Stone Duality for Markov Processes

Dexter Kozen*, Kim G. Larsen†, Radu Mardare† and Prakash Panangaden‡

*Computer Science Department, Cornell University, Ithaca, New York, USA
†Department of Computer Science, University of Aalborg, Aalborg, Denmark
‡School of Computer Science, McGill University, Montreal, Canada

Abstract

We define Aumann algebras, an algebraic analog of probabilistic modal logic. An Aumann algebra consists of a Boolean algebra with operators modeling probabilistic transitions. We prove a Stone-type duality theorem between countable Aumann algebras and countably-generated continuous-space Markov processes. Our results subsume existing results on completeness of probabilistic modal logics for Markov processes.

1. Introduction

For Markov processes, the natural logic is a simple modal logic with probability bounds on the modalities. It is therefore tempting to understand this logic algebraically in the same way that Boolean algebras capture propositional reasoning and the Jonsson-Tarski [1] results give duality for algebras arising from modal logics.

In this paper, we develop a Stone-type duality for continuous-space probabilistic transitions systems and a certain kind of algebra that we have named *Aumann algebras*. These are Boolean algebras with operators that behave like probabilistic modalities. Recent papers [2–4] have established completeness theorems and finite model theorems for similar logics.

A comparison with related work appears in §7. We note here that we go beyond existing completeness results [2–4] in a number of ways. The strong completeness theorems of Goldblatt [3] use a powerful infinitary axiom scheme with an uncountable set of instances and establish the results contingent on the assumption that every consistent set of formulas can be expanded to a maximally consistent set (Lindenbaum's lemma). In our version we show that this assumption can be proved. The key point is that we use different infinitary axioms that have only countably many instances. This allows us to use the Rasiowa–Sikorski lemma [5] to establish our results without needing to assume Lindenbaum's lemma.

Our key results are:

- a description of a new class of algebras that captures, in algebraic form, the probabilistic modal logics used for continuous-state Markov processes;
- a version of the duality for countable algebras and a certain class of countably-generated Markov processes; and
- a complete axiomatization where the infinitary axiom schemes have only uncountably many instances.

The duality is represented in the diagram below. Here $\mathbf{SMP}$ stands for countably based Stone Markov processes and $\mathbf{AA}$ for countable Aumann algebras. The formal definitions are given in §§3–4.

$$\mathbf{SMP} \underset{\mathbb{M}}{\overset{\mathbb{A}}{\rightleftarrows}} \mathbf{AA}^{\mathrm{op}}$$

1.1. A Technical Summary

The duality theorem proved in this paper has some novel features that distinguish it from many others that have appeared in the literature.

We have avoided the assumption that every consistent set of formulas can be expanded to a maximal consistent set axioms [6] by using the Rasiowa–Sikorski lemma (whose proof uses the Baire category theorem) in the following way. In going from the algebra to the dual Markov process, we look at ultrafilters that do *not* respect the infinitary axioms of Aumann algebras. We call these *bad* ultrafilters. We show that these form a meager set (in the standard topological sense) and can be removed without affecting the transition probabilities that we are trying to define. Countability is essential here. In order to show that we do not affect the algebra of clopen sets by doing this, we introduce a distinguished base of clopen sets in the definition of Markov process, which has to satisfy some conditions. We show that this forms an Aumann algebra. We are able to go from a Markov process to an Aumann algebra by using this distinguished base. Morphisms of Markov processes are required to preserve distinguished base elements backwards; that is, if $f : \mathcal{M} \to \mathcal{N}$ and $A \in \mathcal{A}_{\mathcal{N}}$, then $f^{-1}(A) \in \mathcal{A}_{\mathcal{M}}$. Thus we get Boolean algebra homomorphisms in the dual for free.

Removing bad points has the effect of destroying compactness of the resulting topological space. We introduce a new concept called *saturation* that takes the place of compactness. The idea is that a saturated model has *all* the good ultrafilters. The Stone dual of an Aumann algebra is saturated, because it is constructed that way. However, it is possible to have a Markov process that is unsaturated but still represents the same algebra. For example, we removed bad points and could, in principle, remove a few more; as long as the remaining points are still dense, we have not changed the algebra. One can saturate a model by a process akin to compactification. We explicitly describe how to do this below.

2. Background

In this section we present background from measure theory and topology. For proofs we refer the reader to [7] or [8]. We do not discuss the Stone duality theorem; this is discussed elsewhere in this volume.

We use $\mathbb{Q}_0$ to denote the set $\mathbb{Q} \cap [0, 1]$.

Measurable Spaces and Measures

Let M be an arbitrary nonempty set. We assume that the basic notions like *field of sets*, σ-algebra, measurable set and measurable function are know. Similarly with topology, open and closed set and continuous function and the Borel algebra of a topology. We use $[\![M \to N]\!]$ to denote the family of measurable functions from (M, Σ) to (N, Ω).

If $\Omega \subseteq 2^M$, the *σ-algebra generated by* Ω, denoted Ω^σ, is the smallest σ-algebra containing Ω.

