\mu\nu}$	1	M^{-2}
	$g^{\mu\nu}$	1	M^2
	$(-\det g_{\mu\nu})^{1/2}$	1	M^{-n}
	$R^\alpha{}_{\beta\gamma\delta}$	M^2	1
	$R_{\beta\delta}$	M^2	1
	R	M^2	M^2
	R^2	M^4	M^4
	$R_{\alpha\beta}R^{\alpha\beta}$	M^4	M^4
	$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$	M^4	M^4
	Λ	M^4	M^4
	κ	M^2	M^2
	a	1	1
	b	1	1
	c	1	1
Scalar field and coupling constants	ϕ	M	M
	$\partial_\mu \phi$	M^2	M
	j_μ	M^3	M^2
	j^μ	M^3	1
	λ	1	1
	m	M	M
	ξ	1	1
Dirac field and couplings	ψ	$M^{3/2}$	$M^{3/2}$
	γ^μ	1	M
	γ_μ	1	M^{-1}
	j_μ	M^3	M^2
	j^μ	M^3	1
	e	1	1
	m	M	M
Vector field and coupling constants	A_μ	M	1
	A^μ	M	M^2
	$F_{\mu\nu}$	M^2	1
	e	1	1
Lagrangian, actions and stress tensors	$\pounds$	M^4	M^4
	$S = \int d^{nx} \sqrt{g}\ \pounds$	1	1
	$T_\mu{}^\nu$	M^4	M^4
	$T_{\mu\nu}$	M^4	M^2
	$T^{\mu\nu}$	M^4	M^6

function) the renormalization group equations will have some interesting features not usually found in studying the scaling behavior of Green's functions. Most importantly, these RG equations will be nonhomogeneous. Consequently (as will be shown below) this will mean that if a theory has no observed (renormalized) coupling, say, to R^2, at low energies, at high energies such a coupling may appear and become important. Thus, for example, although the present observed cosmological constant seems to be zero, at high energies (such as in the early universe), the matter may couple to the geometry in such a way as to have a large effective cosmological constant.

We begin with the bare matter Lagrangian $\pounds_B{}^{(\text{matter})}$. To this we add the observed gravity Lagrangian $\pounds_R{}^{(\text{gravity})}$:

$$\pounds_R{}^{(\text{gravity})} = \Lambda_R + \kappa_R R + a_R \mathcal{G} + b_R \mathcal{H} + c_R \mathcal{I} \tag{6}$$

We have anticipated the renormalization of $\langle T_\mu{}^\nu \rangle$ and have introduced the last three terms quadratic in the curvature. We use the particular combinations used, e.g., by Collins and Brown [3]. $\mathcal{G}$ is the Gauss-Bonnet form in 4-dimensions,

$$\mathcal{G} = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - 4 R_{\alpha\beta} R^{\delta\beta} + R^2 \tag{7}$$

and $\mathcal{H}$ is the square of the Weyl tensor in n-dimensions,

$$\mathcal{H} = {}^{(n)}C_{\alpha\beta\gamma\delta} {}^{(n)}C^{\delta\beta\gamma\delta} \tag{8}$$

Since we will be studying scaling behavior, it is natural to split off $\mathcal{H}$. $\mathcal{I}$ is simply R^2. As indicated by the $d^n x$, we will use dimensional regularization.

Now, in order to renormalize $T_\mu{}^\nu$ we must add counterterms to the action. These terms will be purely geometrical and can be interpreted as renormalizing the gravitational coupling constants. Thus, in $\pounds^{(\text{geometry})}$ we will have terms, for example, such as

$$\Lambda_R + \text{poles} \equiv \Lambda_B$$

and

$$\kappa_R R + (\text{poles})\, R \equiv \kappa_B R$$

We can then write the full generating function as

$$Z = \int [d\phi] \exp\left(i \int d^n x \sqrt{g}\, (\pounds_B{}^{(\text{matter})} + \pounds_B{}^{(\text{geometry})} \right) \tag{9}$$

By using $\pounds_B{}^{(\text{matter})}$ and $\pounds_B{}^{(\text{geometry})}$, we recover the renormalized field equations

$$\langle T_\mu{}^\nu \rangle_R = \mathbb{G}_\mu{}^\nu{}_R \tag{10}$$

from the functional equation $\delta Z / \delta g^{\mu\nu} = 0$. Here $\mathbb{G}_\mu{}^\nu$ is the generalized Einstein tensor obtained by varying the full quadratic geometric action (6)

$$\mathcal{G}_{\mu\nu} \equiv \frac{2}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}} \int d^n x \sqrt{g}\, \mathcal{L}^{(\text{geometry})}$$

$$= \frac{2}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}} \int d^n x \sqrt{g}\, (\Lambda + \kappa R + a\mathcal{G} + b\mathcal{H} + c\mathcal{I})$$

$$\equiv \Lambda g_{\mu\nu} + \kappa G_{\mu\nu} + a\mathcal{G}_{\mu\nu} + b\mathcal{H}_{\mu\nu} + c\mathcal{I}_{\mu\nu} \tag{11}$$

Thus, we define $\mathcal{G}_{\mu\nu}$, $\mathcal{H}_{\mu\nu}$, and $\mathcal{I}_{\mu\nu}$ as the functional derivatives of the $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{I}$ terms with respect to $g^{\mu\nu}$. $\langle T_\mu{}^\nu \rangle$ in (10) is calculated with a fixed background $g_{\mu\nu}$. A study of (10) will show the validity of this assumption.

