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One-loop renormalization of quantum electrodynamics in curved spacetime

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In this paper we discuss the renormalizability of quantum electrodynamics (QED) in a general curved spacetime. A generating functional is introduced and position-space Feynman rules are obtained. Functional techniques are used to show that a form of Ward's identity can be derived in curved spacetime. A local momentum representation for the scalar and vector propagators is introduced. The one-loop diagrams for the electron and photon self-energy are computed and it is shown that there are no divergences that are not present in flat space. It is shown that this latter result depends crucially on the gauge invariance of the theory and is not merely a trivial consequence of renormalizability of QED in flat spacetime.

I. INTRODUCTION

In this paper we shall consider the renormalization of one-loop diagrams in quantum electrodynamics (QED) in an arbitrary curved background. This problem has been discussed by Drummond and Shore[1] for massless QED in a spherically symmetric spacetime. They used the symmetry to write the electron and photon propagators in terms of the spherical harmonics of the four-sphere and performed their calculations in the transform space. We shall consider the use of a momentumlike representation[2] for the electron and photon propagators which is valid in a general spacetime. This representation was used by Bunch and Parker[2] to discuss the propagator for a scalar field in arbitrary spacetimes. It turns out that this momentum-space expansion is only valid locally but it is sufficient to calculate the divergences that arise in the theory.

This work extends the study of renormalization of interacting field theories in curved spacetimes. The previous situations examined were a massless self-interacting scalar field in de Sitter spacetime,[3] self-interacting scalar fields in arbitrary spacetimes,[2,4-8] and massless QED in de Sitter spacetime. An important conclusion of this work is that while the theories considered are indeed renormalizable in curved spacetime, this does not follow from the fact that the corresponding theories in flat spacetime are renormalizable. Thus we have to examine each theory if we wish to conclude that it is either renormalizable or nonrenormalizable. In addition, there has been much interest in the effect of interactions and vacuum polarization on the so-called conformal anomaly.[9-13] The main motivation, however, for this investigation stems from the fact that the examinations of the ultraviolet divergences that arise in the presence of a classical gravitational field may shed some light on the ultraviolet problem of quantum gravity itself.[14] An additional motivation comes from the recent interest in the role of grand unified theories[15] in explaining the observed baryon-antibaryon asymmetry[16] in the early universe. Such models usually involve applying perturbative techniques in regimes of high curvature so it is necessary to be assured that the curvature does not violate the renormalizability of the theory.

We shall begin by defining the generating functional for QED. As has been argued by Hartle and Hawking[17] this is the natural way to quantize a field theory in curved spacetime since it is a manifestly coordinate-independent procedure. We shall derive the Ward identities for QED by using functional techniques.[18,19] This is essential in our case since we do not have a Fourier transform available in a general curved spacetime so we must express all the Ward identities in terms of the position-space n-point functions (see, for example, Ref. 20).

We will then introduce the momentumlike representation mentioned earlier. Since this representation is only locally valid we will only be able to compute the divergent parts of the relevant Feynman diagrams. Thus the actual amplitudes for physical processes (which will in general depend on the entire geometry of the spacetime) cannot be calculated by our procedure. We will use dimensional regularization to render all divergent integrals finite and the infinities will be displayed as poles at $(n-4)$, where n is the dimensional parameter (see, for example, Ref. 21).

II. THE GENERATING FUNCTIONAL

The generating functional we shall use is very similar to the generating functional in flat spacetime except for the fact that the derivatives appearing in the action and in the field equations are covariant derivatives. We shall use the symbol

∇_μ to denote the covariant derivatives for both tensors and spinors. In discussing spinors in curved spacetime[22] we shall use the symbol $\underline{\gamma}^\mu(x)$ to denote the (spacetime dependent) γ matrices which obey

$$\underline{\gamma}^\mu(x)\underline{\gamma}^\nu(x)+\underline{\gamma}^\nu(x)\underline{\gamma}^\mu(x)=2g^{\mu\nu}(x)\,, \tag{2.1}$$

where $g^{\mu\nu}(x)$ is the spacetime metric. We can introduce the vierbein field $b^\alpha_\mu(x)$ defined by

$$\eta_{\alpha\beta}=b^\mu_\alpha(x)b^\nu_\beta(x)g_{\mu\nu}(x)\,, \tag{2.2}$$

where $\eta_{\alpha\beta}$ is the flat-spacetime metric. In terms of the vierbein the $\underline{\gamma}^\mu(x)$ matrices are related to the usual γ matrices of flat spacetime by

$$\underline{\gamma}_\mu(x)=b^\alpha_\mu(x)\gamma_\alpha\,. \tag{2.3}$$

We will denote the ordinary γ matrices by γ^μ (without the underlining).

The field equations for the photon field $A_\mu(x)$ and the electron field $\psi(x)$ are, in the absence of mutual interaction,

$$[\underline{\gamma}^\mu(x)\nabla_\mu+m_0]\psi(x)=0 \tag{2.4}$$

and

$$\nabla^\nu F_{\mu\nu}\equiv\nabla^\nu(\nabla_\nu A_\mu-\nabla_\mu A_\nu)=0\,. \tag{2.5}$$

In Eq. (2.4), m_0 represents the bare mass of the electron. These field equations can be derived in the usual way from the Lagrangian $\mathfrak{L}=\mathfrak{L}_D+\mathfrak{L}_M$, where

$$\mathfrak{L}_D=\overline{\psi}(\underline{\gamma}^\mu\nabla_\mu+m_0)\psi \tag{2.6}$$

and

$$\mathfrak{L}_M=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\,. \tag{2.7}$$

The interaction between electrons and photons is introduced by adding the interaction term $\mathfrak{L}_I$ to the Lagrangian

$$\mathfrak{L}_I=e_0 j_\mu A^\mu\,, \tag{2.8}$$

where e_0 is the bare charge of the electron and j_μ is the electron-positron current

$$j_\mu=\tfrac{1}{2}[\overline{\psi},\underline{\gamma}_\mu\psi]\,. \tag{2.9}$$

This modifies the field equations to

$$\nabla^\mu F_{\mu\nu}=-e_0 j_\mu \tag{2.10}$$

and

$$(\underline{\gamma}^\mu\nabla_\mu+m_0)\psi=-e_0\underline{\gamma}_\mu\psi A^\mu\,. \tag{2.11}$$

In the functional approach the vacuum-to-vacuum amplitude is given by

$${}_+\langle 0|0\rangle_-=\int[D_\psi D_{\overline{\psi}}DA_\mu]\exp\left\{i\int\mathfrak{L}d\tau(x)\right\}, \tag{2.12}$$

where the factor in square brackets is some appropriate measure on the space of field configurations and $d\tau(x)$ is the covariant volume element. The $|0\rangle_+$ and $|0\rangle_-$ represent the future and past vacuum states, respectively. We shall discuss this point in more detail later in this section; for the moment we assume that we understand what we mean by these vacuums.

A generating functional for Green's functions is obtained by introducing external c-number sources into the Lagrangian and writing the vacuum-to-vacuum amplitude as a functional of these sources, as follows:

$$\begin{aligned}W[j,\eta,\overline{\eta}]&\equiv{}_+\langle 0|0\rangle_-^{J,\eta,\overline{\eta}}\\&=\int[DA_\mu D_\psi D_{\overline{\psi}}]\\&\quad\times\exp\left[i\int(\mathfrak{L}+J^\mu A_\mu+\overline{\eta}\psi+\overline{\psi}\eta)d\tau(x)\right].\end{aligned} \tag{2.13}$$

The source J^μ is an ordinary external c-number source for the electromagnetic field while the η and $\overline{\eta}$ are external c-number spinor sources that anticommute with themselves and with ψ and $\overline{\psi}$. The effect of functionally differentiating W with respect to these sources is to produce factors of A_μ or ψ in the functional integral. As is well known,[23] when the functional integration is performed, one obtains the expectation value of time-ordered products of the field operator. One thus has the correspondence

$$\frac{-i\delta}{\delta J^\mu(x)}\to A_\mu(x)\,,\quad\frac{-i\delta}{\delta\overline{\eta}(x)}\to\psi(x)\,,\quad\frac{-i\delta}{\delta\eta(x)}\to-\overline{\psi}(x)\,. \tag{2.14}$$