Let $\mathbb{R}^+ = \{r \in \mathbb{R} \mid r \geq 0\}$. A nonnegative real-valued function μ defined on a collection of sets (a set function) is *finitely additive* if $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A \cap B = \varnothing$. We say that μ is *countably subadditive* if $\mu(\bigcup_i A_i) \leq \sum_i \mu(A_i)$ for a countable family of measurable sets, and we say that μ is *countably additive* if $\mu(\cup_i A_i) = \sum_i \mu(A_i)$ for a countable *pairwise-disjoint* family of measurable sets. A *measure* on a measurable space $\mathcal{M} = (M, \Sigma)$ is a countably additive set function $\mu : \Sigma \to \mathbb{R}^+$. A measure is a *probability measure* if in addition $\mu(M) = 1$. We use $\Delta(M, \Sigma)$ to denote the set of probability measures on (M, Σ).

A fundamental fact that we use is about extending set functions to measures. This is Theorem 11.3 of [7]. It says that a finitely additive and countably subadditive function on a field of sets can be uniquely extended to a measure on the σ-algebra generated by the field.

We can view $\Delta(M, \Sigma)$ as a measurable space by considering the σ-algebra generated by the sets $\{\mu \in \Delta(M, \Sigma) \mid \mu(S) \geq r\}$ for $S \in \Sigma$ and $r \in [0, 1]$. This is the least σ-algebra on $\Delta(M, \Sigma)$ such that all maps $\mu \mapsto \mu(S) : \Delta(M, \Sigma) \to [0, 1]$ for $S \in \Sigma$ are measurable, where the real interval $[0, 1]$ is endowed with the σ-algebra generated by all rational intervals.

Every topological space has a natural σ-algebra associated with it, namely the one generated by the open sets. This is called the *Borel algebra* of the space, and the measurable sets are called *Borel sets*.

Recall that a topological space is said to be *separable* if it contains a countable dense subset and *second countable* if its topology has a countable base. Second countability implies separability, but not vice versa in general; however, the two concepts coincide for metric spaces. A *Polish space* is the topological space underlying a complete separable metric space.

An *analytic space* is a continuous image of a Polish space in another Polish space. More precisely, if X and Y are Polish spaces and $f : X \to Y$ is continuous, then the image $f(X)$ is an analytic space. Remarkably, one does not get a broader class by allowing f to be merely measurable instead of continuous and by taking the image of a Borel subset of X instead of X.

Analytic spaces enjoy remarkable properties that were crucial in proving the logical characterization of bisimulation [9, 10]. We note that the completeness theorems proved in [2, 11, 12] were established for Markov processes defined on analytic spaces.

The Baire Category Theorem

The Baire category theorem is a topological result with important applications in logic. It can be used to prove the Rasiowa–Sikorski lemma [5] that is central for our paper.

A subset D of a topological space X is *dense* if its closure $\overline{D}$ is all of X. Equivalently, a dense set is one intersecting every nonempty open set. A set $N \subseteq X$ is *nowhere dense* if every nonempty open set contains a nonempty open subset disjoint from N. A set is said to be *of the first category* or *meager* if it is a countable union of nowhere dense sets. A basic fact that we use is that the boundary of an open set is nowhere dense. A *Baire space* is one in which the intersection of countably many dense open sets is dense. For us, the relevant fact is: every compact Hausdorff space is Baire.

Definition 1. *Let $\mathcal{B}$ be a Boolean algebra and let $T \subseteq \mathcal{B}$ be such that T has a greatest lower bound $\bigwedge T$ in $\mathcal{B}$. An ultrafilter (maximal filter) U is said to* respect *T if $T \subseteq U$ implies that $\bigwedge T \in U$.*

If $\mathcal{T}$ is a family of subsets of $\mathcal{B}$, we say that an ultrafilter U respects $\mathcal{T}$ if it respects every member of $\mathcal{T}$.

Theorem 2 (Rasiowa–Sikorski lemma [5])**.** *For any Boolean algebra $\mathcal{B}$ and any countable family $\mathcal{T}$ of subsets of $\mathcal{B}$, each member of which has a meet in $\mathcal{B}$, and for any nonzero $x \in \mathcal{B}$, there exists an ultrafilter in $\mathcal{B}$ that contains x and respects $\mathcal{T}$.*

This lemma was later proved by Tarski in a purely algebraic way. See [3] for a discussion of the role of the Baire category theorem in the proof.

3. Markov Processes and Markovian Logic

Markov processes (MPs) are models of probabilistic systems with a continuous state space and probabilistic transitions [9, 10, 13]. In earlier papers, they were called *labeled* Markov processes to emphasize the fact that there were multiple possible actions, but here we will suppress the labels, as they do not contribute any relevant structure for our results.

Definition 3 (Markov process)**.** *A* Markov process (MP) *is a tuple $\mathcal{M} = (M, \Sigma, \theta)$, where (M, Σ) is an analytic space and $\theta \in [\![M \to \Delta(M, \Sigma)]\!]$.*

In a Markov process $\mathcal{M} = (M, \Sigma, \theta)$, M is the *support set*, denoted by $\mathrm{supp}(\mathcal{M})$, and θ is the *transition function*. For $m \in M$, $\theta(m) : \Sigma \to [0, 1]$ is a probability measure on the state space (M, Σ). For $N \in \Sigma$, the value $\theta(m)(N) \in [0, 1]$ represents the probability of a transition from m to a state in N.

The condition that θ is a measurable function $[\![M \to \Delta(M, \Sigma)]\!]$ is equivalent to the condition that for fixed $N \in \Sigma$, the function $m \mapsto \theta(m)(N)$ is a measurable function $[\![M \to [0, 1]]\!]$ (see e.g. Proposition 2.9 of [13]).