We now study the scaling behavior of $\langle T_\mu{}^\nu \rangle$. As with the Green's functions, we begin by deriving the dependence of $\langle T_\mu{}^\nu \rangle$ on the renormalization parameter μ. We then eliminate the explicit μ dependence by using a naive scaling equation for $\langle T_\mu{}^\nu \rangle$ [12, 13]. We interpret scaling on a curved space-time to mean scaling of the metric $g_{\mu\nu}$ and arrive at a scaling RG equation for $\langle T_\mu{}^\nu \rangle$. We integrate this equation, and then study the high-energy (short-distance) behavior of the field equations (10).

The renormalized $\langle T_\mu{}^\nu \rangle$ is defined to be

$$\langle T_\mu{}^\nu(x, \lambda_R, m_R, \xi_R) \rangle_R \equiv \lim_{n\to 4} (\langle T_\mu{}^\nu(x; \lambda_B, m_B, \xi_B, \mu, n) \rangle - \text{poles in } (n-4)) \tag{12}$$

The left-hand side is a function of the mass parameter μ that is introduced in dimensional regularization. The pole terms in (12) are all proportional to geometric quantities. On dimensional grounds (explained in detail in Section 4) the poles have the following structure:

$$\begin{aligned}\text{poles in } \langle T_\mu{}^\nu \rangle = {} & \mu^{n-4} m_R^4 Y_\Lambda(\lambda_R)\, \delta_\mu{}^\nu + \mu^{n-4} m_R^2 Y_\kappa(\lambda_R, \xi_R)\, G_\mu{}^\nu \\ & + \mu^{n-4} Y_a(\lambda_R)\, \mathcal{G}_\mu{}^\nu + \mu^{n-4} Y_b(\lambda_R)\, \mathcal{H}_\mu{}^\nu \\ & + \mu^{n-4} Y_c(\lambda_R, \xi_R)\, \mathcal{I}_\mu{}^\nu \end{aligned} \tag{13}$$

The Y's are poles in $(n-4)$, the coefficients of which are functions of λ_R, or λ_R and ξ_R, as indicated. The mass dependence is explicitly displayed. These pole terms can be viewed as renormalizations of the gravitational coupling constants. The fact can be seen from the semiclassical field equations, obtained from the functional equation $\delta Z/\delta g^{\mu\nu} = 0$:

$$\langle T_\mu{}^\nu \rangle_B = \mathcal{G}_\mu{}^\nu{}_B$$

or, in terms of the renormalized quantities,

$$\begin{aligned}\langle T_\mu{}^\nu \rangle_R & \equiv \langle T_\mu{}^\nu \rangle_B - \text{poles} \\ & = (\Lambda_B \delta_\mu{}^\nu + \kappa_B G_\mu{}^\nu + a_B \mathcal{G}_\mu{}^\nu + b_B \mathcal{H}_\mu{}^\nu + C_B \mathcal{I}_\mu{}^\nu) - \text{poles}\end{aligned}$$

We now use the explicit form of the pole terms (13) to write

$$\langle T_\mu{}^\nu\rangle_R = (\Lambda_B - \mu^{n-4} m_R^4 Y_\Lambda)\,\delta_\mu{}^\nu + (\kappa_B - \mu^{n-4} m_R^2 Y_\kappa)\,G_\mu{}^\nu + \cdots \equiv \Lambda_R \delta_\mu{}^\nu + \kappa_R G_\mu{}^\nu + \cdots \tag{14}$$

when $n \to 4$. (The dots indicate the quadratic terms.) Thus, the bare gravitational coupling constants are given by

$$\begin{aligned}
\Lambda_B &= \mu^{n-4}(\Lambda_R + m_R^4 Y_\Lambda)\\
\kappa_B &= \mu^{n-4}(\kappa_R + m_R^2 Y_\kappa)\\
a_B &= \mu^{n-4}(a_R + Y_a)\\
b_B &= \mu^{n-4}(b_R + Y_b)\\
c_B &= \mu^{n-4}(c_R + Y_c)
\end{aligned} \tag{15}$$

