Taking it to the limit: Approximate reasoning for Markov processes

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Abstract. We develop a fusion of logical and metrical principles for reasoning about Markov processes. More precisely, we lift metrics from processes to sets of processes satisfying a formula and explore how the satisfaction relation behaves as sequences of processes *and sequences of formulas* approach limits. A key new concept is *dynamically-continuous metric bisimulation* which is a property of (pseudo)metrics. We prove theorems about satisfaction in the limit, robustness theorems as well as giving a topological characterization of various classes of formulas. This work is aimed at providing approximate reasoning principles for Markov processes.

1 Introduction

Probabilistic bisimulation, introduced by Larsen and Skou [LS91] has become the key concept for reasoning about the equivalence of probabilistic and stochastic systems. Labelled Markov processes are the probabilistic analogs of labelled transition systems; they have state spaces that might be continuous but include discrete state spaces as a special case. The theory of probabilistic bisimulation has been extended to stochastic bisimulation relating processes with continuous-state spaces and continuous distributions [DEP02,CLM11a]. These papers provided a characterization of bisimulation using a negation-free logic.

However, it is also widely realized that probabilistic and stochastic bisimulations are too "exact" for most purposes — they only relate processes with identical behaviours. In applications we need instead to know whether two processes that may differ by a small amount in the real-valued parameters (rates or probabilities) have similar behaviours. These motivated the search for a relaxation of the notion of equivalence of processes. Giacalone, Jou and Smolka [GJS90] note that the idea of saying that "processes that are close should have probabilities that are close" does not yield a transitive relation. This leads them to propose that the correct formulation of the "nearness" notion is via a metric.

The metric theory was initiated by Desharnais et al. [D+04] and greatly developed and explored by van Breugel, Worrell and others [vBW01,vB+03]. The key idea was to consider a behavioral *pseudometric*, i.e. a variation of the concept of metric for processes where pairs of distinct processes are at distance 0 whenever the processes are bisimilar. It was hoped that these metrics would provide a quantitative alternative to logic.

Such an alternative did not develop. Work was done on algorithms to compute the metric [vBW01], approximation techniques [D+03] and approximate bisimulation [vB+03] but approximate *reasoning principles* as such did not develop. The present work is a step in that direction. We lift the metric between processes to a metric between logical formulas by standard techniques, using the *Hausdorff metric*; but then we break new ground by exploring the relationship between convergence of processes and of formulas. We thus lay the groundwork for a notion of *approximate reasoning* not by getting rid of the logic but by fusing metric and logical principles. The completeness theorems of [CLM11a,CLM11b] are a powerful impetus for the present paper.

Consider the sequence of stochastic processes represented in Figure 1. The process m has only one state and one self-transition at rate 5; similarly, for each $k \in \mathbb{N}$, the process m_k has one state and one transition at rate $4.\underbrace{9..9}_{k}$. Since the transitions of m_k are always different of the transitions of m, we cannot describe the relation between these processes in terms of bisimulation. Instead, using a behavioral pseudometric, we expect to prove that the sequence $(m_k)_{k\in\mathbb{N}}$ of processes converges to m, since the sequence of rates of m_k converges to the rate of m. We often meet such problems in practice where m is a natural process that we need to analyze, while m_k are increasingly accurate models of m. If, in addition, we have a convergent sequence of logical formulas ϕ_k with limit ϕ such that $m_k \models \phi_k$ for each k, we want to understand whether we can infer $m \models \phi$ — and this is one of the main goals of this paper.

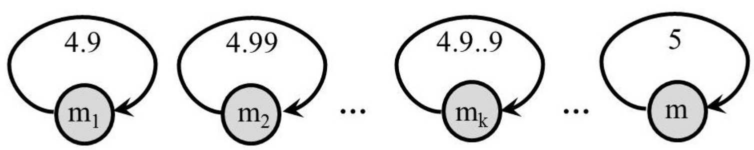

Fig. 1. A sequence of convergent stochastic processes and their limit

In order to address such problems, we identify a general metrical notion that we call *dynamical continuity*. It characterizes the behavioral pseudometrics for which a sequence of processes as the one in our example is convergent; and it allows us to relate the convergence in formulas with convergence of the processes.

Using this concept we can address the above mentioned problem and prove that in general we do not have, at the limit, $m \models \phi$. For the probabilistic case $m \models \phi$ only if ϕ is a *positive* formula. Positive formulas will be defined in the paper; they are restricted, but they suffice for the modal characterization of probabilistic bisimulation. For the stochastic case we have to restrict the set of formulas slightly more, remaining however within a set of formulas that characterize bisimulation. In either case, even if $m \not\models \phi$, there exists a sequence of processes $(n_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} n_k = m$ and $n_k \models \phi$ for each $k \in \mathbb{N}$. So this gives a handle on constructing approximations satisfying prescribed conditions. Along the way we give topological characterizations of various classes of formulas as defining open, closed, G_δ or F_σ sets[1]. All these results hold whenever one

[1] In topology, a G_δ set is a countable intersection of open sets and a F_σ set in a countable union of closed sets.

has a dynamically-continuous metric bisimulation, as it is the case with the behavioral pseudometrics introduced in [D+04,vBW01,vB+03].

The relevance of this work. We prove that the process of extrapolating properties from arbitrary accurate approximations of a system to the system itself – a method widely accepted as valid and used in applications – is not always consistent. Often one constructs better and better approximations of a system, proves properties of these approximations and extrapolates the results to the original system. But can we indeed be sure that if, for instance, the approximants show oscillatory behaviours [BMM09] then the original system also oscillates? The mathematical framework developed in this paper allows us to address such a question and to prove that the answer is **no**, in general: there may exist sequences of arbitrarily accurate approximations of a system showing properties that are not preserved to the limit; and this already happens for fragments of modal probabilistic and stochastic logics, less expressive than CSL or pCTL. We prove that the *preservation to the limit* depends on the logical structure of the property; "negative information" and "approximations from above", for instance, are obstructions to this kind of limiting argument. Moreover, different logics behave differently to the limit. In this paper we show that there is a considerable difference between probabilistic and stochastic logical properties.

To summarize, the main contributions of this paper are
– identifying the notion of a dynamically-continuous metric bisimulation as a general property of behavioural (pseudo)metrics that relax the concept of stochastic/probabilistic bisimulation for Markov processes
– identifying the topologies of logical properties induced by a dynamically-continuous metric bisimulation, both for probabilistic and stochastic logics
– studying the relation between the topology of processes and the topologies of logical properties; we reveal essential differences between probabilistic and stochastic logics.
– theorems that tells when parallel sequences of formulas and processes converge to give satisfaction in the limit
– topological characterizations of various classes of formulas, both for probabilistic and stochastic logics.

2 Preliminaries

In this section we introduce notation and establish terminology. We assume that the basic terminology of topology and measure theory is familiar to the reader. In appendix we collect some basic definitions and the proofs of the major results.

Sets and Measurability. If (M,Σ) is a measurable space with σ-algebra $\Sigma \subseteq 2^M$, we use $\Delta(M,\Sigma)$ to denote the set of measures $\mu : \Sigma \to \mathbb{R}^+$ on (M,Σ) and $\Pi(M,\Sigma)$ to denote the set of probability measures $\mu : \Sigma \to [0,1]$ on (M,Σ).