Given two Markov processes $\mathcal{M}_i = (M_i, \Sigma_i, \theta_i)$, $i = 1, 2$, a surjective measurable function $f : M_1 \to M_2$ is a *zig-zag* if for any $m \in M_1$ and $B \in \Sigma_2$, $\theta_1(m)(f^{-1}(B)) = \theta_2(f(m))(B)$. Such a map is essentially a functional version of bisimulation [9].

Definition 4. *A* span *in a category is a pair of morphisms $f : A \to B$ and $g : A \to C$ with a common domain. Two Markov processes $\mathcal{M}_1, \mathcal{M}_2$ are said to be* bisimilar *if there is a third Markov process $\mathcal{M}$ and a span of zig-zags $f_i : \mathcal{M} \to \mathcal{M}_i$, $i = 1, 2$. Two states $m_i \in \mathrm{supp}(\mathcal{M}_i)$, $i = 1, 2$, are said to be* bisimilar *if there exist a span of zig-zags $f_i : \mathcal{M} \to \mathcal{M}_i$, $i = 1, 2$ and $m \in \mathrm{supp}(\mathcal{M})$ such that $m_i = f_i(m)$, $i = 1, 2$. We write $(\mathcal{M}_1, m_1) \approx (\mathcal{M}_2, m_2)$ to indicate that m_1 and m_2 are bisimilar in this sense.*

In the context of analytic spaces, one can define bisimulation between the states of a Markov process using a relational definition [9, 10, 14].

Markovian logic (ML) is a multi-modal logic for semantics based on MPs [2, 4, 11, 14–17]. In addition to the Boolean operators, this logic is equipped with *probabilistic modal operators* L_r for $r \in \mathbb{Q}_0$ that bound the probabilities of transitions. Intuitively, the formula $L_r\varphi$ is satisfied by $m \in \mathcal{M}$ whenever the probability of a transition from m to a state satisfying φ is at least r.

Definition 5 (Syntax)**.** *The formulas of $\mathcal{L}$ are defined, for a set $\mathcal{P}$ of atomic propositions, by the grammar*

$$\varphi \ ::= \ p \mid \bot \mid \varphi \to \varphi \mid L_r\varphi$$

where p can be any element of $\mathcal{P}$ and r of $\mathbb{Q}_0$.

The Boolean operators $\vee$, $\wedge$, $\neg$, and $\top$ are defined from $\to$ and $\bot$ as usual. For $r_1, \ldots, r_n \in \mathbb{Q}_0$ and $\varphi \in \mathcal{L}$, let

$$L_{r_1 \cdots r_n}\varphi = L_{r_1} \cdots L_{r_n}\varphi.$$

The *Markovian semantics* for $\mathcal{L}$ is defined as follows. For MP $\mathcal{M} = (M, \Sigma, \theta)$, $m \in M$ and an interpretation function $i : M \to 2^{\mathcal{P}}$,

- $\mathcal{M}, m, i \vDash p$ if $p \in i(m)$,
- $\mathcal{M}, m, i \vDash \bot$ never,
- $\mathcal{M}, m, i \vDash \varphi \to \psi$ if $\mathcal{M}, m, i \vDash \psi$ whenever $\mathcal{M}, m, i \vDash \varphi$,
- $\mathcal{M}, m, i \vDash L_r\varphi$ if $\theta(m)([\![\varphi]\!]) \geq r$, where $[\![\varphi]\!] = \{m \in M \mid \mathcal{M}, m, i \vDash \varphi\}$.

For the last clause to make sense, $[\![\varphi]\!]$ must be measurable. This is guaranteed by the fact that $\theta \in [\![M \to \Delta(M, \Sigma)]\!]$ (see e.g. [2]).

Given $\mathcal{M}$ and i, we say that $m \in \mathrm{supp}(\mathcal{M})$ *satisfies* φ if $\mathcal{M}, m, i \vDash \varphi$. We write $\mathcal{M}, m, i \nvDash \varphi$ if not $\mathcal{M}, m, i \vDash \varphi$ and $\mathcal{M}, m, i \vDash \Phi$ if $\mathcal{M}, m, i \vDash \varphi$ for all $\varphi \in \Phi$. We write $\Phi \vDash \varphi$ if for any $\mathcal{M}$ and i, $\mathcal{M}, m, i \vDash \varphi$ whenever $\mathcal{M}, m, i \vDash \Phi$. A formula or set of formulas is *satisfiable* if there exist an MP $\mathcal{M}$, $m \in \mathrm{supp}(\mathcal{M})$ and i that satisfies it. We say that φ is *valid* and write $\vDash \varphi$ if $\neg\varphi$ is not satisfiable.

We now present an axiomatization of ML for Markovian semantics. The system is a Hilbert-style system consisting of the axioms and rules of propositional modal logic and the axioms and rules listed in Table 1. The axioms and the rules are stated for arbitrary $\varphi, \psi \in \mathcal{L}$ and arbitrary $r, s \in \mathbb{Q}_0$.