We now consider the renormalized parameters as functions of the bare ones, e.g.,

$$m_R = m_R(m_B, \lambda_B, \mu, n)$$

and take $\mu(\partial/\partial\mu)$ of (13), with the bare parameters field fixed. In the usual way, we use the chain rule to express this partial derivative in terms of derivatives of $\langle T_\mu{}^\nu\rangle$ looked at again as a function of the renormalized parameters. In the $n \to 4$ limit, we find that

$$\begin{aligned}
&\left[\mu \frac{\partial}{\partial\mu} + \beta(\lambda_R)\frac{\partial}{\partial\lambda_R} - m_R\gamma_m(\lambda_R)\frac{\partial}{\partial m_R} - \xi_R\gamma_\xi(\lambda_R, \xi_R)\frac{\partial}{\partial\xi_R}\right]\\
&\quad\times \langle T_\mu{}^\nu(\lambda_R, m_R, \xi_{R,\mu})\rangle_R = \mu\frac{\partial}{\partial\mu}\Lambda_R\Big|_{\substack{\text{bare parameters}\\ \text{held fixed}}} \delta_\mu{}^\nu + \mu\frac{\partial}{\partial\mu}\kappa_R\Big|\, G_\mu{}^\nu\\
&\quad + \mu\frac{\partial}{\partial\mu}a_R\Big|\,\mathcal{G}_\mu{}^\nu + \mu\frac{\partial b_R}{\partial\mu}\Big|\,\mathcal{H}_\mu{}^\nu + \mu\frac{\partial}{\partial\mu}c_R\Big|\,\mathcal{I}_\mu{}^\nu\\
&\equiv m_R^4\beta_\Lambda(\lambda_R)\,\delta_\mu{}^\nu + m_R^2\beta_\kappa(\lambda_R, \xi_R)\,G_\mu{}^\nu\\
&\quad + \beta_a(\lambda_R, \xi_R)\,\mathcal{G}_\mu{}^\nu + \beta_b(\lambda_R, \xi_R)\,\mathcal{H}_\mu{}^\nu\\
&\quad + \beta_c(\lambda_R, \xi_R)\,\mathcal{I}_\mu{}^\nu \equiv A_\mu{}^\nu(\lambda_R, m_R, \xi_R)
\end{aligned} \tag{16}$$

where β_Λ is defined by

$$m_R^4\beta_\Lambda(\lambda_R) \equiv -\mu\frac{\partial}{\partial\mu}\Big|_{\text{bare parameter fixed}} \Lambda_R$$

β_Λ is given then in terms of the coefficient of the poles in Y_Λ; the other β's are given similarly. We derive the explicit structure of these β's in Section 4.

$\langle T_\mu{}^\nu \rangle$ is a function of the metric $g_{\mu\nu}$ explicitly through the dependence of the operator $T_\mu{}^\nu$ on $g_{\mu\nu}$ and implicitly through the curved space-time propagators that are used to define the expectation value of $T_\mu{}^\nu$. Thus, as $g_{\mu\nu}$ is scaled by a constant α to $g_{\mu\nu}/\alpha^2$ (i.e., as the length scale is reduced by a factor of α) $\langle T_\mu{}^\nu \rangle$ will change. Since the stress tensor $T_\mu{}^\nu$ has a naive mass dimension of 4, we can write the simple differential scaling for $\langle T_\mu{}^\nu \rangle$ in terms of derivatives with respect to the length scale α and the mass parameters m_R and μ:

$$\left[\alpha \frac{\partial}{\partial \alpha} + \mu \frac{\partial}{\partial \mu} + m_R \frac{\partial}{\partial m_R} - 4\right] \langle T_\mu{}^\nu \rangle_R = 0 \tag{17}$$

(See [1] for details of the derivation of the naive scaling equation on curved space-time.) The naive scaling equation is used to eliminate the μ term in (16) to give the final RG equation for the stress tensor:

$$\left[\alpha \frac{\partial}{\partial \alpha} - \beta(\lambda_R) \frac{\partial}{\partial \lambda_R} + m_R(1 + \gamma_m(\lambda_R)) \frac{\partial}{\partial m_R} + \xi_R \gamma_\xi(\lambda_R, \xi_R) \frac{\partial}{\partial \xi_R} - 4\right] \langle T_\mu{}^\nu(\lambda_R, m_R, \xi_R, \mu, \alpha)\rangle_R = A_\mu{}^\nu(\lambda_R, m_R, \xi_R, \mu, \alpha) \tag{18}$$

This equation is linear, but nonhomogeneous. Nonetheless, it can be integrated by introducing the usual effective parameters $\lambda(\alpha)$, $m(\alpha)$, $\xi(\alpha)$ defined by the differential equations [1, 11-13]

$$\alpha \frac{\partial \lambda(\alpha)}{\partial \alpha} = \beta(\lambda(\alpha)), \qquad \text{with } \lambda(\alpha = 1) = \lambda_R \tag{19a}$$

$$\alpha \frac{\partial m(\alpha)}{\partial \alpha} = -m(\alpha)\,[1 + \gamma_m(\lambda(\alpha))], \qquad \text{with } m(\alpha = 1) = m_R \tag{19b}$$

$$\alpha \frac{\partial \xi(\alpha)}{\partial \alpha} = -\xi(\alpha)\,\gamma_\xi(\lambda(\alpha), \xi(\alpha)), \qquad \text{with } \xi(\alpha = 1) = \xi_R \tag{19c}$$