We must now specify more carefully what we mean when we refer to the "vacuum state" in curved spacetime. We shall assume that there are asymptotic regimes in the past and in the future which are either flat or in which there exists some physically motivated definition of particle states. We shall denote the vacuum state in the past by $|\,\text{in}\rangle$ and the vacuum state in the future by $|\,\text{out}\rangle$. Particle states built up from these vacuums will correspond to the particles actually detected by particle detectors located in these asymptotic regimes. It is inconvenient to define an S matrix connecting these early-time states with the late-time states because there will be particles created by the (time-dependent in general) background gravitational field and by the interaction. It turns out to be more convenient to proceed as follows. We set up a Heisenberg picture representation by defining particle states that coincide with those based on the vacuum state $|\text{in}\rangle$ in the distant past. The vacuum state in this picture is denoted by $|H\rangle$. The operators evolve by the full Hamiltonian (excluding the external c-number sources) while the states are evolved only through the coupling to the external sources. The generating functional can now be written as

$$W[J,\eta,\bar{\eta}]=\left\langle H\left|\exp\left[i\int(J_\mu A^\mu+\bar{\eta}\psi+\bar{\psi}\eta)d\tau\right]\right|H\right\rangle . \tag{2.15}$$

An interaction picture can now be set up so that at early times the interaction-picture vacuum coincides with $|\text{in}\rangle$ and $|H\rangle$. The evolution of the interaction-picture states is governed by the interaction Hamiltonian $H_I(=-\mathcal{L}_I)$ with the sources. We shall call the early-time interaction-picture vacuum $|0\rangle_-$ and its time-evolved image $|0\rangle_+$. If we denote the evolution operator of these states by S we have

$$S|0\rangle_-=|0\rangle_+, \tag{2.16a}$$

where

$$S\equiv\exp\left(i\int H_I d\tau\right). \tag{2.16b}$$

The generating functional can now be written as

$$W[J,\eta,\bar{\eta}]=\frac{\langle 0|\exp[i\int(J_\mu A^\mu+\bar{\eta}\psi+\bar{\psi}\eta)]S|0\rangle_-}{{}_+\langle 0|S|0\rangle_-}, \tag{2.17}$$

where the factor S in the denominator cancels the disconnected vacuum diagrams.[20] The interaction-picture operators, however, obey the free field equations so that we can now use the well-known formalism of free quantum fields in curved spacetime.[24-27] The early-time creation and annihilation operators are related to the late-time creation and annihilation operators by a Bogolioubov transformation which contains all the information of particle creation by the background gravitational field. Thus to obtain the outcome of physical scattering experiments we must first use the interaction-picture perturbation scheme to calculate matrix elements between late-time and early-time interaction-picture states; we must then perform a Bogolioubov transformation between the interaction-picture states and the physical late-time particle states based on $|\text{out}\rangle$.

When the generating functional (2.15) is actually calculated[20,28] it will involve the Green's function of the free photon and electron fields. As shown in Ref. 29 (for the scalar field) the choice of a Feynman propagator implicitly constitutes a choice of vacuums both in the past and in the future. Thus the generating functional reflects the choice of vacuums through the boundary conditions on the propagators. Since we are concerned with the divergences of the theory we shall not worry further about this issue.

A gauge is chosen by adding the following gauge-breaking terms to the Lagrangian density[18,30].

$$\mathcal{L}_G=-\frac{1}{2\beta}(\nabla_\mu A^\mu)^2, \tag{2.18}$$

where β is some number. This has the advantage that an entire one-parameter family of covariant gauges can be selected by changing β; selecting $\beta=1$ gives the Feynman gauge while the Landau gauge can be obtained by formally setting $\beta=0$. The field equations for A_μ (in the absence of coupling to the electron field) is now

$$\nabla_\nu F^{\mu\nu}+\frac{1}{\beta}\nabla^\mu(\nabla_\nu A^\nu)=0. \tag{2.19}$$

If we set $\beta=1$ we can simplify this to

$$\nabla_\nu\nabla^\nu A^\mu-\nabla_\nu\nabla^\mu A^\nu+\nabla^\mu\nabla_\nu A^\mu=0 \tag{2.20}$$

or using the definition of the Riemann tensor,[31]

$$\nabla_\nu\nabla^\nu A^\mu-R^\mu{}_\nu A^\nu=0. \tag{2.21}$$

The Feynman rules for the position-space n-point functions can be obtained from $W[J,\eta,\bar{\eta}]$ and are essentially the same as in flat space (see, for example, Ref. 20, Chap. 10); the only difference being that one uses the propagators appropriate to curved spacetime. We recall these rules briefly:

(i) a factor of $S(x,x')$ for every internal electron line joining x to x';

(ii) a factor of $D_{\mu\nu}(x,x')$ for every internal photon line joining x to x';

(iii) a factor of $ie_0\underline{\gamma}_\mu(x)$ for every vertex at x together with an integration over x;

(iv) a factor of (-1) and a trace over spinor indices for every closed electron loop.

III. SCHWINGER EQUATION AND WARD IDENTITIES

The purpose of this section is to show that the photon polarization tensor has vanishing divergence. This will turn out to follow from the Ward identity. We will use Schwinger's functional equation to obtain an expression for the polarization tensor in terms of the exact propagators and the exact vertex function. The statement that the divergence of the polarization tensor vanishes is the position-space analog of the usual momentum-space statement that the polarization tensor is transverse.

We use the generating functional defined in Sec. II by Eq. (2.15). The expectation value of a single field operator is zero, when it is calculated in the state $|H\rangle$, in the limit of the external sources being set equal to zero. We will denote the exact electron propagator in the presence of the external source J by $\hat{S}(x,x')^J$ and we will drop the superscript J when the source is set to zero. The free-electron propagator is denoted by $S(x,x')$. Thus we have

$$\hat{S}(x,x')^J \equiv -i\frac{\langle H|T\psi(x)\bar{\psi}(x')|H\rangle^J}{\langle H|H\rangle^J}$$

$$= -i\frac{{}_+\langle 0|T\psi_{ip}(x)\psi_{ip}(x')S|0\rangle_-^J}{{}_+\langle 0|S|0\rangle_-^J}, \qquad (3.1)$$

where ip stands for interaction picture and S is the evolution operator for the interaction-picture states. The free propagator is

$$S(x,x') = -i\frac{{}_+\langle 0|T\psi_{ip}(x)\bar{\psi}_{ip}(x')|0\rangle_-}{{}_+\langle 0|0\rangle_-}. \qquad (3.2)$$

We shall write the expectation value of A_μ as

$$\langle A_\mu(x)\rangle^J \equiv \frac{\langle H|A_\mu(x)|H\rangle^J}{\langle H|H\rangle^J} = \frac{1}{W[J]}\frac{1}{i}\frac{\delta W[J]}{\delta J^\mu}, \qquad (3.3)$$

where $W[J]$ stands for the generating functional with the external spinor sources set equal to zero. The exact photon propagator in the presence of sources is

$$\hat{D}_{\mu\nu}(x,x')^J \equiv \frac{\delta\langle A_\mu(x)\rangle^J}{\delta J_\nu(x')}$$

$$= i\frac{\langle H|TA_\mu(x)A_\nu(x')|H\rangle^J}{W[J]}$$

$$- i\langle A_\mu(x)\rangle^J\langle A_\nu(x')\rangle^J. \qquad (3.4)$$

When $J=0$, $\langle A_\mu\rangle$ vanishes and we recover the usual exact photon propagator. We shall denote the Dirac operator of Eq. (2) by $\mathfrak{D}$ and the Maxwell operator of Eq. (2) by $M^\mu{}_\nu$.