We organize $\Delta(M,\Sigma)$ and $\Pi(M,\Sigma)$ as measurable spaces: for arbitrary $S \in \Sigma$ and $r > 0$, let $\Theta = \{\mu \in \Delta(M,\Sigma) : \mu(S) \leq r\}$ and $\Omega = \{\mu \in \Pi(M,\Sigma) : \mu(S) \leq r\}$; let $\overline{\Theta}$ and $\overline{\Omega}$ be the σ-algebras generated by Θ and Ω on $\Delta(M,\Sigma)$ and $\Pi(M,\Sigma)$ respectively.

Given two measurable spaces (M, Σ) and (N, Σ'), we use $[\![M \to N]\!]$ to denote the class of measurable mappings from (M, Σ) to (N, Σ').

Given a relation $\Re \subseteq M \times M$, the set $N \subseteq M$ is $\Re$-closed iff $\{m \in M \mid \exists n \in N, (n, m) \in \Re\} \subseteq N$. If (M, Σ) is a measurable space, we denote $\Sigma(\Re) = \{S \in \Sigma \mid S \text{ is } \Re\text{-closed}\}$.

Distances. Let M be a set. A function $d : M \times M \to \mathbb{R}^+$ is a *pseudometric* on M if it satisfies, for arbitrary $x, y, z \in M$, the following axioms.

$$(1)\colon d(x,x) = 0 \qquad (2)\colon d(x,y) \le d(x,z) + d(z,y) \qquad (3)\colon d(x,y) = d(y,x).$$

If d is a pseudometric, (M, d) is a *pseudometric space*.

Given a pseudometric space (M, d), we define the following distances for arbitrary $a \in M$ and $A, B \subseteq M$ with $A \neq \emptyset \neq B$.

$$(1)\colon d^h(a, B) = \inf_{b \in B} d(a,b), \qquad (2)\colon d^{H/2}(A,B) = \sup_{a \in A} \inf_{b \in B} d(a,b),$$

$$(3)\colon d^H(A,B) = max\{\sup_{a \in A} \inf_{b \in B} d(a,b), \sup_{b \in B} \inf_{a \in A} d(a,b)\}.$$

We call d^H the *Hausdorff pseudometric* (associated to d).

Lemma 1. *If (M, d) is a pseudometric space and $\overline{X}$ is the closure of $X \subseteq M$ in the open ball topology $\mathcal{T}_d$, then for arbitrary $A, B \subseteq M$,*
(1): $d^H(A,B) = 0$ iff $\overline{A} = \overline{B}$, (2): $d^H(A,B) = d^H(\overline{A}, B) = d^H(A, \overline{B}) = d^H(\overline{A}, \overline{B})$.

In what follows, we consider for pseudometric spaces (M, d) the following notions of convergence in the open ball topologies $\mathcal{T}_d$ and $\mathcal{T}_{d^H}$ respectively[2]:
– For an arbitrary sequence $(m_k)_{k \in \mathbb{N}}$ of elements of M and an arbitrary $m \in M$, we write $m \in \lim\limits_{k \to \infty} m_k$ (or $\lim\limits_{k \to \infty} m_k \ni m$) to denote that $\lim\limits_{k \to \infty} d(m_k, m) = 0$.
– For an arbitrary sequence $(S_k)_{k \in \mathbb{N}}$ of subsets of M and an arbitrary set $S \subseteq M$, we write $S \in \lim\limits_{k \to \infty} S_k$ (or $\lim\limits_{k \to \infty} S_k \ni S$) to denote that $\lim\limits_{k \to \infty} d^H(S_k, S) = 0$.

Lemma 2. *Let (M, d) be a pseudometric space and $(B_i)_{i \in \mathbb{N}}$ a decreasing sequence of compact subsets of M in the topology $\mathcal{T}_d$. If $\lim\limits_{k \to \infty} B_k \ni A$, then $d^H(A, \bigcap\limits_{i \in \mathbb{N}} B_i) = 0$.*

Lemma 3. *Let (M, d) be a pseudometric space and $(m_k)_{k \in \mathbb{N}} \subseteq M$, $(S_k)_{k \in \mathbb{N}} \subseteq 2^M$ convergent sequences with $m \in \lim\limits_{k \to \infty} m_k$ and $S \in \lim\limits_{k \to \infty} S_k$. If $m_k \in S_k$ for each $k \in \mathbb{N}$, then $d^h(m, S) = 0$. In particular, if S is closed, then $m \in S$.*

3 The Pseudometric Spaces of Processes

In this section we introduce two classes of processes, *discrete-time Markov processes* (DMPs), which are similar to the ones studied in [Pan09,DEP02,Dob07]; and *continuous-time Markov processes* (CMPs) [CLM11a], which are models of stochastic systems

[2] A pseudometric space is not a Hausdorff space and consequently the limits are not unique.

with continuous time transitions. We emphasize that the terms "discrete" and "continuous" refer to time and not to the state space. In this paper we use for both classes the definitions proposed in [CLM11a,CLM11b], which exploits an equivalence between the definitions of Harsanyi type spaces [MV04] and a coalgebraic view of labelled Markov processes [dVR99]. However, with respect to [CLM11a,CLM11b] or to [Pan09,DEP02,Dob07], we do not consider action labels. The subtle issues all involve convergence and other analytical aspects; the labels can easily be added without changing any of these aspects of the theory.

Definition 1 (Processes). *Let* (M, Σ) *be an* analytic space*, where* Σ *is its Borel algebra.*
• *A* discrete Markov kernel *(DMK) is a tuple* $\mathcal{M} = (M, \Sigma, \theta)$*, where* $\theta \in [\![M \to \Pi(M, \Sigma)]\!]$*; if* $m \in M$*,* $(\mathcal{M}, m)$ *is a* discrete Markov process.
• *A* continuous Markov kernel *(CMK) is a tuple* $\mathcal{M} = (M, \Sigma, \theta)$*, where* $\theta \in [\![M \to \Delta(M, \Sigma)]\!]$*; if* $m \in M$*,* $(\mathcal{M}, m)$ *is a* continuous Markov process.
For both types of processes, M *is called the* support set *of* $\mathcal{M}$ *denoted by* $supp(\mathcal{M})$*.*

If m is the current state of a DMP and N is a measurable set of states, the transition function $\theta(m)$ is a probability measure on the state space and $\theta(m)(N) \in [0, 1]$ represents the *probability* of a transition from m to an arbitrary state $n \in N$.

Similarly, if m is the current state of a CMP and N is a measurable set of states, the transition function $\theta(m)$ is a measure on the state space and $\theta(m)(N) \in \mathbb{R}^+$ represents the *rate* of an exponentially distributed random variable that characterizes the duration of a transition from m to an arbitrary state $n \in N$. Indeterminacy in such systems is resolved by races between events executing at different probabilities/rates.

Notice that, in both cases, θ is a measurable mapping between the space of processes and the space of (probabilistic/stochastic) measures. These requirements are equivalent to the conditions on the corresponding two-variable *probabilistic/rate function* used in [Pan09,DEP02,Dob07] to define labelled Markov processes and in [DP03] to define continuous Markov processes (for the proof see, Proposition 2.9 [Dob07]).

The definitions of bisimulation for DMPs and CMPs follow the line of the Larsen-Skou definition of probabilistic bisimulation [LS91].