These equations give λ as a function of α and λ_R:

$$\lambda = \lambda(\alpha, \lambda_R),$$

etc. These relationships can be inverted to give λ_R as a function of α and λ

$$\lambda_R = \lambda_R(\alpha, \lambda)$$

and similarly

$$m_R = m_R(\alpha, \lambda, m)$$

$$\xi_R = \xi_R(\alpha, \lambda, \xi)$$

In terms of these effective parameters, the RG equations for $\langle T_\mu{}^\nu\rangle_R$ is simply

$$\left[\alpha \frac{\partial}{\partial\alpha}\bigg|_{\lambda, m, \xi \text{ held fixed}} - 4\right] \langle T_\mu{}^\nu(\lambda_R(\lambda, \alpha), m_R(m, \lambda, \alpha), \xi_R(\xi, \lambda, \alpha), \alpha)\rangle_R$$
$$= A_\mu{}^\nu(\lambda_R(\lambda, \alpha), m_R(m, \lambda, \alpha), \xi_R(\xi, \lambda, \alpha), \alpha) \quad (20)$$

where again $A_\mu{}^\nu$ represents all the nonhomogeneous terms on the right-hand side of (16). Note that $A_\mu{}^\nu$ depends explicitly on α through its dependence on the metric through $G_\mu{}^\nu$, etc.

We have now derived a scaling equation for $\langle T_\mu{}^\nu\rangle_R$. Next, we derive the scaling of $\mathbb{G}_\mu{}^\nu{}_R$ and study the effective gravitational coupling constants.

§(4): *The Dependence of the Renormalized Gravitational Coupling Constants on μ*

In this section we study the dependence of the renormalized parameters Λ_R, $\kappa_R, \ldots$ on μ. In particular, we will derive expressions for the β functions in (16) in terms of the poles of the bare stress tensor. In the next section we use these results to derive the scaling of the entire geometry term $\mathbb{G}_\mu{}^\nu$ and will study the scaled field equations. Our derivation in this section follows closely that of Collins and Brown [3].

We begin with the cosmological constant terms in $\langle T_\mu{}^\nu\rangle$: $\Lambda\delta_\mu{}^\nu$. Equivalently, we can begin with the cosmological term in £: $\Lambda\sqrt{g}$. Λ will contain counterterms and can be written as

$$\Lambda_B = \Lambda_R + \text{poles}$$

The structure of the pole terms is illuminated by the following observations. First, the physical dimension of Λ (when the space-time dimension n is equal to four) is (mass).[4] (This is the same as the dimension of $T_\mu{}^\nu$ itself.) Thus, the pole terms must have this dimension at $n = 4$. However, the pole terms arise from $£^{(\text{matter})}$, and the only mass parameter there is m. Thus, the poles must be proportional to m^4:

$$\Lambda_B = \Lambda_R + m_R^4 Y_\Lambda$$

where Y_Λ is the sum of pole in $(n - 4)$. For $n \neq 4$, it is convenient to keep the dimension of Λ_R and Y_Λ fixed, although Λ_B will have dimension $(\text{mass})^n$. Thus, we write Λ_B in terms of Λ_R, m_R, Y_Λ, and μ

$$\Lambda_B(n) = \mu^{n-4}(\Lambda_R + m_R^4 Y_\Lambda) \quad (21)$$

Next, we can ask what parameters Y depends on. Since in $£^{(\text{matter})}$, ξ enters only through the term $\xi R\phi^2$, any poles proportional to ξ will also be proportional to R. Thus ξ will appear in the poles of κ and c, but not in Y_Λ. Further, since Y_Λ is dimensionless, if it depended on m, it would be in the combination m/μ. However, in $\pounds^{(m)}$, μ appears only in $\mu^{(n-4)}$ (i.e., as $\ln\mu$) whereas m appears only in powers. Thus there could be no terms like m/μ. Thus we conclude that

$$Y_\Lambda = Y_\Lambda(n; \lambda_R)$$

and that

$$\Lambda_B(n) = \mu^{n-4}(\Lambda_R + m_R^4 Y_\Lambda(n; \lambda_R)) \tag{22}$$

We can now derive the renormalization group equation expressing the dependence of Λ_R on μ.

In order that the singular term Λ_B not appear in the final equation, we choose the bare parameters (along with μ) as the independent variables. Thus

$$\Lambda_R = \Lambda_R(n; \Lambda_B, \lambda_B, m_B, \mu) \tag{23a}$$

$$m_R = m_R(n; \lambda_B, m_B, \mu) \tag{23b}$$

$$\lambda_R = \lambda_R(n; \lambda_B, \mu) \tag{23c}$$

Now we take $\mu(\partial/\partial\mu)$ of both sides of (22), and use the fact that Λ_B and μ are independent to derive the equation

$$-\mu \frac{\partial \Lambda_R}{\partial \mu} = (n-4)\,\Lambda_R + m_R^4 \left[(n-4)\,Y_\Lambda + 4\,\frac{\mu}{m_R}\,\frac{\partial m_R}{\partial \mu}\,Y_\Lambda + \mu\,\frac{\partial \lambda_R}{\partial \mu}\,\frac{\partial Y_\Lambda}{\partial \lambda_R}\right] \tag{24}$$