$\hat{S}^J$ and $\langle A_\mu\rangle^J$ obey the differential equations[20]

$$\mathfrak{D}\hat{S}(x,x')^J = \delta(x,x')$$

$$+\frac{ie_0\langle H|T\gamma_\mu\delta(x)\psi(x)\bar{\psi}(x')A_\mu(x)|H\rangle^J}{W[J]} \qquad (3.5)$$

and

$$M^\nu_\mu\langle A_\nu(x)\rangle^J = -J_\mu(x) + e_0\,\mathrm{tr}\underline{\gamma}_\mu(x)S(x,x')^J. \qquad (3.6)$$

We will rewrite these as functional differential equations as follows. Recalling the correspondences of Eq. (2) we can easily check that the second term on the right-hand side of Eq. (3.5) is obtained by functionally differentiating S^J with respect to J; we will also get a term involving $\delta W[J]/\delta J$ which is just $\langle A_\mu\rangle$. Using this and rearranging terms we obtain from Eq. (3.5)

$$\left(\mathfrak{D} + ie_0\underline{\gamma}^\mu(x)\frac{\delta}{\delta J^\mu(x)} + e_0\underline{\gamma}^\mu(x)\langle A_\mu(x)\rangle^J\right)\hat{S}(x,x')^J = \delta(x,x'). \qquad (3.7)$$

If we functionally differentiate Eq. (3.6) with respect to J we obtain

$$M^\nu{}_\mu\hat{D}_{\nu\sigma}(x,x')^J = -g_{\mu\sigma}\delta(x,x') + e_0\,\mathrm{tr}\underline{\gamma}_\mu\frac{\delta}{\delta J^\sigma(x')}\hat{S}(x,x')^J. \qquad (3.8)$$

This last pair of equations are Schwinger's functional differential equations for the exact propagators.

We will now derive an expression for $\delta\hat{S}^J/\delta J$ in terms of the proper vertex function. First we introduce a (formal) inverse of $\hat{S}(x,x')^J$,

$$\int\hat{S}(x,x')^J\hat{S}^{-1}(x'',x')^J d\tau(x'')$$

$$= \int\hat{S}^{-1}(x,x'')^J\hat{S}(x'',x')d\tau(x'') = \delta(x,x'). \qquad (3.9)$$

If we functionally differentiate Eq. (3.9) with respect to $\langle A_\mu\rangle$ and simplify we obtain

$$\frac{\delta\hat{S}(x,x')^J}{\delta\langle A_\mu(y)\rangle} = -\int\hat{S}(x,z)^J\frac{\delta\hat{S}^{-1}(z,z')^J}{\delta\langle A_\mu(y)\rangle}$$

$$\times\hat{S}(z,x')^J d\tau(z)d\tau(z'). \qquad (3.10)$$

However, the chain rule for functional derivatives is

$$\frac{\delta\hat{S}(x,x')^J}{\delta J^\nu(y)} = \int\frac{\delta S(x,x')^J}{\delta\langle A_\mu(y')\rangle}\frac{\delta\langle A_\mu(y')\rangle}{\delta J^\nu(y)}d\tau(y')$$

$$= \int\frac{\delta S(x,x')^J}{\delta\langle A_\mu(y')\rangle}\hat{D}_{\mu\nu}(y',y)^J d\tau(y'). \qquad (3.11)$$

Inserting Eq. (3.10) into Eq. (3.11) we obtain

$$\frac{\delta\hat{S}(x,x')^J}{\delta J^\nu(y)} = -\int\hat{S}(x,z)^J\frac{\delta\hat{S}^{-1}(z,z')}{\delta\langle A_\mu(y')\rangle}$$

$$\times\hat{S}(z',x')^J\hat{D}_{\mu\nu}(y',y)^J d\tau(y',z',z). \qquad (3.12)$$

If we denote

$$\frac{\delta\hat{S}^{-1}(z,z')^J}{\delta\langle A_\mu(y')\rangle}$$

by $\Gamma^\mu(z,z';y)$ then Eq. (3.12) asserts that in the limit $J\to 0$, Γ^μ is the exact three-point function with the external exact propagator removed which is precisely the proper vertex function.

We now introduce the quantity $\pi_{\mu\nu}(x,x')$ by

$$\pi_{\mu\nu}(x,x') = e_0{}^2\,\mathrm{tr}\int\underline{\gamma}_\mu(x)\hat{S}(x,y)\Gamma_\nu(y,y';x')$$

$$\times\hat{S}(y',x)d\tau(y)d\tau(y'). \qquad (3.13)$$

We can use Eq. (3.12) in Eq. (3.8) and invert the latter by observing that $-D_{\mu\nu}(x,x')$ is the Green's function for M, the Maxwell operator. Taking the limit $J\to 0$ and using the definition of $\pi_{\mu\nu}(x,x')$ we obtain the integral equation for $\hat{D}_{\mu\nu}(x,x')$,

$$\hat{D}_{\mu\nu}(x,x')=D_{\mu\nu}(x,x')+e_0^2\,\mathrm{tr}\int D_{\mu\sigma}(x,y)\pi^{\sigma\lambda}(y,y')\times\hat{D}_{\lambda\nu}(y',x')d\tau(y)d\tau(y')\,. \tag{3.14}$$

This last equation shows that $\pi_{\mu\nu}$ is the sum of all proper self-energy insertions in the photon propagator.[20]

We shall now derive a version of Ward's identity[19] and use it to show that $\pi_{\mu\nu}$ has vanishing divergence. First we note that if we change A_μ by $\nabla_\mu\lambda(x)$, where $\lambda(x)$ is some function, then the change in the generating functional is the same as if we had changed $\bar{\psi}(x)$ to $e^{-ie_0\lambda(x)}\bar{\psi}(x)$. The change in $\hat{S}^{-1}$ due to the first change is easily written in terms of Γ as follows:

$$\begin{aligned}\delta\hat{S}^{-1}(x,x')&=-e_0\int\Gamma^\mu(x,x';y)\delta\langle A_\mu(y)\rangle d\tau(y)\\&=-e_0\int\Gamma^\mu(x,x';y)\nabla_\mu\lambda(y)d\tau(y)\\&=+e_0\int[\nabla_\mu\Gamma^\mu(x,x';y)]\lambda(y)d\tau(y)\,,\end{aligned} \tag{3.15}$$

where we have used the definition of Γ in the first step and integration by parts in the last step. However, $\delta\hat{S}^{-1}$ can be obtained directly from the transformation of ψ and $\bar{\psi}$, thus

$$\begin{aligned}\delta\hat{S}^{-1}(x,x')&=ie_0[\lambda(x)-\lambda(x')]\hat{S}^{-1}(x,x')\\&=ie_0\int[\delta(x,y)-\delta(x',y)]\times\hat{S}^{-1}(x,x')\lambda(z)d\tau(z)\,.\end{aligned} \tag{3.16}$$

Comparing the two expressions which must be valid for any choice of λ we conclude

$$\nabla_\mu\Gamma^\mu(x,x';y)=i[\delta(x,y)-\delta(x',y)]\hat{S}^{-1}(x,x') \tag{3.17}$$

which is a form of Ward's identity.

Now we can compute $\nabla_\nu\pi^{\mu\nu}(x,x')$ as follows:

$$\begin{aligned}\nabla_\nu\pi^{\mu\nu}(x,x')&=e_0^2\,\mathrm{tr}\int\underline{\gamma}^\mu(x)\hat{S}(x,y)\nabla_\nu\Gamma^\nu(y,y';x')\times\hat{S}(y',s)d\tau(y)d\tau(y')\\&=ie_0^2\,\mathrm{tr}\int\underline{\gamma}^\mu\hat{S}(x,y)[\delta(y,x')-\delta(y',x')]\times\hat{S}^{-1}(y,y')\hat{S}(y',x)d\tau(y)d\tau(y')\\&=0\,,\end{aligned} \tag{3.18}$$

where we have used (3.13) in the first step and (3.17) in the second step. This shows that the gauge transverse character of the polarization tensor is preserved in curved spacetime. The point of going through the functional derivation of standard results is to emphasize that they are valid in curved spacetime despite the nonavailability of the standard momentum-space results.

IV. MOMENTUM-SPACE REPRESENTATION

In order to calculate the singular parts of diagrams we will use an approximate momentum-space representation for the photon and electron propagators. This is an extension of a technique developed and used by Bunch and Parker[2] for scalar fields. In Ref. 2 the nature of the approximation is discussed and its equivalence to the proper-time representation is shown. The technique correctly gives the singular parts of products of propagators but in general fails to give the finite remainder one needs to compute the amplitudes for physical processes.