Definition 2 (Bisimulation). *Given the DMK (CMK)* $\mathcal{M} = (M, \Sigma, \theta)$*, a* bisimulation relation *on* $\mathcal{M}$ *is a relation* $\mathfrak{R} \subseteq M \times M$ *such that whenever* $(m, n) \in \mathfrak{R}$*, for any* $C \in \Sigma(\mathfrak{R})$*,* $\theta(m)(C) = \theta(n)(C)$*. Two processes* $(\mathcal{M}, m)$ *and* $(\mathcal{M}, n)$ *are* bisimilar*, written* $m \sim_{\mathcal{M}} n$*, if they are related by a bisimulation relation.*

The bisimulation relation between processes with different Markov kernels is defined by taking the disjoint union of the two [Pan09,DEP02,CLM11a,CLM11b]. For this reason, in what follows we use $\sim$ without extra indices to denote the largest bisimulation relation. We call the largest bisimulation of DMPs *probabilistic bisimulation* and the largest bisimulation of CMPs *stochastic bisimulation*.

As we have already underlined in the introduction, the concept of bisimulation for probabilistic or stochastic processes is very strict. We can however relax it by introducing a behavioral pseudometric [D+04,Pan09] which, formally, is a distances between processes that measure their similarity in terms of quantitative behaviour: the kernel of a behavioral pseudometric is a bisimulation. Moreover, we expect that a behavioral pseudometric can prove that a sequence of processes as $(m_k)_{k\in\mathbb{N}}$ represented in Figure 1 is convergent to m. Hereafter in this section we identify a sufficient condition satisfied by any behavioral pseudometric that can prove such a convergence.

Before proceeding with the definition, recall that the convergences are in the corresponding open ball topologies, as defined in the preliminary section. In addition, we define the *kernel* of d as being the set $ker(d) = \{(m,n) \in M \times M \mid d(m,n) = 0\}$.

Definition 3 (Dynamically-continuous metric bisimulation). *Given the DMK (CMK) $\mathcal{M} = (M, \Sigma, \theta)$, a pseudometric $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ is*
– a metric bisimulation *if $ker(d) = \sim$*
– dynamically-continuous *if whenever $(m_k)_{k\in\mathbb{N}} \in \mathcal{M}$ with $\lim_{k\to\infty} m_k \ni m$ (in $\mathcal{T}_d$), for any $S \in \Sigma(\sim)$ there exists a decreasing sequence $(S_k)_{k\in\mathbb{N}} \subseteq \Sigma(\sim)$ of compact sets in the topology $\mathcal{T}_d$ such that $\lim_{k\to\infty} S_k \ni S$ (in $\mathcal{T}_{d^H}$) and $\lim_{k\to\infty} \theta(m_k)(S_k) = \theta(m)(S)$.*

This new concept seems very natural to us and in fact all the behavioral pseudometrics defined in [D+04,vBW01,vB+03] are dynamically-continuous metric bisimulations, due to properties of the Kantorovich metric. Notice also the coinductive nature of this definition, which is reminiscent of the general definition of bisimulation.

Example 1. Let us now convince ourselves that any behavioral pseudometric that can prove the convergence in Figure 1 is indeed a dynamically-continuous metric bisimulation. Formally, in Figure 1 we have represented the CMK $\mathcal{M} = (M, \Sigma, \theta)$ where $M = \{m, m_1, ..m_k, ..\}$, $\Sigma = 2^M$, $\theta(m_k)(\{m_k\}) = 4.\underbrace{9..9}_{k}$ for $k \in \mathbb{N}$ and $\theta(m)(\{m\}) = 5$.

If in the open ball topology we can prove that $\lim_{k\to\infty} m_k \ni m$, then for each $k \in \mathbb{N}$, $S_k = \{m, m_k, m_{k+1}, ..\}$ is a compact set – since it is closed and bounded in a complete pseudometric space; $S_k \supseteq S_{k+1}$ and $\lim_{k\to\infty} S_k \ni \{m\}$. Because $\theta(m_k)$ is a measure, $\theta(m_k)(S_k) = \theta(m_k)(\{m_k\}) + \theta(m_k)(S_{k+1})$. But $\theta(m_k)(S_{k+1}) = 0$, hence $\theta(m_k)(S_k) = \theta(m_k)(\{m_k\})$. This implies that $\lim_{k\to\infty} \theta(m_k)(S_k) = \lim_{k\to\infty} \theta(m_k)(\{m_k\}) = \theta(m)(\{m\})$ and verifies the second condition of the previous definition. All these arguments motivate our choice for the definition of dynamically-continuous metric bisimulation. □

4 Markovian Logics

In this section we present two logics for Markovian processes: the *discrete Markovian Logic* (DML) for semantics based on DMPs, similar to the logics introduced in the literature, for example in [Aum99,FH94,LS91]; and the *continuous Markovian logic* (CML) for semantics based on CMPs [CLM11a]. In addition to the boolean operators, these logics are endowed with *probabilistic/stochastic modal operators* that approximate the probabilities/rates of transitions. For $r \in \mathbb{Q}^+$, $L_r\phi$ characterizes $(\mathcal{M}, m)$ whenever the probability/rate of the transition from m to the class of states satisfying ϕ is *at least* r; symmetrically, $M_r\phi$ is satisfied when this probability/rate is *at most* r.

Definition 4 (Syntax). *The formulas of $\mathcal{L}$ are introduced by the following grammar.*

$$\mathcal{L}: \quad \phi := \top \mid \neg\phi \mid \phi \wedge \phi \mid L_r\phi \mid M_r\phi, \qquad r \in \mathbb{Q}_+.$$

We isolate the fragment $\mathcal{L}[0,1] \subseteq \mathcal{L}$ defined only for $r \in [0,1] \cap \mathbb{Q}^+$. $\mathcal{L}$ contains the well-formed formulas of CML, $\mathcal{L}[0,1]$ contains the well-formed formulas of DML. As usual, both logics use all the boolean operators including $\perp = \neg\top$.

The major difference between the two logics is reflected in their semantics. The semantics of DML is defined for DMPs, while the semantics of CML is defined in terms of CMPs. The satisfiability relations is similar for the two logics. It assumes a fixed structure $\mathcal{M} = (M, \Sigma, \theta)$ that represents a DMK when it refers to DML, and a CMK when it refers to CML; and $m \in M$ is an arbitrary process.

$m \models \top$ always,
$m \models \neg\phi$ iff it is not the case that $m \models \phi$,
$m \models \phi \wedge \psi$ iff $m \models \phi$ and $m \models \psi$,
$m \models L_r\phi$ iff $\theta(m)([\![\phi]\!]) \geq r$,
$m \models M_r\phi$ iff $\theta(m)([\![\phi]\!]) \leq r$,

where $[\![\phi]\!] = \{m \in M \mid m \models \phi\}$.

When it is not the case that $m \models \phi$, we write $m \not\models \phi$.

The semantics of $L_r\phi$ and $M_r\phi$ are well defined only if $[\![\phi]\!]$ is measurable. This is guaranteed by the fact that θ is a measurable mapping – for the proof see [CLM11a].

In spite of their apparent similarities, the two logics are very different at the provability level. In Appendix we present, in Table 1, a complete axiomatization for DML [Zho07] and in Table 2 a complete axiomatization for CML [CLM11a]. The key differences between the two logics consists of the relation between $L_r\phi$ and $M_r\phi$. In the discrete logic the two are related by De Morgan dualities, stating that the probability of a transition to a state satisfying ϕ depends of the probability of a transition to some state satisfying $\neg\phi$: $\vdash L_r\phi \leftrightarrow M_{1-r}\neg\phi$ and $\vdash M_r\phi \leftrightarrow L_{1-r}\neg\phi$. In the continuous case, the two modal operators are independent [CLM11a]. We will see in the following sections that this difference is deeply reflected in the topologies of the two spaces of formulas.