(A1) $\vdash L_0\varphi$

(A2) $\vdash L_r\top$

(A3) $\vdash L_r\varphi \to \neg L_s\neg\varphi,\ r+s>1$

(A4) $\vdash L_r(\varphi\wedge\psi)\wedge L_s(\varphi\wedge\neg\psi) \to L_{r+s}\varphi,\ r+s\leq 1$

(A5) $\neg L_r(\varphi\wedge\psi)\wedge\neg L_s(\varphi\wedge\neg\psi) \to \neg L_{r+s}\varphi,\ r+s\leq 1$

(R1) $\dfrac{\vdash \varphi\to\psi}{\vdash L_r\varphi\to L_r\psi}$

(R2) $\{L_{r_1\cdots r_n r}\psi \mid r<s\} \vdash L_{r_1\cdots r_n s}\psi$

Table 1. Axioms of $\mathcal{L}$

If $\Phi \subseteq \mathcal{L}$ and $\varphi \in \mathcal{L}$, we write $\Phi \vdash \varphi$ and say that Φ *derives* φ if φ is provable from the axioms and the extra assumptions Φ. We write $\vdash \varphi$ if $\varnothing \vdash \varphi$.

A formula or set of formulas is *consistent* if it cannot derive $\perp$. We say that $\Phi \subseteq \mathcal{L}$ is *maximally consistent* if it is consistent and it has no proper consistent extensions. The set Φ is *filtered* if for all $\varphi, \psi \in \Phi$ there exists $\rho \in \Phi$ with $\vdash \rho \to \varphi\wedge\psi$.

The (strong) completeness of this logic is proved in [4, 11] by assuming Lindenbaum's lemma and using the following stronger version of (R1) for filtered $\Phi \subseteq \mathcal{L}$ proposed in [6]:

$$\frac{\Phi\vdash\varphi}{L_r\Phi\vdash L_r\varphi}$$

where $L_r\Phi = \{L_r\psi \mid \psi\in\Phi\}$. A consequence of our duality theorem is (strong) completeness of ML with the axiomatization in Table 1.

The logical equivalence induced by ML on the class of MPs coincides with bisimulation equivalence [9, 10]. The proof requires that the state space be an analytic space.

Theorem 6 (Hennessy-Milner)**.** *Given two MPs $\mathcal{M}_i$ and $m_i \in \mathrm{supp}(\mathcal{M}_i)$, $i=1,2$, $(\mathcal{M}_1,m_1) \approx (\mathcal{M}_2,m_2)$ iff for all $\varphi\in\mathcal{L}$,*

$$\mathcal{M}_1,m_1\models\varphi \ \Leftrightarrow\ \mathcal{M}_2,m_2\models\varphi.$$

4. Aumann Algebras

In this section we introduce an algebraic version of Markovian logic consisting of Boolean algebra with operators F_r for $r\in\mathbb{Q}_0$ corresponding to the operators L_r of ML. We call this Aumann Algebra (AA) in honor of Robert Aumann, who has made fundamental contributions to probabilistic logic [15].

4.1. Definition of Aumann Algebras

Definition 7 (Aumann algebra)**.** *An* Aumann algebra (AA) *is a structure $\mathcal{A} = (A, \to, \perp, \{F_r\}_{r\in\mathbb{Q}_0}, \leq)$ where*

- *$(A,\to,\perp,\leq)$ is a Boolean algebra;*
- *for each $r\in\mathbb{Q}_0$, $F_r : A\to A$ is a unary operator; and*
- *the axioms in Table 2 hold for all $a,b\in A$ and $r,s,r_1,\ldots,r_n\in\mathbb{Q}_0$.*

The Boolean operations $\vee$, $\wedge$, $\neg$, and $\top$, are defined from $\to$ and $\perp$ as usual.

Morphisms of Aumann algebras are Boolean algebra homomorphisms that commute with the operations F_r. The category of Aumann algebras and Aumann algebra homomorphisms is denoted **AA**.

We abbreviate $F_{r_1}\cdots F_{r_n}a$ by $F_{r_1\cdots r_n}a$.

(AA1) $\top \leq F_0 a$

(AA2) $\top \leq F_r\top$

(AA3) $F_r a \leq \neg F_s\neg a,\ r+s>1$

(AA4) $F_r(a\wedge b)\wedge F_s(a\wedge\neg b) \leq F_{r+s}a,\ r+s\leq 1$

(AA5) $\neg F_r(a\wedge b)\wedge\neg F_s(a\wedge\neg b) \leq \neg F_{r+s}a,\ r+s\leq 1$

(AA6) $a\leq b \Rightarrow F_r a\leq F_r b$

(AA7) $(\bigwedge_{r<s} F_{r_1\cdots r_n r}a) = F_{r_1\cdots r_n s}a$

Table 2. Aumann algebra

The operator F_r is the algebraic counterpart of the logical modality L_r. The first two axioms state tautologies, while the third captures the way F_r interacts with negation. Axioms (AA4) and (AA5) assert finite additivity, while (AA6) asserts monotonicity.

The most interesting axiom is the infinitary axiom (AA7). It asserts that $F_{r_1\cdots r_n s}a$ is the greatest lower bound of the set $\{F_{r_1\cdots r_n r}a \mid r<s\}$ with respect to the natural order $\leq$. In SMPs, it will imply countable additivity.

The following lemma establishes some basic consequences.