The photon field in the Feynman gauge satisfies

$$\nabla_\nu\nabla^\nu A^\mu-R^\mu{}_\nu A^\nu=0 \tag{4.1}$$

and the equation for the photon propagator is

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta D^\mu{}_{\mu'}(x,x')-R^\mu{}_\nu D^\nu{}_{\mu'}(x,x')=-\delta(x,x')\delta^\mu_{\mu'}\,, \tag{4.2}$$

where the derivatives act at x and the Ricci tensor is at the point x as well. We will rewrite this equation in Riemann normal coordinates[32,33] with origin at the point x'. For convenience we will define $\bar{D}^\mu{}_{\mu'}(x,x')$ by

$$\begin{aligned}D^\mu{}_{\mu'}(x,x')&=g^{-1/4}(x)\bar{D}^\mu{}_{\mu'}(x,x')g^{-1/4}(x')\\&=g^{-1/4}(x)\bar{D}^\mu{}_{\mu'}(x,x')\,,\end{aligned} \tag{4.3}$$

where we have used the fact that $g(x')=1$ since x' is the origin of the coordinate system. The Christoeffel symbols in the Riemann normal coordinates are[33]

$$\Gamma^\sigma_{\alpha\beta}=-\tfrac{1}{3}(R^\sigma{}_{\alpha\beta\gamma}+R^\sigma{}_{\beta\alpha\gamma})y^\gamma\,, \tag{4.4}$$

where y^α represents the coordinates of the point x and the Riemann tensor is evaluated at x'. The expansions are carried out up to terms that include two derivatives of the metric; this is because it turns out to be sufficient to compute the divergences in the diagrams we shall consider.

We use (4.4) to expand the covariant derivatives in Eq. (4.2) to give the equation for $\bar{D}^\mu{}_{\mu'}(x,x')$:

$$\begin{aligned}&\eta^{\alpha\beta}\partial_\alpha\partial_\beta\bar{D}^\mu{}_{\mu'}(y)+\tfrac{1}{6}R\bar{D}^\mu{}_{\mu'}(y)\\&\quad-\tfrac{4}{3}R^\mu{}_\nu\bar{D}^\nu{}_{\mu'}(y)-\tfrac{1}{3}R^\sigma{}_\gamma y^\gamma\partial_\sigma\bar{D}^\mu{}_{\mu'}(y)\\&\quad+\tfrac{1}{3}R^\alpha{}_\gamma{}^\beta{}_\delta y^\gamma y^\delta\partial_\alpha\partial_\beta\bar{D}^\mu{}_{\mu'}(y)-\tfrac{2}{3}R^\mu{}_\lambda{}^\alpha{}_\gamma y^\gamma\partial_\alpha\bar{D}^\lambda{}_{\mu'}(y)\\&\quad+\tfrac{2}{3}R^{\mu\alpha}{}_{\lambda\gamma}y^\gamma\partial_\alpha\bar{D}^\lambda{}_\mu(y)+\cdots=-\delta(y)\delta^\mu_{\mu'}\,,\end{aligned} \tag{4.5}$$

where ∂_α is $\partial/\partial y^\alpha$ and $\eta^{\alpha\beta}$ is the Minkowski metric. The momentum-space approximation is defined by introducing the quantity $\bar{D}^{\mu}{}_{\mu'}(k)$ defined by the (formal) equation

$$\bar{D}^{\mu}{}_{\mu'}(x,x')=\int\frac{d^nk}{(2\pi)^n}e^{iky}\bar{D}^{\mu}{}_{\mu'}(k)\,, \tag{4.6}$$

where $ky=\eta^{\alpha\beta}k_\alpha y_\beta$. This integral has no clear meaning since we have not specified what the variable k is nor have we defined the measure d^nk. Nevertheless we shall continue to use it with the understanding that it is valid for small y (large k) and hence correctly captures the short-distance (ultraviolet) divergences in the theory. Intuitively one can think of k as representing an effective wave vector (in the spirit of the WKB approximation) that approaches the "true" wave vector for larger and larger values of k. $\bar{D}^{\mu}{}_{\mu'}(k)$ is implicitly a function of x'. In flat spacetime $\bar{D}^{\mu}{}_{\mu'}(k)$ would be independent of x and x' separately since translational invariance would tell us that $\bar{D}^{\mu}{}_{\mu'}(x,x')$ depends only on $(x-x')$.

The quantity $\bar{D}^{\mu}{}_{\mu'}(k)$ is assumed to have an expansion of the form

$$\bar{D}^{\mu}{}_{\mu'}(k)=\bar{D}^{\mu}_{0\mu'}+\bar{D}^{\mu}_{1\mu'}+\bar{D}^{\mu}_{2\mu'}+\cdots\,, \tag{4.7}$$

where $\bar{D}^{\mu}_{i\mu'}$ is assumed to have a geometrical coefficient involving i derivatives of the metric. On dimensional grounds $\bar{D}^{\mu}_{i\mu'}(k)$ must be of order $k^{-(2+i)}$ so that Eq. (4.7) is an asymptotic expansion in large k.

It is easy to see from Eq. (4.5) that

$$\bar{D}^{\mu}_{0\mu'}(k)=\delta^{\mu}{}_{\mu'}/k^2 \tag{4.8}$$

and

$$\bar{D}^{\mu}_{1\mu'}(k)=0\,. \tag{4.9}$$

The equation satisfied by $\bar{D}^{\mu}_{2\mu'}$ is

$$\begin{aligned}\eta^{\alpha\beta}\partial_\alpha\partial_\beta\bar{D}^{\mu}_{2\mu'}(y)&+\tfrac{1}{6}R\bar{D}^{\mu}_{0\mu'}(y)\\&-\tfrac{4}{3}R^{\mu}{}_{\nu}\bar{D}^{\nu}_{0\mu'}(y)-\tfrac{1}{3}R^{\sigma}{}_{\gamma}y^{\gamma}\partial_\sigma\bar{D}^{\mu}_{0\mu'}(y)\\&+\tfrac{1}{3}R^{\alpha}{}_{\gamma}{}^{\beta}{}_{\delta}y^{\gamma}y^{\delta}\partial_\alpha\partial_\beta\bar{D}^{\mu}_{0\mu'}(y)-\tfrac{2}{3}R^{\mu}{}_{\lambda}{}^{\alpha}{}_{\gamma}y^{\gamma}\partial_\alpha\bar{D}^{\lambda}_{0\mu'}(k)\\&-\tfrac{2}{3}R^{\mu\alpha}{}_{\lambda\gamma}y^{\gamma}\partial_\alpha\bar{D}^{\lambda}_{0\mu'}(y)+\cdots=0\,,\end{aligned} \tag{4.10}$$

where by $\bar{D}^{\mu}_{2\mu'}(y)$ we mean

$$\int\frac{d^nk}{(2\pi)^n}e^{iky}\bar{D}^{\mu}_{2\mu'}(k)\,. \tag{4.11}$$

The equation for $\bar{D}^{\mu}_{2\mu'}$ can be solved by using the k representation. To simplify the expressions we note that y_α can be written as $-i\partial/\partial k^\alpha$ when we use (4.6) to replace $\bar{D}^{\mu}{}_{\mu'}(y)$ by $\bar{D}^{\mu}{}_{\mu'}(k)$ and the resulting expressions can be integrated by parts. The expression we finally obtain is

$$\bar{D}^{\mu}_{2\mu'}(y)=\int\frac{dk^n}{(2\pi)^n}e^{iky}\times\left(\frac{\tfrac{1}{6}R\delta^{\mu}{}_{\mu'}-\tfrac{2}{3}R^{\mu}{}_{\nu}\delta^{\nu}{}_{\mu'}}{k^4}-\frac{\tfrac{4}{3}R^{\mu\beta}{}_{\nu\gamma}k^{\gamma}k_{\beta}\delta^{\nu}{}_{\mu'}}{k^6}\right). \tag{4.12}$$