There exist strong relations between logical equivalence and bisimulation both for the probabilistic and for the stochastic cases. In [DP03,Pan09] it was shown that the logical equivalence induced by $\mathcal{L}[0,1]$ on the class of DMPs coincides with probabilistic bisimulation. A similar result holds for CML, [CLM11a].

5 The Topological Space of Logical Formulas

Since a dynamically-continuous metric bisimulation is a relaxation of the bisimulation relation, in what follows we try to identify similar logical characterization results for dynamically-continuous metric bisimulation. In order to do this, we organize the space of the logical formulas as a pseudometric space, by identifying a logical formula with the set of its models and using the Hausdorff distance.

Formally, assume that the space $\mathcal{M}$ of the continuous (or discrete) Markov kernel is a pseudometric space defined by $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$. The Hausdorff pseudometric d^H associated to d is a distance between the sets $[\![\phi]\!]$ of models, for arbitrary $\phi \in \mathcal{L}$ (or $\phi \in \mathcal{L}[0,1]$ respectively). Consequently, we can define, for arbitrary $\phi, \psi \in \mathcal{L}$ (or $\phi, \psi \in \mathcal{L}[0,1]$ respectively), a distance δ by $\delta(\phi, \psi) = d^H([\![\phi]\!], [\![\psi]\!])$.

Proposition 1. *$(\mathcal{L}, \delta)$ and $(\mathcal{L}[0,1], \delta)$ are pseudometric spaces.*

5.1 The topology of Discrete Markovian Logic

In this subsection we concentrate on the discrete Markovian logic. Let $\mathcal{M} = (M, \Sigma, \theta)$ be the universal DMP organized as a pseudometric space by the behavioural pseudometric d. Let $(\mathcal{L}[0,1], \delta)$ be the pseudometric space of logical formulas.

To understand deeper the relation between the induced topologies, in what follows we isolate the following fragments of $\mathcal{L}[0,1]$.

$$\mathcal{L}[0,1]^+ : \quad f := \top \mid f \vee f \mid f \wedge f \mid L_r\phi \mid M_r\phi, \quad \phi \in \mathcal{L}[0,1],$$
$$\mathcal{L}[0,1]^- = \{\neg f \mid f \in \mathcal{L}[0,1]^+\}.$$

As in the preliminaries, we use $\mathcal{T}_d$ to denote the open ball topology, $\overline{X}$ and X^{int} to denote the closure and the interior of $X \subseteq \mathcal{M}$, respectively.

Proposition 2. *Let d is a dynamically-continuous metric bisimulation on $\mathcal{M}$.*
1. If $\phi \in \mathcal{L}[0,1]^+$, then $[\![\phi]\!]$ is a closed set in the topology $\mathcal{T}_d$.
2. If $\phi \in \mathcal{L}[0,1]^-$, then $[\![\phi]\!]$ is an open set in the topology $\mathcal{T}_d$.
3. $\overline{[\![\neg M_r\phi]\!]} = [\![L_r\phi]\!]$ and $\overline{[\![\neg L_r\phi]\!]} = [\![M_r\phi]\!]$.
4. $[\![M_r\phi]\!]^{int} = [\![\neg L_r\phi]\!]$ and $[\![L_r\phi]\!]^{int} = [\![\neg M_r\phi]\!]$.

At this point we want to understand more about the kernel of δ and its relation to provability. Since the axiomatic system of DML is complete, the next theorem follows from the definition of Hausdorff distance.

Theorem 1. *If $\phi, \psi \in \mathcal{L}[0,1]$ and $\vdash \phi \leftrightarrow \psi$, then $\delta(\phi, \psi) = 0$.*

The next results and the example will show that actually the reverse of Theorem 1 is not true: not all the formulas at distance zero are logically equivalent.

Theorem 2. *Let d be a dynamically-continuous metric bisimulation and $\phi, \psi \in \mathcal{L}[0,1]$ such that $\delta(\phi, \psi) = 0$.*
1. If $\phi \in \mathcal{L}[0,1]^+$, then $\vdash \psi \to \phi$.
2. If $\phi, \psi \in \mathcal{L}[0,1]^+$, then $[\delta(\phi, \psi) = 0$ iff $\vdash \phi \leftrightarrow \psi]$.

Proof. 1. $\delta(\phi, \psi) = 0$ is equivalent to $d^H(\llbracket\phi\rrbracket, \llbracket\psi\rrbracket) = 0$, which is equivalent, as stated in Lemma 1, to $\overline{\llbracket\phi\rrbracket} = \overline{\llbracket\psi\rrbracket}$. If $\phi \in \mathcal{L}[0,1]^+$, by Proposition 2, $\llbracket\phi\rrbracket$ is closed. Hence, $\overline{\llbracket\psi\rrbracket} = \llbracket\phi\rrbracket$. This implies that $\llbracket\psi\rrbracket \subseteq \llbracket\phi\rrbracket$, i.e., $\models \psi \to \phi$, which is equivalent to $\vdash \psi \to \phi$. □

Example 2. There exist logical formulas that are at distance zero without being logically equivalent:

$$\delta(L_r\phi, \neg M_r\phi) = 0, \quad \vdash \neg M_r\phi \to L_r\phi \quad \text{but} \quad \nvdash L_r\phi \to \neg M_r\phi.$$

To prove these, observe that for any model $m \in \mathcal{M}$, if $m \models \neg M_r\phi$, then $m \models L_r\phi$. This guarantees that the closure of $\llbracket\neg M_r\phi\rrbracket$ is included in $\llbracket L_r\phi\rrbracket$. Observe that if $\theta(m)(\llbracket\phi\rrbracket) = r$, then $m \models L_r\phi$, but $m \not\models \neg M_r\phi$.
Suppose that there exists a model $m \in \mathcal{M}$ such that $m \in \llbracket L_r\phi\rrbracket$ and $d^h(m, \llbracket\neg M_r\phi\rrbracket) > 0$. Then, $\theta(m)(\llbracket\phi\rrbracket) \geq r$ and $\theta(m)(\llbracket\phi\rrbracket) < r$ - impossible. Hence the closure of $\llbracket\neg M_r\phi\rrbracket$ coincides with $\llbracket L_r\phi\rrbracket$ and this proves that $\delta(L_r\phi, \neg M_r\phi) = 0$. □

The next theorem states that whenever $(m_k)_{k\in\mathbb{N}}$ is a sequence of increasingly accurate approximations of m, if $m_k \models \phi_k$ for each k, we cannot guarantee that m satisfies the limit ϕ of $(\phi_k)_{k\in\mathbb{N}}$; but, there exists a sequence of approximations of m satisfying ϕ.