Lemma 8. *Let $\mathcal{A} = (A,\to,\perp,\{F_r\}_{r\in\mathbb{Q}_0},\leq)$ be an Aumann algebra. For all $a,b\in A$ and $r,s\in\mathbb{Q}_0$,*

(i) *$F_r\perp = \perp$ for $r>0$;*
(ii) *if $r\leq s$, then $F_s a\leq F_r a$;*
(iii) *if $a\leq\neg b$ and $r+s>1$, then $F_r a\leq\neg F_s b$.*

As expected, the formulas of Markovian logic modulo logical equivalence form a free Aumann algebra that is countable. Define $\equiv$ on formulas by: $\varphi\equiv\psi$ if $\vdash\varphi\to\psi$ and $\vdash\psi\to\varphi$. Let $[\varphi]$ denote the equivalence class of φ modulo $\equiv$, and let $\mathcal{L}/{\equiv} = \{[\varphi] \mid \varphi \in \mathcal{L}\}$. By (R1), the modality L_r is well defined on $\equiv$-classes. The Boolean operators are also well defined by considerations of propositional logic.

Theorem 9. *The structure*

$$(\mathcal{L}/{\equiv}, \to, [\bot], \{L_r\}_{r\in\mathbb{Q}_0}, \leq)$$

is an Aumann algebra, where $[\varphi] \leq [\psi]$ *iff* $\vdash \varphi \to \psi$.

5. Stone Markov Processes

In our duality theory, we work with Markov processes constructed from certain zero-dimensional Hausdorff spaces. We call such structures *Stone–Markov processes* (SMPs).

5.1. MPs with Distinguished Base

We restrict our attention to Markov processes $(M, \mathcal{A}, \theta)$, where $\mathcal{A}$ is a distinguished countable base of clopen sets that is closed under the set-theoretic Boolean operations and the operations

$$F_r(A) = \{m \mid \theta(m)(A) \geq r\}, \quad r \in \mathbb{Q}_0.$$

The measurable sets Σ are the Borel sets of the topology generated by $\mathcal{A}$. Morphisms of such spaces are required to preserve the distinguished base; thus a morphism $f : \mathcal{M} \to \mathcal{N}$ is a continuous function such that

- for all $m \in M$ and $B \in \Sigma_{\mathcal{N}}$,
$$\theta_{\mathcal{M}}(m)(f^{-1}(B)) = \theta_{\mathcal{N}}(f(m))(B);$$
- for all $A \in \mathcal{A}_{\mathcal{N}}$, $f^{-1}(A) \in \mathcal{A}_{\mathcal{M}}$.

5.2. Saturation

Unlike Stone spaces, SMPs are not topologically compact, but we do postulate a completeness property that is a weak form of compactness, which we call *saturation*. One can saturate a given SMP by a completion procedure that is reminiscent of Stone–Čech compactification. Intuitively, one adds points to the structure without changing the represented algebra. An MP is *saturated* if it is maximal with respect to this operation.

One can define the saturation by completing by a certain family of ultrafilters of called *good ultrafilters*. These are ultrafilters respecting the infinitary condition (AA7) in the definition of Aumann algebras (§4). All principal ultrafilters of an SMP are already good, and one must only add the rest. The details of this construction are given in §6. One can give a more conceptual definition of saturation; we will do this in the full paper.

5.3. Definition of SMP

Definition 10 (Stone–Markov Process)**.** *A Markov process* $\mathcal{M} = (M, \mathcal{A}, \theta)$ *with distinguished base is a* Stone–Markov process (SMP) *if it is saturated.*

The morphisms of SMPs are just the morphisms of MPs with distinguished base as defined above.

The category of SMPs and SMP morphisms is denoted **SMP**.

6. Stone Duality

In this section we describe the duality between SMPs and countable AAs. This is in the spirit of the classical Stone representation theorem [18], or, more precisely, the representation theorem of Jonsson and Tarski [1] for Boolean algebras with operators. Here the details are somewhat different, as we must deal with measure theory.

6.1. From AAs to SMPs

For this subsection, we fix an arbitrary countable Aumann algebra

$$\mathcal{A} = (A, \to, \bot, \{F_r\}_{r\in\mathbb{Q}_0}, \leq).$$

Let $\mathcal{U}^*$ be the set of all ultrafilters of $\mathcal{A}$. The classical Stone construction gives a Boolean algebra of sets isomorphic to $\mathcal{A}$ with elements

$$(\!|a|\!)^* = \{u \in \mathcal{U}^* \mid a \in u\}, \ a \in \mathcal{A}$$
$$(\!|\mathcal{A}|\!)^* = \{(\!|a|\!)^* \mid a \in A\}.$$

The sets $(\!|a|\!)^*$ generate a Stone topology τ^* on $\mathcal{U}^*$, and the $(\!|a|\!)^*$ are exactly the clopen sets of the topology.

Let $\mathcal{F}$ be the set of elements of the form $\alpha^r = F_{t_1\cdots t_n r}a$ for $a \in A$ and $t_1, \ldots, t_n, r \in \mathbb{Q}_0$. As before, we consider this term as parameterized by r; that is, if $\alpha^r = F_{t_1\cdots t_n r}a$, then α^s denotes $F_{t_1\cdots t_n s}a$. The set $\mathcal{F}$ is countable since A is. Axiom (AA7) asserts all infinitary conditions of the form

$$\alpha^s = \bigwedge_{r<s} \alpha^r. \tag{1}$$

for $\alpha^s \in \mathcal{F}$. Let us call an ultrafilter u *bad* if it violates one of these conditions in the sense that for some $\alpha^s \in \mathcal{F}$, $\alpha^r \in u$ for all $r < s$ but $\alpha^s \notin u$. Otherwise, u is called *good*. Let $\mathcal{U}$ be the set of good ultrafilters of $\mathcal{A}$.