For the spinor field we use the result reported in Ref. 2. To simplify the expressions we need the following results in Riemann normal coordinates.[34] The spinor connection Λ_μ is given by

$$\Lambda_\mu=\tfrac{1}{16}[\gamma_\alpha,\gamma_\beta]R^{\alpha\beta}{}_{\mu\nu}y^{\nu}+O(y^2)\,, \tag{4.13}$$

the vierbein field is

$$b^{\alpha}_{\mu}(y)=\delta^{\alpha}_{\mu}-\tfrac{1}{6}\gamma_\alpha\eta^{\alpha\lambda}R_{\mu\nu\lambda\sigma}y^{\nu}y^{\sigma}+O(y^3)\,, \tag{4.14}$$

and finally the γ matrices appropriate to curved spacetime are

$$\underline{\gamma}_\mu(y)=b^{\alpha}_{\mu}(y)\gamma_\alpha=\gamma_\mu\eta^{\alpha\lambda}R_{\mu\nu\lambda\sigma}y^{\nu}y^{\sigma}+O(y^3)\,. \tag{4.15}$$

The spinor derivative appearing in Dirac's equation is

$$\begin{aligned}\underline{\gamma}^{\mu}\nabla_\mu=\underline{\gamma}^{\mu}\partial_\mu-\underline{\gamma}^{\mu}\Lambda_\mu=\gamma^{\mu}\partial_\mu&+\tfrac{1}{6}R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\gamma_\nu y^{\alpha}y^{\beta}\partial_\mu\\&-\tfrac{1}{16}\gamma^{\mu}[\gamma_\alpha,\gamma_\beta]R^{\alpha\beta}{}_{\mu\epsilon}y^{\epsilon}+\cdots\,.\end{aligned} \tag{4.16}$$

The equation for the electron propagator $S(x,x')$ is[2]

$$\gamma(\underline{\gamma}^{\mu}\nabla_\mu+m)S(x,x')=\delta(x,x')1\,, \tag{4.17}$$

where 1 is the unit matrix in spinor space and γ is a matrix defined by

$$\gamma\gamma_\alpha\gamma^{-1}=-\gamma^{\dagger}_{\alpha}\,. \tag{4.18}$$

The Pauli adjoint of a spinor field ψ is given by

$$\bar{\psi}=\psi^{T}\gamma\,. \tag{4.19}$$

The momentum-space expression[2,35] for S (to order k^{-4}) is then

$$\begin{aligned}S(x,x')=i\int\frac{d^nk}{2\pi^n}\bigg[&\frac{ik_\alpha\gamma^\alpha-m}{k^2+m^2}+\frac{\tfrac{1}{4}R(ik_\alpha\gamma^\alpha-m)}{(k^2+m^2)^2}\\&-\frac{\tfrac{2}{3}R_{\sigma\tau}k^{\sigma}k^{\tau}(i\gamma_\alpha k^\alpha-m)}{(k^2+m^2)^3}\\&+\frac{i\tfrac{1}{8}R^{\alpha\beta}{}_{\mu\nu}[\gamma_\alpha,\gamma_\beta]\gamma^{\mu}k^{\nu}}{(k^2+m^2)^2}\bigg]\gamma^{-1}\,.\end{aligned} \tag{4.20}$$

V. RENORMALIZATION

In this section we shall calculate the divergences that arise in one-loop diagrams contributing to the electron and photon self-energy. We will use dimensional regularization to render divergent integrals finite and the divergences will be displayed as poles at $n=4$, where n is the dimensional parameter (see, for example, Ref. 21). It is for this reason that we have written the momentum representation of the previous section as n-dimensional integrals.

Anticipating the divergences, we introduce the following renormalization constants and renormalized quantities:

$$\psi_R = Z_2^{-1/2}\psi\,, \tag{5.1a}$$

$$A_{\mu R} = Z_3^{-1/2} A_\mu\,, \tag{5.1b}$$

$$\Gamma_{\mu R} = Z_1^{-1}\Gamma_\mu\,, \tag{5.1c}$$

$$M_R = Z_4^{-1} M_0\,, \tag{5.1d}$$

where the subscript R denotes a renormalized quantity and the Z's denote the (infinite)renormalization constants. The renormalization constants themselves are regarded as functions of the renormalized charge e_R. Thus we may write

$$Z_i = 1 + \sum_{\nu=1}^{\infty} Z_i^{(\nu)} e_R^{\nu}\,, \tag{5.2}$$

where the divergences show up as poles at $n=4$ appearing in the quantities $Z_i^{(\nu)}$.

The renormalized charge is physically given by the low-energy limit of electron-photon (Compton) scattering. If we insert the renormalized propagators into the expression for Γ we obtain the following relation between the bare charge e_0 and the renormalized charge e_R and the Z_i's:

$$e_0 = Z_1 Z_2^{-1} Z_3^{-1/2} e_R\,. \tag{5.3}$$

According to the Ward identity Eq. (3.17), which holds for bare fields and renormalized fields, we must have

$$Z_1 = Z_2^{-1} \tag{5.4}$$

and hence Eq. (5.3) reduces to

$$e_0 = Z_3^{-1/2} e_R\,. \tag{5.5}$$

Thus we conclude that charge and vertex renormalization is entirely fixed by the electron and photon wave-function renormalization.

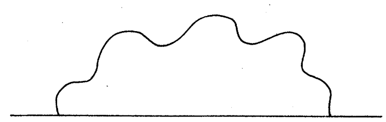

FIG. 1. One-loop correction to the electron propagator.

A. The electron propagator

The diagram corresponding to the one-loop correction to the electron propagator is shown in Fig. 1. The electron propagator including the one-loop correction is

$$\hat{S}(x,x') = S(x,x') - e_R^2 \int d\tau(u)\, d\tau(u')[S(x,u)\underline{\gamma}_\mu(u) D^\mu{}_\nu(u,u')\underline{\gamma}^\nu(u') S(u',u) S(u',x')]\,, \tag{5.6}$$

where $\hat{S}$ stands for the corrected electron propagator. To evaluate the divergent part of the integral in Eq. (5.6) a normal coordinate system is set up with origin at u and the momentum-space expansions of the preceding section is used for S and $D^\mu{}_\nu$.

Concentrating on the second term in Eq. (5.6) we have

$$-e_R^2 \int d^n u\, d^n u'\, g^{1/2}(u') S(x,u)\gamma_\mu \bar{D}^\mu{}_\nu(u,u') \times g^{1/4}(u')\underline{\gamma}^\nu(u') S(u',u) S(u',x')\,, \tag{5.7}$$

where the fact that in normal coordinates about u the covariant volume element is just $d^n u$ and $\underline{\gamma}_\mu(u)$ is γ_μ has been used. Using the momentum representation Eq. (4.12) for $\bar{D}$ and (4.20) for S leads to the following expression for (5.7):

$$-e_R^2 \int d^n u\, d^n u'\, S(x,u)\gamma_\mu (1 - \tfrac{1}{4} R_{\lambda\tau} y^\lambda y^\tau) \int \frac{d^n k}{(2\pi)^n} e^{iky} \left[\frac{\delta^\mu_\nu}{k^2} + \frac{\frac{1}{6} R\, \delta^\mu_\nu - \frac{2}{3} R^\mu{}_\alpha \delta^\alpha_\nu}{k^4} - \frac{4}{3}\left(\frac{R^{\mu\beta}{}_{\alpha\gamma} k^\gamma k_\beta \delta^\alpha_\nu}{k^6} \right) \right]$$
$$\times \int \frac{d^n q}{(2\pi)^n} e^{iqy} \left[\frac{(i\gamma^\alpha q_\alpha - m)}{q^2 + m^2} + \frac{\frac{1}{4} R (i\gamma^\alpha q_\alpha - m)}{(q^2+m^2)^2} \right] (\gamma^\nu + \tfrac{1}{6} R^\nu{}_\sigma{}^\epsilon{}_\delta \gamma_\epsilon y^\sigma y^\delta) S(u',x')\,, \tag{5.8}$$

where y^α are the normal coordinates of u'; the factor of $(1 - \frac{1}{4} R_{\lambda\tau} y^\lambda y^\tau)$ represents $g^{3/4}(u')$ and $(\gamma^\nu + \frac{1}{6} R^\nu{}_\sigma{}^\epsilon{}_\delta \gamma_\epsilon y^\sigma y^\delta)$ is $\underline{\gamma}^\nu(u')$ in normal coordinates. All the expansions have terms involving more than two derivatives of the metric truncated. We introduce the new variable $p_\alpha \equiv k_\alpha + q_\alpha$ and eliminate q in favor of p in Eq. (5.8). Writing out only the p and k integrals of Eq. (5.8) we have

$$\int \frac{d^n k}{(2\pi)^n} e^{ipy} \int \frac{d^n k}{(2\pi)^n} \left[\frac{\delta^\mu_\nu}{k^2} + O(k^{-4}) \right] \left[\frac{i\gamma^\alpha (p-k)_\alpha}{(p-k)^2 + m^2} + O(k^{-3}) \right]. \tag{5.9}$$

It is clear from power counting that only the product of the first two terms will give rise to a divergent k integration. Thus the effect of all the curvature corrections to the flat spacetime propagator is to make only finite correction to the electron self-energy.