Since Λ_R is finite, $\mu(\partial\Lambda_R/\partial\mu)$ will be finite. We further assume that it is analytic in n at $n = 4$. We expand both sides of (24) in powers of $(n-4)$ and equate coefficients. The fact that $\mu(\partial\Lambda_R/\partial\mu)$ contains no poles puts restriction on the coefficients of the poles Y_Λ. In particular, we write Y_Λ as

$$Y_\Lambda = \sum_{\nu=1}^{\infty} \frac{e_\nu(\lambda_R)}{(n-4)^\nu} \tag{25}$$

We can now use the RG equations for λ_R and m_R [11-13]:

$$\mu\,\frac{\partial \lambda_R}{\partial \mu} = \lambda_R(n-4) + \beta(\lambda_R) \tag{26a}$$

$$\mu\,\frac{\partial m_R}{\partial \mu} = -m_R\,\gamma_m(\lambda_R) \tag{26b}$$

to determine the structure of the RG equation for Λ_R:

$$\mu\,\frac{\partial \Lambda_R}{\partial \mu} = -(n-4)\,\Lambda_R + m_R^4 \beta_\Lambda \tag{27}$$

where

$$\beta_\Lambda = -\left(e_1 + \lambda_R \frac{de_1}{d\lambda_R}\right) \tag{28}$$

This, then, is one equation determining how the constant Λ_R would change as the unobserved mass parameter μ is changed, with the bare parameters held fixed. The β_Λ derived here is the term that appears in the RG equation for $\langle T_\mu{}^\nu \rangle_R$ (16).

Having derived the RG equation for the effective cosmological constant in detail, we now give a brief account of the RG equations for the gravitational constant κ, and for the coupling constants, a, b, and c which give the strength of the coupling of the stress tensor of matter to the quadratic terms in the curvature.

The usual gravitational coupling constant κ had dimensions $(\text{mass})^{n-2}$. Thus we can write, in analogy to (22),

$$\kappa_B = \mu^{n-4}(\kappa_R + m_R^2 Y_\kappa(\lambda, \xi)) \tag{29}$$

Y_κ may depend on ξ, but only linearly, since κ_B contains those poles in $T_{\mu\nu}{}^{(\text{matter})}$ proportional to R. Thus, we could write

$$Y_\kappa(\lambda, \xi) = Y_\kappa{}^{(1)}(\lambda) + \xi Y_\kappa{}^{(2)}(\lambda) \tag{30}$$

If we write

$$Y_\kappa = \sum_{\nu=1}^{\infty} \frac{f_\nu(\lambda, \xi)}{(n-4)^\nu} \tag{31}$$

then we can write for the RG equation for κ:

$$\mu \frac{\partial \kappa}{\partial \mu} (\kappa_B, \lambda_B, m_B, \xi_B, \mu, n) = -\kappa(n-4) + \beta_\kappa m_R^2 \tag{32}$$

where

$$\beta_\kappa = -\left(f_1 + \lambda_R \frac{df_1}{d\lambda_R}\right) \tag{33}$$

Finally we discuss the scaling behavior of the couplings to the terms quadratic in the curvature. We write

$$a_B = \mu^{n-4}(a_R + Y_a(\lambda_R)) \tag{34a}$$

$$b_B = \mu^{n-4}(b_R + Y_b(\lambda_R)) \tag{34b}$$

$$c_B = \mu^{n-4}(c_R + Y_c(\lambda_R, \xi_R)) \tag{34c}$$

(Y_c can have terms linear and quadratic in ξ_R). The β functions are again given in terms of the residues of the poles of the Y's as

$$\beta_a = -\left(j_1 + \lambda_R \frac{dj_1}{d\lambda_R}\right) \tag{35a}$$

$$\beta_b = -\left(k_1 + \lambda_R \frac{dk_1}{d\lambda_R}\right) \tag{35b}$$

$$\beta_c = -\left(l_1 + \lambda_R \frac{\partial l_1}{\partial \lambda_R}\right) \tag{35c}$$

for

$$Y_a \equiv \sum_{\nu=1}^{\infty} \frac{j(\lambda_R)}{(n-4)^\nu} \tag{36a}$$

$$Y_b \equiv \sum_{\nu=1}^{\infty} \frac{k(\lambda_R)}{(n-4)^\nu} \tag{36b}$$

$$Y_c \equiv \sum_{\nu=1}^{\infty} \frac{l(\lambda_R, \xi_R)}{(n-4)^\nu} \tag{36c}$$

§(5): *The RG Equation for $\mathbb{G}_\mu{}^\nu$ and the Effective Gravitational Coupling Constants*

We now study the scaling behavior of the gravitational coupling constants which enter the semiclassical field equations. We first derive a scaling equation for $\mathbb{G}_\mu{}^\nu$, combine it with the scaling equation for $\langle T_\mu{}^\nu \rangle$ (20), and then integrate the result. Finally, we identify the parameters that enter the scaled semiclassical field equations as the effective coupling constants.