Theorem 3. *If d is a dynamically-continuous metric bisimulation and $(\phi_k)_{k\in\mathbb{N}} \subseteq \mathcal{L}[0,1]$, $(m_k)_{k\in\mathbb{N}} \subseteq \mathcal{M}$ are two convergent sequences such that $\lim_{k\to\infty} \phi_k \ni \phi$, $\lim_{k\to\infty} m_k \ni m$ and for each $k \in \mathbb{N}$, $m_k \models \phi_k$, then there exists a convergent sequence $(n_k)_{k\in\mathbb{N}} \subseteq \mathcal{M}$ such that $\lim_{k\to\infty} n_k \ni m$ and $n_k \models \phi$ for each $k \in \mathbb{N}$.*

Proof. If we apply Lemma 3 for $S_k = \llbracket\phi_k\rrbracket$ and $S = \llbracket\phi\rrbracket$, we obtain that $d^h(m, \llbracket\phi\rrbracket) = 0$ which implies that there exists a sequence $(n_k)_{k\in\mathbb{N}} \subseteq \llbracket\phi\rrbracket$ such that $\lim_{k\to\infty} n_k = m$. □

There exist, however, properties that can be "taken to the limit".

Theorem 4. *Let d be a dynamically-continuous metric bisimulation and $(\phi_k)_{k\in\mathbb{N}} \subseteq \mathcal{L}[0,1]$, $(m_k)_{k\in\mathbb{N}} \subseteq \mathcal{M}$ two convergent sequences such that $\lim_{k\to\infty} \phi_k \ni \phi$, $\lim_{k\to\infty} m_k \ni m$ and $m_k \models \phi_k$ for each $k \in \mathbb{N}$. If $\phi \in \mathcal{L}[0,1]^+$, then $m \models \phi$.*

Proof. As in Theorem 3, $d^h(m, \llbracket\phi\rrbracket) = 0$. Since $\phi \in \mathcal{L}[0,1]^+$, Proposition 2 guarantees that $\llbracket\phi\rrbracket$ is closed and using the second part of Lemma 3 we obtain $m \in \llbracket\phi\rrbracket$. □

5.2 The topology of the continuous logic

In this subsection we investigate similar problems for the case of CMPs and continuous Markovian logic. Hereafter, let $\mathcal{M}$ be the universal CMP organized as a pseudometric space by the behavioural pseudometric d. Let $(\mathcal{L}, \delta)$ be the pseudometric space of logical formulas.

Lemma 4. *For arbitrary $\phi \in \mathcal{L}$,*

1. $\overline{[\![\neg M_r\phi]\!]} = [\![L_r\phi]\!]$,
2. $[\![M_r\phi]\!]^{int} = [\![\neg L_r\phi]\!]$,
3. $[\![M_r\phi]\!] = \bigcap_{k\in\mathbb{N}} [\![\neg L_{r+\frac{1}{k}}\phi]\!]$
4. $[\![L_r\phi]\!] = \bigcap_{k\in\mathbb{N}} [\![\neg M_{r-\frac{1}{k}}\phi]\!]$.

In the following examples we show that, unlike in the probabilistic case, $[\![M_r\psi]\!]$ is sometimes neither open nor closed in $\mathcal{T}_d$.

Example 3. We return to the stochastic system described in Example 1 and represented in Figure 1. Notice that, for each $k \in \mathbb{N}$, $m_k \models M_0L_5\top$ meaning that each m_k cannot do a transition to a state (which is equivalent with "it does it at rate 0") where from it is possible to do a transition at rate at least 5. But at the limit, $m \models \neg M_0L_5\top$ since $\theta(m)([\![L_5\top]\!]) = 5 > 0$. Consequently, $[\![M_0L_5\top]\!]$ is not closed in $\mathcal{T}_d$, since we found a sequence of processes from $[\![M_0L_5\top]\!]$ with a limit outside $[\![M_0L_5\top]\!]$.

To prove that sometimes $[\![M_r\psi]\!]$ is not open either, consider the same processes as before only that for each $k \in \mathbb{N}$, $\theta(m_k)(\{m_k\}) = r_k$, where $(r_k)_{k\in\mathbb{N}} \in \mathbb{Q}^+$ is a strictly decreasing sequence with limit 5. In this case, for each $k \in \mathbb{N}$, $m_k \models \neg M_5\top$, since $\theta(m_k)(\{m_k\}) > 5$. However, to the limit we have $m \models M_5\top$ proving that $[\![\neg M_5\top]\!]$ is not closed in $\mathcal{T}_d$, hence, $[\![M_5\top]\!]$ is not open. $\square$

To understand this topology more deeply, we isolate the following fragments of $\mathcal{L}$.

$\mathcal{L}^+ :\quad f := \top \mid f \wedge f \mid f \vee f \mid L_r\phi \mid M_r\phi,\qquad \mathcal{L}^- = \{\neg f \mid f \in \mathcal{L}^+\},$

$\mathcal{L}_0 :\quad f := \top \mid f \wedge f \mid \neg f \mid L_r\phi,$

$\mathcal{L}_0^+ :\quad f := \top \mid f \wedge f \mid f \vee f \mid L_r\phi,\qquad \mathcal{L}_0^- = \{\neg f \mid f \in \mathcal{L}_0^+\},$

where in the previous definitions $\phi \in \mathcal{L}[0,1]^+$.

The next lemma marks essential differences between the topology of DML formulas and the topology of CML formulas. Recall that, in topology, a G_δ set is a countable intersection of open sets and a F_σ set is a countable union of closed sets.

Theorem 5. *If d is a dynamically-continuous metric bisimulation, then*
1. If $\phi \in \mathcal{L}_0^+$, then $[\![\phi]\!]$ is a closed set in the topology $\mathcal{T}_d$.
2. If $\phi \in \mathcal{L}_0^-$, then $[\![\phi]\!]$ is an open set in the topology $\mathcal{T}_d$.
3. If $\phi \in \mathcal{L}^+$, then $[\![\phi]\!]$ is a G_δ set in the topology $\mathcal{T}_d$.
4. If $\phi \in \mathcal{L}^-$, then $[\![\phi]\!]$ is a F_σ set in the topology $\mathcal{T}_d$.

As for DML, logical equivalence is a subset of the kernel of δ.

Theorem 6. *If $\vdash \phi \leftrightarrow \psi$, then $\delta(\phi,\psi) = 0$.*

However, Theorem 2 does not hold for CML. Instead we have the following weaker result relying on the fact that $[\![\phi]\!]$ is closed whenever $\phi \in \mathcal{L}_0^+$.

Theorem 7. *Let d be a dynamically-continuous metric bisimulation and $\phi,\psi \in \mathcal{L}$ such that $\delta(\phi,\psi) = 0$.*

1. If $\phi \in \mathcal{L}_0^+$, then $\vdash \psi \to \phi$.
2. If $\phi, \psi \in \mathcal{L}_0^+$, then $[\delta(\phi, \psi) = 0$ iff $\vdash \phi \leftrightarrow \psi]$.

A similar result to Theorem 3 holds for CML.

Theorem 8. *If d is a dynamically-continuous metric bisimulation and $(\phi_k)_{k\in\mathbb{N}} \subseteq \mathcal{L}$, $(m_k)_{k\in\mathbb{N}} \subseteq \mathcal{M}$ are two convergent sequences such that $\lim_{k\to\infty} \phi_k \ni \phi$, $\lim_{k\to\infty} m_k \ni m$ and $m_k \models \phi_k$ for each $k \in \mathbb{N}$, then there exists a convergent sequence $(n_k)_{k\in\mathbb{N}} \subseteq \mathcal{M}$ such that $\lim_{k\to\infty} n_k \ni m$ and $n_k \models \phi$ for each $k \in \mathbb{N}$.*

Theorem 4 does not hold for CML. But since $[\![\phi]\!]$ is closed whenever $\phi \in \mathcal{L}_0^+$, we have a weaker version of it that does not involve the operators of type M_r.