If only the divergent terms are retained in (5.8) we get

$$\hat{S}(x,x')=S(x,x')-e_R{}^2\int d^nu\,d^nu'\,S(x,u)\gamma_\mu\int\frac{d^np}{(2\pi)^n}e^{ipy}\int\frac{d^nk}{(2\pi)^n}\left(\frac{\delta^\mu_\nu}{k^2}\right)\left[\frac{i\gamma^\alpha(p_\alpha-k_\alpha)-m}{(p-k)^2+m^2}\gamma^\nu S(u',x')\right]. \tag{5.10}$$

The k integral of Eq. (5.10) is precisely the integral that one calculates for the one-loop electron self-energy in flat spacetime.[21] According to the standard result in flat spacetime the k integral [denoted $\Sigma(p,n)$] can be written in the form[21]

$$\Sigma(p,n)=A(n)+(i\gamma^\alpha p_\alpha+m)B(n)+\Sigma_f(p)\,, \tag{5.11}$$

where A and B are constants that diverge at $n=4$ and Σ_f is the finite part of $\Sigma(p,n)$. Using the form (5.11) in (5.10) leads to

$$\hat{S}(x,x')=S(x,x')-e_R{}^2\int d^nu\,d^nu'\,S(x,u)\left\{\int\frac{d^np}{(2\pi)^n}e^{ipy}[A+B(ip^\alpha\gamma_\alpha+m)]\right\}. \tag{5.12}$$

The Σ_f has been dropped since we are only interested in the divergences. We can rewrite Eq. (5.12) in the form

$$\begin{aligned}\hat{S}(x,x')&=S(x,x')-e_R{}^2\int d^nu\,d^nu'\,S(x,u)\left\{[A+B(\gamma^\alpha\nabla_\alpha+m)]\int\frac{d^np}{(2\pi)^n}e^{ipy}\right\}S(u',x')\\&=S(x,x')-e_R{}^2\int d^nu\,d^nu'\,S(x,u)[A+B(\gamma^\alpha\nabla_\alpha+m)]\delta(y)S(u',x')\,.\end{aligned} \tag{5.13}$$

We have used the covariant derivative because it coincides with the partial derivative at the origin of the normal coordinates. We can now integrate by parts to make $\gamma^\alpha\nabla_\alpha$ act on S and use the fact that S satisfies the Dirac equation with a δ-function source. The δ functions can be used to undo the u,u' integrations giving

$$\begin{aligned}\hat{S}(x,x')=&S(x,x')-e_R{}^2\int d^nu\,s(x,u)(A+2Bm)S(u,x')\\&+e_R{}^2BS(x,x')\,.\end{aligned} \tag{5.14}$$

The first term is just the usual electron mass renormalization and is canceled by adding a term of the form $e_R{}^2Z_4^{(2)}m\bar{\psi}\psi$ to the Lagrangian while the second term corresponds to rescaling the fields ψ and $\bar{\psi}$ so that the free propagator is also scaled. Thus the second term is removed by the familiar electron wave-function renormalization.

The momentum-space expansion we have used shows that the electron mass and wave-function renormalizations are unaffected by spacetime curvature. This is to be contrasted with the calculation of Drummond and Shore[1] whose method applies to conformally invariant QED (massless electrons) in a conformally flat spacetime. In their method it is necessary to perform all the integrations and compute the renormalization constants before it becomes clear that the curvature does not affect the renormalization. The method used in the preceding calculation adheres fairly closely to the flat-spacetime formalism so that the similarities and differences between the two procedures are manifest. Thus a simple power-counting argument suffices to establish the result that the renormalization is unaffected by spacetime curvature. The procedure used by Drummond and Shore,[1] however, allows them to calculate the finite renormalized propagator, which cannot be obtained in the present framework.

B. The photon propagator

The diagram corresponding to the one-loop correction to the photon propagator is shown in Fig. 2. The discussion is conveniently carried out in terms of the polarization tensor $\pi_{\alpha\beta}$ introduced in Sec. III. The one-loop contribution to $\pi_{\alpha\beta}$ is

$$\pi_{\alpha\beta}(x,x')=e_R{}^2\,\mathrm{tr}\underline{\gamma}_\alpha(x)S(x,x')\underline{\gamma}_\beta(x')S(x',x)\,. \tag{5.15}$$

Divergences arise when x is in the vicinity of x' so we introduce a normal coordinate system with origin at x and use the momentum-space representation for $S(x,x')$. The expression for $\pi_{\alpha\beta}$ is

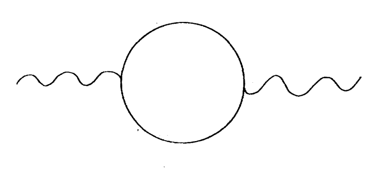

FIG. 2. One-loop correction to the photon propagator.

$$\pi_{\alpha\beta}(x,x')=\mathrm{tr}e_R{}^2\gamma_\alpha\int\frac{d^nk}{(2\pi)^n}\frac{d^nq}{(2\pi)^n}\left[\frac{(i\gamma_\sigma k^\sigma-m)}{k^2+m^2}-\frac{\frac{1}{4}R(i\gamma_\sigma k^\sigma-m)}{(k^2+m^2)^2}-\frac{\frac{2}{3}R_{\epsilon\tau}k^\epsilon k^\tau(i\gamma_\sigma k^\sigma-m)}{(k^2+m^2)^3}\right.$$
$$\left.-i\frac{\frac{1}{8}R^{\lambda\tau}{}_{\mu\nu}[\gamma_\lambda,\gamma_\tau]\gamma^\mu k^\nu}{(k^2+m^2)^2}\right]\gamma_\beta\left(\frac{i\gamma_\sigma q^\sigma-m}{q^2+m^2}+\cdots\right)e^{i(k+q)y}\,; \tag{5.16}$$

as before the y^α are the normal coordinates of x'. The spacetime-dependent $\underline{\gamma}$ matrices have been replaced with ordinary γ matrices because all spinor indices are being traced over and the results are independent of the representation of the γ matrices. As in the discussion of the electron propagator we define $p_\alpha=q_\alpha+k_\alpha$ and rewrite Eq. (5.16) as

$$\pi_{\alpha\beta}(x,x')=\mathrm{tr}e_R{}^2\gamma_\alpha\int\frac{d^np}{(2\pi)^n}e^{ipy}\int\frac{d^nk}{(2\pi)^n}\left[\frac{i\gamma_\sigma k^\sigma-m}{k^2+m^2}+\cdots\right]\gamma_\beta\left[\frac{i\gamma_\sigma(p-k)^\sigma-m}{(p-k)^2+m^2}+\cdots\right]. \tag{5.17}$$

By power counting, it is easy to see that in the k integration of Eq. (5.17) those terms that arise from the products of the first terms in each of the square brackets diverges. Similarly, products of terms that contain the curvature linearly and one of the first terms in either of the square brackets diverge whereas those terms that arise as products of two terms, each of which is linear in the curvature, produce a finite (at $n=4$) contribution to the k integration.