Were we to study the coupling of a quantum field to an external field in flat space-time, we would ask how the high-frequency modes of the quantum current coupled to the high-frequency modes of the external field, as compared to the coupling of the low-frequency modes. For the coupling of the quantum stress tensor to classical gravity, however, we lack the convenient Fourier representation. Still, we do have some notion of scaling for the curved space-time and thus ask how $\langle T_\mu{}^\nu \rangle_R$—when evaluated for the metric $g_{\mu\nu}/\alpha^2$—couples to the geometry of that metric as compared to the coupling for $\alpha = 1$. The answer comes from the scaled semiclassical field equations.

We begin with the dependence of $\mathbb{G}_\mu{}^\nu$ on the renormalization parameter. Although $\mathbb{G}$ has been renormalized, we find it to be essentially independent of μ.

This fact is not surprising, however, in light of the fact that it is really $\langle T_\mu{}^\nu \rangle$ that has been renormalized and hence carries all of the nontrivial scaling information.

Consider, then, the "renormalized" geometry

$$\mathbb{G}_\mu{}^\nu{}_R(\Lambda_R, \ldots) \equiv \lim_{n \to 4} (\mathbb{G}_\mu{}^\nu(\Lambda_B, \ldots) + \text{poles}) \tag{37}$$

Taking $\mu(\partial/\partial\mu)$ of (37) with bare parameters fixed gives

$$\mu \left.\frac{\partial}{\partial\mu}\right|_{\text{bare parameters fixed}} \mathbb{G}_\mu{}^\nu{}_R(\Lambda_R, \ldots \mu) = A_\mu{}^\nu(\lambda_R, \cdots \mu) \tag{38}$$

where $A_\mu{}^\nu$ is the same inhomogeneous term that appears in the RG equation for $\langle T_\mu{}^\nu \rangle_R$ (16) and (20). We can use the chain rule and the definition of the β functions (16) to conclude that

$$\mu \left.\frac{\partial}{\partial\mu}\right|_{\text{renormalized parameters fixed}} \mathbb{G}_\mu{}^\nu(\Lambda_R, \ldots, \mu) = 0$$

This trivial dependence of $\mathbb{G}_\mu{}^\nu$ on μ is a consequence of the external-field nature of $\mathbb{G}_\mu{}^\nu$ and hence its dependence on $\Lambda_R, \ldots$ being multiplicative.

Thus $\mathbb{G}_\mu{}^\nu(\Lambda_R, \ldots)$ scales naively. Taking account of the physical dimensions of $\Lambda_R, \ldots$, we can write the differential scaling equation for $\mathbb{G}_\mu{}^\nu$:

$$\left(\alpha \frac{\partial}{\partial\alpha} + 4\Lambda_R \frac{\partial}{\partial\Lambda_R} + 2\kappa_R \frac{\partial}{\partial\kappa_R} - 4\right) \mathbb{G}_\mu{}^\nu \left(\Lambda_R, \kappa_R, a_R, b_R, c_R; \frac{g_{\mu\nu}}{\alpha^2}\right) = 0 \tag{39}$$

We now investigate the effective parameters that enter the scaled semiclassical field equations. Combining (39) and (20) and integrating the result gives the high-energy (large-α) behavior

$$\langle T_\mu{}^\nu(\Lambda_R, m_R, \xi_R; \alpha)\rangle_R = \mathbb{G}_\mu{}^\nu{}_R(\Lambda_R, \kappa_R, a_R, b_R, c_R; \alpha) \tag{40}$$

is equivalent to

$$\alpha^4 \langle T_\mu{}^\nu(\lambda(\alpha), m(\alpha), \xi(\alpha), \alpha = 1)\rangle_R$$
$$= \alpha^4 \mathbb{G}_\mu{}^\nu(\Lambda_{\text{eff}}(\alpha), \kappa_{\text{eff}}(\alpha), a_{\text{eff}}(\alpha), b_{\text{eff}}(\alpha)\, c_{\text{eff}}(\alpha); \alpha = 1) \tag{41}$$

That is, the field equations at large α are equivalent to a scaled-up version of small-α field equations, only with effective coupling constants. $\lambda(\alpha)$, $m(\alpha)$, and $\xi(\alpha)$ are defined by (19). The effective gravitational couplings are given by

$$\Lambda(\alpha) = \Lambda_R + \int_1^\alpha \frac{d\alpha'}{(\alpha')^5} m_R^4(m, \alpha)\, \beta_\Lambda(\lambda_R(\lambda, \alpha)) \tag{42a}$$

$$\kappa(\alpha) = \kappa_R + \int_1^\alpha \frac{d\alpha'}{(\alpha')^3}\, m_R^2(m, \alpha)\, \beta_\kappa(\lambda_R(\lambda, \alpha), \xi_R(\xi, \lambda, \alpha)) \tag{42b}$$

$$\left.\begin{matrix} a(\alpha) \\ b(\alpha) \\ c(\alpha) \end{matrix}\right\} = \left.\begin{matrix} a_R \\ b_R \\ c_R \end{matrix}\right\} + \int_1^\alpha \frac{d\alpha'}{(\alpha')}\, \beta_{\left\{\begin{smallmatrix} a \\ b \\ c \end{smallmatrix}\right\}} (\lambda_R(\lambda, \alpha)\, \xi_R(\xi, \lambda, \alpha)) \qquad \begin{matrix} (42c) \\ (42d) \\ (42e) \end{matrix}$$