Theorem 9. *Let d be a dynamically-continuous metric bisimulation and $(\phi_k)_{k\in\mathbb{N}} \subseteq \mathcal{L}$, $(m_k)_{k\in\mathbb{N}} \subseteq \mathcal{M}$ two convergent sequences such that $m_k \models \phi_k$ for each $k \in \mathbb{N}$, $\lim_{k\to\infty} \phi_k \ni \phi$ and $\lim_{k\to\infty} m_k \ni m$. If $\phi \in \mathcal{L}_0^+$, then $m \models \phi$.*

6 Conclusions

The main contributions of the present paper are the following results:
– The definition of dynamically-continuous metric bisimulation which is the correct distance-based counterpart of the concept of probabilistic/stochastic bisimulation.
– The definition of the topology of logical formulas canonically induced by the behavioural pseudometrics.
– Theorems that establish when parallel sequences of (probabilistic or stochastic) processes and formulas converge to give satisfaction *in the limit*; these results reveal important differences between the probabilistic and stochastic Markovian logics.
–Theorems regarding the relationships between logical formulas being at zero distance and logical equivalence/provability.
– Topological characterization of various classes of formulas.

There are many new things to explore. We currently prepare a coalgebraic presentation of this work that helped us understanding a *metric analogue of Stone duality* for Markov processes. These results are in preparation and are directly inspired by the present work. One topic that we have not understood to our satisfaction is the precise relationship between the kernel of the Hausdorff metric on formulas and provability. We would like to find a definition of the logical distance independent of the semantics and, in this sense, we exploit our previous works on completeness of Markovian logics. Another topic that we are currently explore is the relation with the approximation theory for Markov processes that, we believe, can highly benefit from this work.

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Appendix

In this appendix we have collected some basic definitions from topology and measure theory and the proofs of the major results presented in the paper.

(A1): $\vdash L_0\phi$
(A2): $\vdash L_r\top$
(A3): $\vdash L_r\phi \leftrightarrow M_{1-r}\neg\phi$
(A4): $\vdash L_r\phi \to \neg L_s\neg\phi,\ r+s>1$
(A5): $\vdash L_r(\phi\wedge\psi)\wedge L_s(\phi\wedge\neg\psi)\to L_{r+s}\phi,\ r+s\leq 1$
(A6): $\vdash \neg L_r(\phi\wedge\psi)\wedge\neg L_s(\phi\wedge\neg\psi)\to\neg L_{r+s}\phi,\ r+s\leq 1$
(R1): If $\vdash\phi\to\psi$ then $\vdash L_r\phi\to L_r\psi$
(R2): $\{\neg M_r\psi \mid r<s\}\vdash L_s\psi$

Table 1: The axiomatic system of DML

(B1): $\vdash L_0\phi$
(B2): $\vdash L_{r+s}\phi\to\neg M_r\phi,\ s>0$
(B3): $\vdash \neg L_r\phi\to M_r\phi$
(B4): $\vdash \neg L_r(\phi\wedge\psi)\wedge\neg L_s(\phi\wedge\neg\psi)\to\neg L_{r+s}\phi$
(B5): $\vdash \neg M_r(\phi\wedge\psi)\wedge\neg M_s(\phi\wedge\neg\psi)\to\neg M_{r+s}\phi$
(S1): If $\vdash\phi\to\psi$ then $\vdash L_r\phi\to L_r\psi$
(S2): $\{L_r\psi \mid r<s\}\vdash L_s\psi$
(S3): $\{M_r\psi \mid r>s\}\vdash M_s\psi$
(S4): $\{L_r\psi \mid r>s\}\vdash\perp$

Table 2: The axiomatic system of CML

Distances. Let (M,d) be a pseudometric space. The *open ball* with center $x\in M$ and radius $e>0$ is the set $\{y\in M \mid d(x,y)<e\}$. The class of open balls forms a basis for a topology called *the open ball topology* of d that we denote by $\mathcal{T}_d$.

Lemma. If d is a pseudometric, then the Hausdorff distance d^H is a pseudometric.

Proof: For arbitrary $A\subseteq M$, $d^{H/2}(A,A)=\sup_{a\in A}\inf_{a'\in A}d(a,a')=0$ since $\inf_{a'\in A}d(a,a')=0$. Consequently, $d^H(A,A)=0$.

Consider arbitrary $A,B,C\subseteq M$ and arbitrary $a\in A$, $b\in B$, $c\in C$. Then the triangle inequality for d guarantees that $d(a,c)\leq d(a,b)+d(b,c)$. Because $\inf_{c\in C}d(a,c)\leq d(a,c)$ for arbitrary $c\in C$, we obtain that $d^h(a,C)\leq d(a,b)+d(b,c)$. There exists $(b_k)_{k\in\mathbb{N}}\subseteq B$ such that $\lim_{k\to\infty}d(a,b_k)=d(a,B)$. We also have, for each k, $d^h(a,C)\leq d(a,b_k)+d(b_k,c)$ and going to the limit, we obtain that $d^h(a,C)\leq d^h(a,B)+\lim_{k\to\infty}d(b_k,c)$. Now we obtain $d^h(a,C)\leq d^h(a,B)+\inf_{c\in C}\lim_{k\to\infty}d(b_k,c)$, which is equivalent to $d^h(a,C)\leq d^h(a,B)+\lim_{k\to\infty}d^h(b_k,C)$. But because $(b_k)_{k\in\mathbb{N}}\subseteq B$, $\lim_{k\to\infty}d^h(b_k,C)\leq\sup_{b\in B}d^h(b,C)\leq d^{H/2}(B,C)$. Consequently, $d^h(a,C)\leq d^h(a,B)+d^{H/2}(B,C)$ implying further $d^{H/2}(A,C)\leq d^{H/2}(A,B)+d^{H/2}(B,C)$. Similarly, $d^{H/2}(C,A)\leq d^{H/2}(B,A)+d^{H/2}(C,B)$ implying further $d^H(A,C)\leq d^H(A,B)+d^H(B,C)$. □

Analytic Spaces. A *Polish space* is the topological space underlying a complete, separable metric space; i.e. it has a countable dense subset or equivalently a countable basis of open sets. An *analytic space* is the image of a Polish space under a continuous function from one Polish space to another. A good exposition of analytic spaces is contained in [Dud89] or [Arv76].

In this paper we will make use of analytic spaces to introduce Markov processes. The point of working with analytic spaces is that they have some remarkable properties that are needed for the proof of the logical characterization of bisimulation including the striking fact that one can characterize bisimulation with a very limited modal logic that has no infinite branching nor any kind of negative construct like negation, implication or even just the constant False [DEP02,CLM11a].

Proof (Lemma 1). 1. $d^H(A,B)=0$ iff $max\{d^{H/2}(A,B),d^{H/2}(B,A)\}=0$ iff $[d^{H/2}(A,B)=0$ and $d^{H/2}(B,A)=0]$ iff $[\sup\limits_{a\in A}\inf\limits_{b\in B} d(a,b)=0$ and $\sup\limits_{b\in B}\inf\limits_{a\in A} d(a,b)=0]$ iff [for arbitrary $a\in A$ and $b\in B$, $d(a,B)=d(b,A)=0$] iff [for arbitrary $a\in A$ and $b\in B$, $a\in\overline{B}$ and $b\in\overline{A}$] iff $\overline{A}=\overline{B}$.