Let $\tau = \{B \cap \mathcal{U} \mid B \in \tau^*\}$ be the subspace topology on $\mathcal{U}$, and let

$$(\!|a|\!) = \{u \in \mathcal{U} \mid a \in u\} = (\!|a|\!)^* \cap \mathcal{U}$$
$$(\!|\mathcal{A}|\!) = \{(\!|a|\!) \mid a \in A\}.$$

Then τ is countably generated by the sets $(\!|a|\!)$ and all $(\!|a|\!)$ are clopen in the subspace topology.

Lemma 11. *If $\overline{S}$ denotes the closure of S in τ^*, then*

$$(\!|\neg\alpha^s|\!)^* = \overline{\bigcup_{r<s} (\!|\neg\alpha^r|\!)^*}.$$

Proof: ($\supseteq$):From Lemma 8 and (AA6) we have that $\neg\alpha^r \leq \neg\alpha^s$ for any $r < s$. Consequently, for any $u \in \mathcal{U}^*$, $u \ni \neg\alpha^r$ implies $u \ni \neg\alpha^r$.
($\subseteq$): It is sufficient to prove that for every open set $B \in \tau^*$, if $B \cap (\!|\neg\alpha^s|\!) \neq \varnothing$, then $B \cap (\!|\neg\alpha^r|\!) \neq \varnothing$ for some $r < s$. Proving this for $B \in (\!|\mathcal{A}|\!)^*$ is sufficient.

Let $B = (\!|a|\!)^*$ and suppose that $(\!|a|\!)^* \cap (\!|\neg\alpha^s|\!) \ni u$. Then $a \wedge \neg\alpha^s \in u$, implying $a \wedge \neg\alpha^s \neq \bot$. Applying Rasiowa-Sikorski lemma we obtain that there exists $v \in \mathcal{U}$ such that $a \wedge \neg\alpha^s \in v$. Consequently, $a, \neg\alpha^s \in v$ and since v is a good ultrafilter, there exists $r < s$ such that $\neg\alpha^r \in v$. Hence $v \in (\!|a|\!)^* \cap (\!|\neg\alpha^s|\!) \neq \varnothing$. □

The next lemma asserts that the set $\mathcal{U}^* \setminus \mathcal{U}$ of bad ultrafilters is meager. We will use this to argue that $\mathcal{U}$ is dense in $\mathcal{U}^*$, therefore no $(\!|a|\!)$ vanishes as a result of dropping the bad points from $(\!|a|\!)^*$. It will follow that $(\!|\mathcal{A}|\!)$ and $(\!|\mathcal{A}|\!)^*$ are isomorphic as Boolean algebras.

Lemma 12. *The set $\mathcal{U}^* \setminus \mathcal{U}$ is of first category in the Stone topology τ^*. In particular, $\mathcal{U}$ is dense in $\mathcal{U}^*$.*

Proof: We must prove that $\mathcal{U}^* \setminus \mathcal{U}$ is a countable union of nowhere dense sets. Since A is countable, the set $\mathcal{F}$ is countable as well. Each bad ultrafilter $u \in \mathcal{U}^* \setminus \mathcal{U}$ violates at least one constraint (1), thus

$$\mathcal{U}^* \setminus \mathcal{U} = \bigcup_{\alpha^s \in \mathcal{F}} U_{\alpha^s},$$

where

$$\begin{aligned} U_{\alpha^s} &= \{u \in \mathcal{U}^* \mid \alpha^s \notin u \text{ and } \forall r < s\ \alpha^r \in u\} \\ &= (\!|\neg\alpha^s|\!)^* \setminus \bigcup_{r<s} (\!|\neg\alpha^r|\!)^*. \end{aligned}$$

Now we argue that each U_{α^s} is nowhere dense. In τ^*, $(\!|\neg\alpha^s|\!)^*$ is a closed set while $\bigcup_{r<s} (\!|\neg\alpha^r|\!)^*$ is a countable union of open sets, hence open. Applying Lemma 11, U_{α^s} is the boundary of an open set, hence nowhere dense. □

Since $(\mathcal{U}, \tau)$ is a subspace of the Stone space $(\mathcal{U}^*, \tau^*)$ and every subspace of a zero-dimensional (resp. Hausdorff) is again so, we obtain the following result.

Proposition 13. *The space $(\mathcal{U}, \tau)$ is a zero-dimensional Hausdorff space.*

6.2. Construction of $\mathbb{M}(\mathcal{A})$

We can now form a Markov process $\mathbb{M}(\mathcal{A}) = (\mathcal{U}, \Sigma, \theta)$, where Σ is the σ-algebra generated by $(\!|\mathcal{A}|\!)$. To define the measure $\theta(u)$ for an ultrafilter $u \in \mathcal{U}$, we need to prove some additional results.

Lemma 14. *For all $a \in A$ and $u \in \mathcal{U}$, the set*

$$\{r \in \mathbb{Q}_0 \mid F_r a \in u\}$$

is nonempty and closed downward in the natural order on $\mathbb{Q}_0$.