The various integrals in Eq. (5.17) will be evaluated separately. The first (and most divergent) integral contributing to Eq. (5.17) is

$$I^{(1)\alpha}{}_\beta\equiv\mathrm{tr}e_R{}^2\gamma^\alpha\int\frac{d^nk}{(2\pi)^n}\frac{(i\gamma_\sigma k^\sigma-m)\gamma_\beta[i\gamma^\tau(p-k)_\tau-m]}{(k^2+m^2)[(p-k)^2+m^2]}. \tag{5.18}$$

This integral is identical to the photon self-energy integral in flat spacetime[21] and the result is well known. Using the calculation of Ref. 21 we have

$$I^{(1)\alpha}{}_\beta=e_R{}^2(p^2\delta^\alpha_\beta-p^\alpha p_\beta)\frac{\Gamma(2-n/2)}{12\pi^2}F_{21}\left(2-\frac{n}{2},2;\frac{5}{2}\frac{-p^2}{4m}\right). \tag{5.19}$$

The divergence manifests itself through the pole at $n=4$ in the Γ function. The divergent part of $I^{(1)\alpha}{}_\beta$ can thus be written as

$$I^{(1)\alpha}{}_\beta(\mathrm{div})=-\frac{e_R{}^2}{6\pi^2(n-4)}(p^2\delta^\alpha_\beta-p^\alpha p_\beta), \tag{5.20}$$

where the Γ function and the hypergeometric function were expanded and only those terms that diverged at $n=4$ were retained. When $I^{(1)\alpha}{}_\beta$ is inserted into Eq. (5.17) we have the term

$$\frac{-e_R{}^2}{6\pi^2(n-4)}\int\frac{dp}{(2\pi)^n}e^{ipy}(p^2\delta^\alpha_\beta-p^\alpha p_\beta)+\text{other terms}$$
$$=\frac{+e_R{}^2}{6\pi^2(n-4)}(\eta^{\mu\nu}\partial_\mu\partial_\nu\delta^\alpha-\partial^\alpha\partial_\beta)\delta(y)$$
$$+\text{other terms}\,. \tag{5.21}$$

This can be rewritten in terms of covariant derivatives by using the following identities valid for Riemann normal coordinates:

$$\partial^\alpha\partial_\beta\delta(y)=\nabla^\alpha\nabla_\beta\delta(y)-\tfrac{1}{3}R^\alpha{}_\beta\delta(y)\,, \tag{5.22a}$$

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\delta^\alpha\delta(y)=(\Box-\tfrac{1}{3}R)\delta^\alpha_\beta\delta(y)\,. \tag{5.22b}$$

Using these identities in the expression Eq. (5.21) we have

$$\frac{e_R{}^2}{6\pi^2(n-4)}(\Box\delta^\alpha_\beta-\nabla^\alpha\nabla_\beta)\bar\delta(x,x')$$
$$+\frac{e_R{}^2}{18\pi^2(n-4)}(R^\alpha{}_\beta-R\delta^\alpha_\beta)\delta(x,x')\,. \tag{5.23}$$

We have omitted the other terms for brevity and have used the covariant δ function denoted $\bar\delta(x,x')$:

$$\bar\delta(x,x')\equiv g^{-1/2}\delta(y)=\delta(y)\,. \tag{5.24}$$

We now consider the effect of these contributions to $\pi_{\alpha\beta}$ on the photon self-energy. We recall that the wave equation satisfied by the free photon propagator is

$$(\Box\delta^\alpha_\beta+\nabla^\alpha\nabla_\beta-\nabla_\beta\nabla^\alpha)D^\beta{}_\nu(x,x')=-\bar\delta(x,x')\delta^\alpha_\nu\,. \tag{5.25}$$

We are using the Feynman gauge and have replaced the Riemann tensor by a pair of skewed covariant derivatives. If the first terms of Eq. (5.23) are inserted for π in Eq. (3.14) we get

$$\hat D_{\mu\nu}(x,x')=D_{\mu\nu}(x,x')+\frac{e_R{}^2}{6\pi^2(n-4)}$$
$$\times\int D_{\mu\sigma}(x,u)(\Box\delta^\sigma_\lambda-\nabla^\sigma\nabla_\lambda)\bar\delta(u,u') \tag{5.26}$$
$$\times D^\lambda{}_\nu(u',x')d\tau(u)d\tau(u')\,.$$

We now integrate by parts twice so that the derivatives act on the u' argument of $D^\lambda{}_\nu(u',x')$ and use the wave equation to simplify. The final result, after using the δ functions to remove as many of the integrations as possible, is

$$\hat D_{\mu\nu}(x,x')=D_{\mu\nu}(x,x')-\frac{e_R{}^2}{6\pi^2(n-4)}D_{\mu\nu}(x,x')$$
$$-\frac{e_R{}^2}{6\pi^2(n-4)}\int D_{\mu\sigma}(x,u)\nabla^\sigma\nabla_\lambda D^\lambda{}_\nu(u,x')d\tau(u)\,. \tag{5.27}$$

The last term in this expression corresponds to an (infinite) change in the gauge-fixing term of Eq. (2.18) thus this divergence is removed by merely changing the β of Eq. (2.18) so that the renormalized propagator is also in the Feynman gauge. The second term on the right-hand side of Eq. (5.27) is the photon wave-function renormalization. If we write Eq. (5.27) in terms of the renormalized fields we have $A_{R\mu}=Z_3^{-1/2}A_\mu$ and $D_{R\mu\nu}=Z_3^{-1}D_{\mu\nu}$. Thus if we define

$$Z_3^{(2)}=-\frac{1}{6\pi^2(n-4)} \text{ and } Z_3^{(1)}=0 \tag{5.28}$$

we will remove the divergence produced by the second term. Both these renormalizations, of the gauge-fixing parameter and of the photon field, are the same as in flat spacetime and in de Sitter spacetime.[1]

We are now left with the remaining terms arising from Eq. (5.17), all of which contain the curvature, and the remaining term of Eq. (5.23) which also contains the curvature. The survival of any such term would force us to introduce new couplings between the photon field and the curvature into the Lagrangian. One could then renormalize the associated coupling constants to eliminate the divergences which depend on the curvature.

The next term in Eq. (5.17) that can be divergent is

$$I^{(2)\alpha}{}_\beta=\operatorname{tr}e_R{}^2\frac{R}{4}\gamma^\alpha\int\frac{d^nk}{(2\pi)^n}\frac{(i\gamma_\sigma k^\sigma-m)\gamma_\beta[i\gamma_\delta(p-k)^\delta-m]}{(k^2+m^2)^2[(p-k)^2+m^2]}\,. \tag{5.29}$$

By power counting we see that there is a divergence at $n=4$ only if there are two powers of k in the numerator. Thus the divergent part of $I^{(2)}$ is

$$I^{(2)}_{\alpha\beta}(\text{div})=\frac{e_R{}^2R}{4}\operatorname{tr}\gamma_\alpha\int\frac{d^nk}{(2\pi)^n}\frac{\gamma_\sigma k^\sigma\gamma_\beta\gamma_\delta k^\delta}{(k^2+m^2)^2[(p-k)^2+m^2]}\,. \tag{5.30}$$

We combine the denominators using the formula[35]

$$\frac{1}{abc}=2\int_0^1dx\int_0^x dy[ax+by+c(1-x-y)]^{-3} \tag{5.31}$$

and we perform the traces using[35]

$$\operatorname{tr}\gamma_\alpha\gamma_\sigma\gamma_\beta\gamma_\delta=4(g_{\alpha\sigma}g_{\beta\delta}-g_{\alpha\beta}g_{\sigma\delta}+g_{\alpha\delta}g_{\beta\sigma})\,. \tag{5.32}$$

We obtain in place of Eq. (5.30) the equation

$$I^{(2)}_{\alpha\beta}=2e_R{}^2R\int_0^1dx\int_0^x dy\int\frac{d^nk}{(2\pi)^n}\frac{(2k_\alpha k_\beta-k^2g_{\alpha\beta})}{[k^2+m^2+p^2(1-x-y)(x+y)]^3}\,, \tag{5.33}$$

where we have made the change of variables

$$k\to k-(1-x-y)p\,.$$

The k integration can now be done using[21]

$$\int\frac{d^nk}{(2\pi)^n}\frac{k_\mu k_\nu}{(k^2+H)^3}=\frac{\frac{1}{2}g_{\mu\nu}\Gamma(2-n/2)}{(4\pi)^2\Gamma(3)}+\text{finite}\,. \tag{5.34}$$