2. We have the triangle inequalities $d^H(A,\overline{B})\leq d^H(A,B)+d^H(B,\overline{B})$ and $d^H(A,B)\leq d^H(A,\overline{B})+d^H(\overline{B},B)$. From 1 we have that $d^H(B,\overline{B})=d^H(\overline{B},B)=0$ implying $d^H(A,\overline{B})=d^H(A,B)$. The other equalities can be proved in the same way. □

Proof (Lemma 2). Notice that $(2^M,d^H)$ is a pseudometric space and, hence, not a Hausdorff space, so a convergent sequence might have more than one limit. In fact if a sequence of sets converges to some X and $d^H(X,Y)=0$ then the sequence will also converge to Y.

To prove our result, we prove for the beginning that for arbitrary $A\subseteq M$, $d^{H/2}(A,\bigcap\limits_{i\in\mathbb{N}}B_i)=\sup\limits_{i\in\mathbb{N}} d^{H/2}(A,B_i)$.

Let $B=\bigcap\limits_{i\in\mathbb{N}}B_i$. Consider arbitrary $a\in A$. Since $B_i\supseteq B_{i+1}\supseteq B$ for each i, the sequence $d^h(a,B_i)$ is increasing and $d^h(a,B)\geq d^h(a,B_i)$. Hence, $d^h(a,B_i)$ is convergent and $\lim\limits_{i\to\infty} d^h(a,B_i)=\sup\limits_{i\in\mathbb{N}} d^h(a,B_i)\leq d^h(a,B)$. Since $d^h(a,B_i)=\inf\limits_{b\in B_i} d(a,b)$ and B_i is compact, there exists $b_i\in B_i$ such that $d(a,b_i)=d^h(a,B_i)$. Consequently, $\lim\limits_{i\to\infty} d^h(a,B_i)=\lim\limits_{i\to\infty} d(a,b_i)$.

Let $b\in\lim\limits_{i\to\infty} b_i$, which exists (for a subsequence) since B_i are compact. Clearly, $b\in B_1$ and $(b_i)_{i\in\mathbb{N}}\subseteq B_1$. Suppose $b\notin B$. Then, there exists $k\in\mathbb{N}$ such that $b\in B_k$ and for any $i\geq 1$, $b\notin B_{k+i}$. Then, for all $i\geq k$, $b_i\in B_k\setminus B_{k+1}$, which is impossible since $b_i\in B_i$ converges to b.

Consequently, $b\in B$, $\lim\limits_{i\to\infty} d(a,b_i)=d(a,b)$ and since $d(a,b)\geq d^h(a,B)$, we obtain $\lim\limits_{i\to\infty} d^h(a,B_i)\geq d^h(a,B)$.

Hence, $\lim\limits_{i\to\infty} d^h(a,B_i)=d^h(a,B)$ implying $\sup\limits_{a\in A}\lim\limits_{i\to\infty} d^h(a,B_i)=\sup\limits_{a\in A} d^h(a,B)$ and further $d^{H/2}(A,B)=\sup\limits_{i\in\mathbb{N}} d^{H/2}(A,B_i)$.

Now we return to the result we want to prove.

Consider $x \in B$. This means that for any k, $x \in B_k$. We have $d^h(x, A) \le d^H(B_k, A)$ and the latter sequence of numbers converges to 0 by assumption. Therefore $d^h(x, A) = 0$, i.e. $x \in \overline{A}$. We have proved that $B \subseteq \overline{A}$ and hence $d^{H/2}(B, \overline{A}) = 0$.

We prove now that $d^{H/2}(\overline{A}, B) = 0$.
Since $d^H(A, B_k)$ converges to 0 by assumption, using Lemma 1, we obtain that $d^H(\overline{A}, B_k)$ converges to 0 and further that $d^{H/2}(\overline{A}, B_k)$ converges to 0. This implies that for any $a \in \overline{A}$, $d^h(a, B_k)$ converges to 0, since $d^h(a, B_k) \le d^{H/2}(\overline{A}, B_k)$.
Consequently, the sequence $d^h(a, B_k)$ of positive numbers converges to 0; moreover, this sequence is increasing, since $(B_k)_{k\in\mathbb{N}}$ is decreasing. Hence, for each $k \in \mathbb{N}$, $d^h(a, B_k) = 0$ and this is true for each $a \in \overline{A}$. From here we get that $d^{H/2}(\overline{A}, B_k) = 0$, for all $k \in \mathbb{N}$. Using 1, $d^{H/2}(\overline{A}, B) = \sup_{k\in\mathbb{N}} d^{H/2}(\overline{A}, B_k) = 0$.
We have proved that $d^{H/2}(B, \overline{A}) = d^{H/2}(\overline{A}, B) = 0$ which implies $d^H(\overline{A}, B) = 0$. Using Lemma 1, $d^H(A, B) = 0$. □

Proof (Lemma 3). Observe that $d^H(S', S'') = max\{\sup_{m'\in S'} d^h(m', S''), \sup_{m''\in S''} d^h(m'', S')\}$.
Because $m_k \in S_k$ implies $d_h(m_k, S) \le d^H(S_k, S)$ and because $\lim_{k\to\infty} d^H(S_k, S) = 0$, we obtain that $\lim_{k\to\infty} d_h(m_k, S) = 0$.
On the other hand, $\lim_{k\to\infty} m_k = m$ implies $\lim_{k\to\infty} d_h(m_k, S) = d_h(m, S)$, hence $d_h(m, S) = 0$. If S is closed, $d_h(m, S) = 0$ implies $m \in S$. □

Proof (Proposition 2). 1. Induction on $\phi \in \mathcal{L}[0, 1]^+$. The Boolean cases are trivial, since the entire universe $\mathcal{M} = [\![\top]\!]$ and the intersection and the union of two closed sets are closed.

[The case $\phi = L_r\psi$ for some $\psi \in \mathcal{L}$]: Suppose that $\lim_{k\to\infty} m_k = m$ and for each $k \in \mathbb{N}$, $m_k \models L_r\psi$. Because d is a dynamically-continuous metric bisimulation and $[\![\psi]\!] \in \Sigma(\sim)$, there exists a decreasing sequence $(S_k)_{k\in\mathbb{N}} \subseteq \Sigma(\sim)$ of compact sets in $\mathcal{T}_d$ such that $\lim_{k\to\infty} S_k \ni [\![\psi]\!]$ and $\lim_{k\to\infty} \theta(m_k)(S_k) = \theta(m)([\![\psi]\!])$. From Lemma 2, $d^H([\![\psi]\!], \bigcap_{k\in\mathbb{N}} S_k) = 0$ and using Lemma 1, $\overline{[\![\psi]\!]} = \bigcap_{k\in\mathbb{N}} S_k$. Hence, $[\![\psi]\!] \subseteq \overline{[\![\psi]\!]} \subseteq S_k$ for any k. Since $m_k \models L_r\psi$, $\theta(m_k)([\![\psi]\!]) \ge r$ implying $\theta(m_k)(S_k) \ge \theta(m_k)([\![\psi]\!])$ and further, $\theta(m_k)(S_k) \ge r$. Hence, $\lim_{k\to\infty} \theta(m_k)(S_k) \ge r$, implying $\theta(m)([\![\psi]\!]) \ge r$, i.e., $m \models L_r\psi$.