Proof: The set contains at least 0 by (AA1). Downward closure follows from Lemma 8(ii). □

It follows that $\{r \in \mathbb{Q}_0 \mid \neg F_r a \in u\}$ is closed upward. Thus we can define the function $\theta : \mathcal{U} \to (\!|\mathcal{A}|\!) \to [0, 1]$ by

$$\begin{aligned} \theta(u)((\!|a|\!)) &= \sup\{r \in \mathbb{Q}_0 \mid F_r a \in u\} \\ &= \inf\{r \in \mathbb{Q}_0 \mid \neg F_r a \in u\}. \end{aligned}$$

Note that $\theta(u)((\!|a|\!))$ is not necessarily rational. In the following, we use the extension theorem to show that θ can be uniquely extended to a transition function. This will allow us to construct a Markov process on the space of good ultrafilters.

Lemma 15. *The set $(\!|\mathcal{A}|\!)$ is a field of sets, and for all $u \in \mathcal{U}$, the function $\theta(u)$ is finitely additive.*

Proof: That the set $(\!|\mathcal{A}|\!)$ is a field of sets is immediate from the Stone representation theorem and the fact that $(\!|\mathcal{A}|\!)$ is dense in $(\!|\mathcal{A}|\!)^*$.

To show finite additivity, suppose $a, b \in A$ and $(\!|a|\!) \cap (\!|b|\!) = \varnothing$. Then $a \wedge b = 0$. We wish to show that

$$\theta(u)((\!|a \vee b|\!)) = \theta(u)((\!|a|\!)) + \theta(u)((\!|b|\!)).$$

It suffices to show the inequality in both directions. For $\leq$, by the definition of θ, it suffices to show

$$\begin{aligned} &\sup\{t \mid F_t(a \vee b) \in u\} \\ &\leq \inf\{r \mid \neg F_r a \in u\} + \inf\{s \mid \neg F_s b \in u\} \\ &= \inf\{r + s \mid \neg F_r a \in u \text{ and } \neg F_s b \in u\} \\ &= \inf\{r + s \mid \neg F_r a \wedge \neg F_s b \in u\}; \end{aligned}$$

that is, if $F_t(a \vee b) \in u$ and $\neg F_r a \wedge \neg F_s b \in u$, then $t \leq r + s$. But

$$\begin{aligned} \neg F_r a \wedge \neg F_s b &= \neg F_r((a \vee b) \wedge a) \wedge \neg F_s((a \vee b) \wedge \neg a) \\ &\leq \neg F_{r+s}(a \vee b) \quad \text{by (AA5)}, \end{aligned}$$

thus $\neg F_{r+s}(a \vee b) \in u$, and $t \leq r + s$ follows from the characterization of Lemma 14.

The inequality in the opposite direction is similar, using (AA4). We need to show

$$\inf\{t \mid \neg F_t(a \vee b) \in u\} \geq \sup\{r+s \mid F_r a \wedge F_s b \in u\};$$

that is, if $\neg F_t(a \vee b) \in u$ and $F_r a \wedge F_s b \in u$, then $t \geq r+s$. But

$$\begin{aligned} F_r a \wedge F_s b &= F_r((a \vee b) \wedge a) \wedge F_s((a \vee b) \wedge \neg a) \\ &\leq F_{r+s}(a \vee b) \quad \text{by (AA4)}, \end{aligned}$$

thus $F_{r+s}(a \vee b) \in u$, and again $r+s \leq t$ by Lemma 14. □

The following is the key technical lemma where we use the fact that we have removed the bad ultrafilters.

Lemma 16. *For $u \in \mathcal{U}$, $\theta(u)$ is continuous from above at $\varnothing$ relatively to the field $(\!|\mathcal{A}|\!)$.*

Proof: We prove that if $u \in \mathcal{U}$ (it is a good ultrafilter) and $b_0 \geq b_1 \geq \cdots$ with $\bigcap_i (\!|b_i|\!) = \varnothing$, then

$$\inf_i \theta(u)((\!|b_i|\!)) = 0.$$

Consider the countable set $\mathcal{F}$ of elements of the form $\alpha^r = F_{t_1 \cdots t_n r} a$ for $a \in A$ and rational $t_1, \ldots, t_n, r \geq 0$, parameterized by r. If $r < s$, then $\alpha^s \leq \alpha^r$. Using (AA4),

$$\theta(u)((\!|\alpha^r \wedge \neg\alpha^s|\!)) \leq \theta(u)((\!|\alpha^r|\!)) - \theta(u)((\!|\alpha^s|\!)). \quad (2)$$

Since u is good, $F_t\alpha^r \in u$ for all $r < s$ iff $F_t\alpha^s \in u$, therefore

$$\theta(u)((\!|\alpha^s|\!)) = \inf_{r<s} \theta(u)((\!|\alpha^r|\!)). \quad (3)$$

Let $\varepsilon > 0$ be an arbitrarily small positive number. For each $\alpha \in \mathcal{F}$ and $s \in \mathbb{Q}_0$, choose $\varepsilon_\alpha^s > 0$ such that $\sum_{\alpha \in \mathcal{F}} \sum_{s \in \mathbb{Q}_0} \varepsilon_\alpha^s = \varepsilon$. By (2) and (3), we can choose $r_\alpha^s < s$ such that

$$\theta(u)((\!|\alpha^{r_\alpha^s} \wedge \neg\alpha^s|\!)) \leq \theta(u)((\!|\alpha^{r_\alpha^s}|\!)) - \theta(u)((\!|\alpha^s|\!)) \leq \varepsilon_\alpha^s.$$