The reason for the different powers of α in these integrals is that the geometric terms also scale. Thus, for example, consider the $G_\mu{}^\nu$ part of $\mathbb{G}_\mu{}^\nu$. It scales with $g_{\mu\nu}$ according to

$$G_\mu{}^\nu(g_{\mu\nu}/\alpha^2) = \alpha^2 G_\mu{}^\nu(g_{\mu\nu})$$

Note that the naive scaling of κ (i.e., α^2), has been factored out of the definition of κ_{eff}. Thus $(\alpha^2 \kappa_{\text{eff}}(\alpha))\, G_\mu{}^\nu(\alpha)$ scales as α^4.

Having defined the effective parameters in the scaled semiclassical field equations, we now evaluate them explicitly for the free scalar field. Note that although

$$\lambda(\alpha) = \lambda_R$$

$$m(\alpha) = \alpha^{-1} m_R$$

$$\xi(\alpha) = \xi_R$$

for the free field, the gravitational constants still have a nontrivial scaling behavior.

§(6): *Scaling Behavior of the Free Field Stress Tensor*

In this section we examine the scaling behavior of the coefficients of the counterterms in the semiclassical gravity Lagrangian for a free neutral, scalar field with a $\xi R\phi^2$ coupling to the curvature scalar. Unlike the case of Green's functions, the stress-tensor insertions require nontrivial renormalization even for a free field. Thus we can study nontrivial scaling behavior without the calculational problems associated with interacting fields. We will report the results of calculations for the interacting field in a subsequent publication.

We use the coefficients Λ, κ, a, b, and c defined in equation (6). There is also a term in R which we have left out since it does not contribute to the stress tensor. (The R term that does appear in the stress tensor arises from varying the quadratic curvature terms in £.) For the free field case these coefficients are well known (see, for example, [5]). The results are

$$\Lambda_{\text{div}} = \frac{1}{4\pi^2(n-4)}\, \frac{m^4}{n(n-2)} \tag{43a}$$

$$\kappa_{\text{div}} = \frac{1}{8\pi^2(n-4)} \frac{(\xi - 1/6)\, m^2}{(n-2)} \tag{43b}$$

$$a_{\text{div}} = \frac{1}{16\pi^2(n-4)} \left(\frac{n-6}{n-3}\right) \frac{1}{720} \tag{43c}$$

$$b_{\text{div}} = \frac{1}{16\pi^2(n-4)} \left(\frac{n-2}{n-3}\right) \frac{1}{240} \tag{43d}$$

$$c_{\text{div}} = \frac{1}{16\pi^2(n-4)} \frac{1}{2} \left(\xi - \frac{1}{6}\right)^2 \tag{43e}$$

It is interesting to note that for the conformally invariant case κ_{div} and c_{div} vanish but the others do not.

We begin by studying the scaling of Λ, the cosmological constant. The appropriate RG equation for this case is equation (42a). However, since we are dealing with the free field we can drop the terms involving λ_R to obtain

$$\Lambda(\alpha) = \Lambda_R + \int_1^\alpha \frac{d\alpha'}{(\alpha')^5}\, m_R^4(m, \alpha)\, \beta_\Lambda \tag{44}$$

For the free scalar field, we have, using (43a) and (28)

$$e_1 = \frac{1}{4\pi^2(n-2)\, n} \tag{45}$$

and hence

$$\beta_\Lambda = -\frac{1}{4\pi^2 n(n-2)} \tag{46}$$

Thus

$$\Lambda(\alpha) = \Lambda_R - \int_1^\alpha \frac{d\alpha'}{\alpha'} (\alpha')^{-4} m_R^4 \frac{1}{4\pi^2(n)(n-2)} \tag{47}$$

We can express m_R as $\alpha m(\alpha)$ in equation (47) to get

$$\Lambda(\alpha) = \Lambda_R - \frac{m^4}{4\pi^2 n(n-2)} \int_1^\alpha \frac{d\alpha'}{\alpha'}$$

$$= \Lambda_R - \frac{m^4(\alpha)}{4\pi^2 n(n-2)} \frac{\ln\alpha}{\alpha^4} \tag{48}$$

Again using $m_R = \alpha m(\alpha)$ and at $n = 4$ we get

$$\Lambda(\alpha) = \Lambda_R - \frac{m_R^4}{32\pi^2\alpha^4} \ln(\alpha) \tag{49}$$

It is interesting to note that if we have a massive field theory ($m_R \neq 0$), then even if $\Lambda_R = 0$ we have an "effective" cosmological constant at $\alpha \neq 1$. However, as $\alpha \to \infty$, $\Lambda_{\text{eff}} \to \Lambda_R$.