We obtain finally

$$I^{(2)}_{\alpha\beta}(\text{div})=\frac{e_R{}^2Rg_{\alpha\beta}}{8\pi^2(n-4)}\,. \tag{5.35}$$

As was shown the calculations are straightforward extensions of the usual calculations for QED in flat spacetime. The calculations are somewhat tedious as they involve taking traces of up to six γ matrices and combining four denominators using the formula[35]

$$(abcd)^{-1}=6\int_0^1dx\int_0^x dy\int_0^y dz[ax+by+cz+d(1-x-y-z)]^{-4}\,. \tag{5.36}$$

We shall not go through the rest of the calculation but merely quote the results:

$$I^{(3)\alpha}{}_\beta(\text{div})=-\frac{e_R{}^2(R^\alpha{}_\beta-R\delta^\alpha_\beta)}{18\pi^2(n-4)}\,, \tag{5.37}$$

$$I^{(4)}_{\alpha\beta}(\text{div})=-\frac{e_R{}^2Rg_{\alpha\beta}}{8\pi^2(n-4)}\,. \tag{5.38}$$

Each of these terms is multiplied by the integral over p which yields a δ function. If we combine all the curvature-dependent divergent terms in π, i.e., Eqs. (5.38), (5.37), (5.35) and the second term in (5.23), they cancel completely. Thus in a general curved spacetime the photon self-energy does not generate any new divergences that are not present in flat spacetime.

That this cancellation occurs is not fortuitous but results from gauge invariance. As has been in Sec. III the Ward identity leads to the following condition on $\pi_{\mu\nu}(x,x')$:

$$\nabla^\mu\pi_{\mu\nu}(x,x')=0\,. \tag{5.39}$$

The divergent part of $\pi_{\mu\nu}(x,x')$ must itself satisfy (5.39). The divergences in $\pi_{\mu\nu}$ can consist of the following pieces:

$$\pi_{\mu\nu}(x,x')(\text{div})=K^{(0)}g_{\mu\nu}\tilde{\delta}(x,x')+K^{(1)}g_{\mu\nu}\tilde{\delta}(x,x')+K^{(2)}\nabla_\mu\nabla_\nu\tilde{\delta}(x,x')+S_{\mu\nu}(x)\tilde{\delta}(x,x')\,, \tag{5.40}$$

where the appearance of δ functions and their derivatives express the fact that the divergences are ultraviolet, i.e., they arise only in the coincidence limit $x=x'$ so that they are associated with the short-distance behavior of the propagators. The quantities $K^{(0)}$, $K^{(1)}$, and $K^{(2)}$ are (infinite at $n=4$) constants while $S_{\alpha\beta}(x)$ is a tensor containing curvature terms that also is infinite at $n=4$. The $K^{(0)}$ term expresses the possibility of there being a quadratically divergent momentum integral while the other three terms correspond to the possibility of there being logarithmically divergent momentum integrations. Intuitively, taking two derivatives of the δ function (as in the $K^{(1)}$ and $K^{(2)}$ terms) or two derivatives of the metric (as in the $S^{\alpha\beta}$ terms) adds two powers of the momentum to the denominator of the momentum integrals. There are no preferred vector fields one can use to construct linearly divergent terms with the correct index structure. Since $\pi_{\alpha\beta}$ is quadratically divergent in flat spacetime and spacetime curvature can only add terms which are less divergent than the highest-order divergence in flat spacetimes it follows that the expression Eq. (5.40) exhausts all the possible divergences in $\pi_{\mu\nu}(x,x')$ in a general curved spacetime.

As has been shown by explicit calculation of the one-loop diagram the terms involving $K^{(0)}$ and $S_{\alpha\beta}$ in Eq. (5.40) are not present, while those involving $K^{(1)}$ and $K^{(2)}$ are present. In view of Eq. (5.39) we have

$$\nabla_\mu[K^{(0)}g^{\mu\nu}\tilde{\delta}(x,x')]=K^{(0)}g^{\mu\nu}\nabla_\mu\tilde{\delta}(x,x')=0 \quad (5.41)$$

which implies that $K^{(0)}$ itself must vanish. This is the position-space analog of the statement that gauge invariance eliminates the necessity of photon mass renormalization. For the term involving $S^{\mu\nu}$ we have

$$\nabla_\mu[S^{\mu\nu}\tilde{\delta}(x,x')]=0\,. \quad (5.42)$$

If we regard this expression to be a distribution acting on smooth test functions of compact support $f(x)$ and $g_\nu(x')$ we obtain using Eq. (5.42)

$$\begin{aligned}0&=\int\int d\tau(x)d\tau(x')[\nabla_\mu(S^{\mu\nu}(x)\tilde{\delta}(x,x'))f(x)g_\nu(x')]\\&=-\int\int d\tau(x)d\tau(x')[S^{\mu\nu}(x)\tilde{\delta}(x,x')(\nabla_\mu f(x))g_\nu(x')]\\&=-\int d\tau(x)[S^{\mu\nu}(x)(\nabla_\mu f(x))g_\nu(x)]\\&=\int d\tau(x)[\nabla_\mu(S^{\mu\nu})fg_\nu]+\int d\tau(x)[S^{\mu\nu}f\nabla_\mu g_\nu]\,; \quad (5.43)\end{aligned}$$

the first equality is merely Eq. (5.42), the second follows from integration by parts, in the third the δ function has been used to do the x' integration, and the fourth equality follows by integration by parts. Since Eq. (5.43) is true for any choice of f and g we can specify them arbitrarily. Suppose $S^{\mu\nu}$ does not vanish at some point then we can choose g_ν so that it is zero at that point and $\nabla_\mu g_\nu$ is not zero at that point. Now f can be chosen so that it is sharply peaked around the point in question so that on the right-hand side of Eq. (5.43) the first integral drops out while the second is nonzero which contradicts Eq. (5.42). Hence the only way one can satisfy Eq. (5.42) is to have $S^{\alpha\beta}\equiv 0$. Thus we conclude that there are no divergent quantities in which the curvature tensor appears.

VI. CONCLUSION

In the preceding pages it was shown that quantum electrodynamics at the one-loop level is renormalizable in a general curved spacetime. It is important to note that the fact that the theory is renormalizable is not a simple consequence of the fact that the corresponding theory is renormalizable in flat spacetime. One cannot *a priori* assert that new couplings between the fields and curvature tensor will not appear in the process of renormalization[4,5,7] and indeed in the present case it is the gauge invariance of the theory that prevents such couplings from appearing.

The general lesson to be learned is that one must examine a given theory in curved spacetime before it is used to compute physically interesting quantities. While the results of this paper only apply to the one-loop case the arguments presented at the end of Sec. V are nonperturbative and strongly suggest that the theory is renormalizable to all orders of perturbation. Recent results by Bunch[36,37] show how one can generalize the momentum-space representation for the scalar field and prove renormalizability of $\lambda\phi^4$ theory to all orders in the coupling constant.

In the course of the discussion and calculations no mention has been made of the topology of the spacetime. Clearly the key requirement for the implementation of the above scheme is the existence of a geodesically convex neighborhood around every point in spacetime. This is a local requirement and is clearly satisfied in any "reasonable," singularity-free spacetime whatever the global topology might be. This fits in with the notion that the ultraviolet infinities reflect short distance or local problems with field theory. The infrared divergences of QED on the other hand do depend on the global structure of spacetime. For example in a compact universe one would not have photons with arbitrarily long wavelengths and there would be no infrared divergence. If the spacetime topology is nontrivial on very small scales,[9] for example, if the spatial slices were cylinders whose radius was of the order of a Compton wavelength of an electron, then one can expect that the renormalizability will be destroyed and recent results of Ford[9] indicate that this is indeed the case.

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