[The case $\phi = M_r\psi$ for some $\psi \in \mathcal{L}$]: from the soundness of $\vdash M_r\phi \leftrightarrow L_{1-r}\neg\phi$ we obtain $[\![M_r\phi]\!] = [\![L_{1-r}\neg\phi]\!]$. Now we can use the fact that $[\![L_{1-r}\neg\phi]\!]$ is open.

2. It is a direct consequence of 1, since $[\![\neg\psi]\!]$ is the complement of $[\![\psi]\!]$.

3. Because $\models \neg M_r\phi \to L_r\phi$, $[\![\neg M_r\phi]\!] \subseteq [\![L_r\phi]\!]$.
Consider a sequence $(m_k)_{k\in\mathbb{N}}$ of elements of $\mathcal{M}$ such that $\lim_{k\to\infty} m_k = m \in \mathcal{M}$ and for each

k, $m_k \models \neg M_r\phi$. To prove that $\overline{[\![\neg M_r\phi]\!]} = [\![L_r\phi]\!]$, we need to verify that $m \models L_r\psi$.
Since d is a dynamically-continuous metric bisimulation, there exists a decreasing sequence of compact sets $(S_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} S_k \ni [\![\psi]\!]$ and $\lim_{k\to\infty} \theta(m_k)(S_k) = \theta(m)([\![\psi]\!])$.
Because $S_k \supseteq [\![\psi]\!]$ (see the proof of 1.), $\theta(m_k)(S_k) \geq \theta(m_k)([\![\psi]\!])$ and from $m_k \models \neg M_r\psi$ we get $\theta(m_k)([\![\psi]\!]) > r$. Hence, $\theta(m_k)(S_k) > r$, implying $\lim_{k\to\infty} \theta(m_k)(S_k) \geq r$. This is equivalent to $\theta(m)([\![\psi]\!]) \geq r$, i.e., $m \models L_r\psi$.
To prove the other equality, notice that using axiom (A3) we have $\overline{[\![\neg L_r\phi]\!]} = \overline{[\![\neg M_{1-r}\neg\phi]\!]}$. We apply the first equality and obtain $\overline{[\![\neg M_{1-r}\neg\phi]\!]} = [\![L_{1-r}\neg\phi]\!]$ and using again axiom (A3), $[\![L_{1-r}\neg\phi]\!] = [\![M_r\phi]\!]$.

4. It is a direct consequence of 3. □

Proof (Lemma 4). 1. Because $\models \neg M_r\phi \to L_r\phi$, $[\![\neg M_r\phi]\!] \subseteq [\![L_r\phi]\!]$.
Consider a sequence $(m_k)_{k\in\mathbb{N}}$ of elements of $\mathcal{M}$ such that $\lim_{k\to\infty} m_k = m \in \mathcal{M}$ and for each k, $m_k \models \neg M_r\phi$. To prove that $\overline{[\![\neg M_r\phi]\!]} = [\![L_r\phi]\!]$, we need to verify that $m \models L_r\psi$.
Since d is a dynamically-continuous metric bisimulation, there exists a decreasing sequence $(S_k)_{k\in\mathbb{N}}$ of compact elements of $\Sigma(\sim)$ such that $\lim_{k\to\infty} S_k \ni [\![\psi]\!]$ and $\lim_{k\to\infty} \theta(m_k)(S_k) = \theta(m)([\![\psi]\!])$.
Because $S_k \supseteq [\![\psi]\!]$, $\theta(m_k)(S_k) \geq \theta(m_k)([\![\psi]\!])$ and from $m_k \models \neg M_r\psi$ we get $\theta(m_k)([\![\psi]\!]) > r$. Hence $\theta(m_k)(S_k) > r$, implying $\lim_{k\to\infty} \theta(m_k)(S_k) \geq r$. This is equivalent to $\theta(m)([\![\psi]\!]) \geq r$, i.e., $m \models L_r\psi$.

2. It is a direct consequence of 1.

3. It is a consequence of the fact that $[0, r] = \bigcap_{k\in\mathbb{N}} [0, r + \frac{1}{k})$.

4. This follows from $[r, \infty) = \bigcap_{k\in\mathbb{N}} (r - \frac{1}{k}, \infty)$. □

Proof (Theorem 5). 1. Induction on $\phi \in \mathcal{L}_0^+$. The Boolean cases are trivial and use the fact that the entire universe $\mathcal{M} = [\![\top]\!]$, hence it is closed and that the intersection and the union of two closed sets are closed.

The case $\phi = L_r\psi$ for some $\psi \in \mathcal{L}$: Suppose that $\lim_{k\to\infty} m_k = m$ and for each $k \in \mathbb{N}$, $m_k \models L_r\psi$; we have to prove that $m \models L_r\psi$.
Because d is a dynamically-continuous metric bisimulation, $\lim_{k\to\infty} m_k = m$ and $[\![\psi]\!] \in \Sigma(\sim)$, there exists a decreasing sequence $(S_k)_{k\in\mathbb{N}} \subseteq \Sigma(\sim)$ of compact sets in $\mathcal{T}_d$ such that $\lim_{k\to\infty} S_k \ni [\![\psi]\!]$ and $\lim_{k\to\infty} \theta(m_k)(S_k) = \theta(m)([\![\psi]\!])$.
From Lemma 2, $d^H([\![\psi]\!], \bigcap_{k\in\mathbb{N}} S_k) = 0$ and using Lemma 1, $\overline{[\![\psi]\!]} = \bigcap_{k\in\mathbb{N}} S_k$, since $\bigcap_{k\in\mathbb{N}} S_k$ is closed. Hence, $[\![\psi]\!] \subseteq \overline{[\![\psi]\!]} \subseteq S_k$ for any k.

Since $m_k \models L_r\psi$, $\theta(m_k)([\![\psi]\!]) \geq r$. $[\![\psi]\!] \subseteq S_k$ implies $\theta(m_k)(S_k) \geq \theta(m_k)([\![\psi]\!])$ and further, $\theta(m_k)(S_k) \geq r$. Hence, $\lim_{k\to\infty} \theta(m_k)(S_k) \geq r$, implying $\theta(m)([\![\psi]\!]) \geq r$, i.e., $m \models L_r\psi$.

2. It is a direct consequence of 1 since $\psi \in \mathcal{L}_0^-$ iff $\neg\psi \in \mathcal{L}_0^+$ and $[\![\neg\psi]\!]$ is the complement of $[\![\psi]\!]$.

3. Induction on $\phi \in \mathcal{L}^+$. The Boolean cases are trivial, since the union and the intersection of two countable interactions of open sets is a countable intersection of open sets.

The case $\phi \in \mathcal{L}_0^+$: follows from 1, since any closed set in a pseudometrizable space is G_δ.

The case $\phi = M_r\psi$ for some $\psi \in \mathcal{L}$: From Lemma 4, $[\![M_r\psi]\!] = \bigcap_{k\in\mathbb{N}} [\![\neg L_{r+\frac{1}{k}}\psi]\!]$. Because $[\![\neg L_{r+\frac{1}{k}}\psi]\!]$ are open, we obtain that $[\![M_r\psi]\!]$ is a G_δ set.

4. This is a consequence of 3, since $[\![\neg\psi]\!]$ is the complement of $[\![\psi]\!]$. $\square$