Now we turn to the behavior of κ with respect to scaling. The equation for $\kappa(\alpha)$ is (42b)

$$\kappa(\alpha) = \kappa_R + \int_1^\alpha \frac{d\alpha'}{(\alpha')^3} m_R^2(m, \alpha)\, \beta_\kappa(\lambda_R(\lambda, \alpha), \xi_R(\xi, \lambda, \alpha))$$

Using equations (43b) and (33) we get (at $n = 4$)

$$f_1 = \frac{(\xi_R - 1/6)}{16\pi^2} \tag{50}$$

and

$$\beta_\kappa = -\frac{(\xi_R - 1/6)}{16\pi^2} \tag{51}$$

Then (42b) becomes

$$\kappa(\alpha) = \kappa_R + \int_1^\alpha \frac{d\alpha'}{(\alpha')^3} \frac{(\xi_R - 1/6)}{16\pi^2} m_R^2 \tag{52}$$

Using the scaling of m and ξ, we get

$$\kappa(\alpha) = \kappa_R - \frac{(\xi_R - 1/6)\, m_R^2}{16\pi^2 \alpha^2}$$

Again we notice that as $\alpha \to \infty$, $\kappa_{\text{eff}} \to \kappa_R$, i.e., the naive scaling dominates. As in the case of Λ, for $\alpha \neq 1$ there is a deviation from naive scaling. This correction to naive scaling vanishes for a massless theory and also for a massive theory with $\xi_R = 1/6$, the conformally invariant coupling.

To complete the discussion we examine the scaling behavior of a, b, and c. These coefficients are important because they signal the presence of non-Einsteinian terms in the semiclassical Lagrangian. Using equations (6) and (35) we have at $n = 4$

$$Y_a = \frac{-1}{5760\pi^2(n - 4)} \tag{53a}$$

$$Y_b = \frac{1}{1920\pi(n - 4)} \tag{53b}$$

$$Y_c = \frac{(\xi_R - 1/6)^2}{32\pi^2(n - 4)}. \tag{53c}$$

The corresponding β functions (36) are

$$\beta_a = -\frac{1}{5760\pi^2} \tag{54a}$$

$$\beta_b = \frac{1}{1920\pi^2} \tag{54b}$$

$$\beta_c = \frac{(\xi_R - 1/6)^2}{32\pi^2} \tag{54c}$$

Since none of these β functions depends on α we can immediately integrate the corresponding RG equations to get the scaled constants [10]

$$a(\alpha) = a_R + \beta_a \ln\alpha \tag{55a}$$

$$b(\alpha) = b_R + \beta_b \ln\alpha \tag{55b}$$

$$c(\alpha) = c_R + \beta_c \ln\alpha \tag{55c}$$

Unlike the two preceding cases these three constants do not have their asymptotic behavior dominated by the naive scaling of the renormalized constants. Thus even if the renormalized constants vanish we will have effective high-energy values of these constants that are not zero. Even for the conformally coupled field $\beta_c = 0$ but β_a and β_b will not vanish.

§(7): *Conclusions*

We have seen that even the free field stress tensor possesses nontrivial scaling behavior in the high-energy limit. Thus one must take seriously the non-Einsteinian terms when one is calculating the back-reaction of particles created by quantum processes in the early universe.

It would be interesting to obtain the effect of introducing interacting fields on the scaling of the stress tensor components. Clearly the β functions would not have the simple structure that they have in the free field case. Collins and Brown [3] have shown that the β functions of the non-Einsteinian terms are not affected until fifth order in the coupling constant for conformally invariant fields. However, an examination of second-order diagrams in massive $\lambda\phi^4$ theory indicates that one would get modifications to β_Λ and β_κ at that order [6].

The significance of these results may be seen in two ways. First, we note that if κ_R and Λ_R are set to zero we get nonzero effective values of κ and Λ for $\alpha \neq 1$. This is basically saying that if one changes the "cutoff" on the size of quantum fluctuations one sees an induced κ and Λ. This lends support to the induced gravity scheme advocated by Adler [14].

Secondly we observe that a, b, c get very large as α goes to infinity. In the functional integral approach to quantum gravity this would imply that fluctuations in the quadratic terms are strongly suppressed at high energies. Thus one would expect that a perturbative treatment of these tersm would be a good approximation at high energies. In other words the higher loop quantum corrections are small compared to the semiclassical terms. In this approximation if one studies the graviton propagator there will be $1/\kappa^2$ terms coming from R and $1/\kappa^4$ terms coming from terms quadratic in the curvature. At high energies the propagator will look like $1/\kappa^4$ since κ^2 terms will be small compared to κ^4 terms. Thus the theory will be approximately scale invariant at high energies. Smolin [10] has obtained similar results for gravity interacting with spinor fields, though he is only able to show that the coefficient b diverges logarithmically. Our results may be viewed as an extension of Smolin's and as support for his view that "the Planck scale is a crossover point between the dynamics at longer distances described by the Einstein Lagrangian and the dynamics at shorter distances which will be described by an effective Lagrangian" [10].

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