The assumption $\bigcap_i (\!|b_i|\!) = \varnothing$ implies that $\bigcap_i (\!|b_i|\!)^*$ contains only bad ultrafilters. The set of good ultrafilters is

$$\bigcap_{\alpha \in \mathcal{F}} \bigcap_{s \in \mathbb{Q}_0} \left(\bigcup_{r<s} (\!|\neg\alpha^r|\!)^* \cup (\!|\alpha^s|\!)^* \right). \quad (4)$$

Thus $\bigcap_i (\!|b_i|\!) = \varnothing$ is equivalent to the condition

$$\left(\bigcap_{\alpha \in \mathcal{F}} \bigcap_{s \in \mathbb{Q}_0} \left(\bigcup_{r<s} (\!|\neg\alpha^r|\!)^* \cup (\!|\alpha^s|\!)^* \right) \right) \cap \bigcap_i (\!|b_i|\!)^* = \varnothing.$$

From this it follows that

$$\left(\bigcap_{\alpha \in \mathcal{F}} \bigcap_{s \in \mathbb{Q}_0} \left((\!|\neg\alpha^{r_\alpha^s}|\!)^* \cup (\!|\alpha^s|\!)^* \right) \right) \cap \bigcap_i (\!|b_i|\!)^* = \varnothing.$$

Since the space of ultrafilters is compact in the presence of the bad ultrafilters and $(\!|a|\!)^*$ is a clopen for any $a \in A$, there exist finite sets $C_0 \subseteq \mathcal{F}$ and $S_0 \subseteq \mathbb{Q} \cap [0,1]$ and $j \in \mathbb{N}$ such that

$$\bigcap_{\alpha \in C_0} \bigcap_{s \in S_0} (\!|\neg\alpha^{r_\alpha^s} \vee \alpha^s|\!)^* \cap (\!|b_j|\!)^* = \varnothing,$$

or in other words,

$$\begin{aligned} (\!|b_j|\!)^* &\subseteq \bigcup_{\alpha \in C_0} \bigcup_{s \in S_0} (\!|\alpha^{r_\alpha^s} \wedge \neg\alpha^s|\!)^* \\ &= (\!| \bigvee_{\alpha \in C_0} \bigvee_{s \in S_0} (\alpha^{r_\alpha^s} \wedge \neg\alpha^s) |\!)^* \end{aligned}$$

Thus in the Boolean algebra $\mathcal{A}$,

$$b_j \leq \bigvee_{\alpha \in C_0} \bigvee_{s \in S_0} (\alpha^{r_\alpha^s} \wedge \neg\alpha^s). \quad (5)$$

Consequently,

$$\begin{aligned} \theta(u)((\!|b_j|\!)) &\leq \theta(u)((\!| \bigvee_{\alpha \in C_0} \bigvee_{s \in S_0} (\alpha^{r_\alpha^s} \wedge \neg\alpha^s)|\!)) \\ &\leq \sum_{\alpha \in C_0} \sum_{s \in S_0} \theta(u)((\!|\alpha^{r_\alpha^s} \wedge \neg\alpha^s|\!)) \\ &\leq \sum_{\alpha \in C_0} \sum_{s \in S_0} \varepsilon_\varphi^s \quad \leq \varepsilon. \end{aligned}$$

As $\varepsilon > 0$ was arbitrary, $\inf_i \theta(u)((\!|b_i|\!)) = 0$. □

Since $(\!|\mathcal{A}|\!)$ is a field, the previous results and the extension theorem imply that for all $u \in \mathcal{U}$, the set function $\theta(u)$ can be uniquely extended to a measure on the σ-algebra Σ generated by $(\!|\mathcal{A}|\!)$.

Now we are ready to prove that $\mathbb{M}(\mathcal{A})$ is a Stone Markov process.

Theorem 17. *If $\mathcal{A}$ is a countable Aumann algebra, then $\mathbb{M}(\mathcal{A}) = (\mathcal{U}, (\!|\mathcal{A}|\!), \theta)$ is a Stone Markov process.*

Proof: We first prove that the space of good ultrafilters is analytic. Since any second-countable Stone space is Polish, the set of all ultrafilters (good and bad) is Polish. The good ultrafilters form a Borel set in the space of all ultrafilters—in fact, a $G_{\sigma\delta}$ Borel set as given by (4)—and since any Borel set in a Polish space is analytic, we obtain that the space of good ultrafilters is analytic.

The space is saturated, since all possible good ultrafilters are present, and the set $\{(\!|a|\!) \mid u \in (\!|a|\!)\}$ is just u.

To conclude that $\mathbb{M}(\mathcal{A})$ is a Markov process, it remains to verify that θ is a measurable function. Let $a \in A$, $r \in \mathbb{R} \cap [0,1]$, and $(r_i)_i \subseteq \mathbb{Q}_0$ an increasing sequence with supremum r. Let $X = \{\mu \in \Delta(\mathcal{U}, \Sigma) \mid$

$\mu(\llparenthesis a \rrparenthesis) \geq r\}$. It suffices to prove that $\theta^{-1}(X) \in \Sigma$. But

$$\begin{aligned}\theta^{-1}(X) &= \{u \in \mathcal{U} \mid \theta(u)(\llparenthesis a \rrparenthesis) \geq r\} \\ &= \bigcap_i \{u \in \mathcal{U} \mid \theta(u)(\llparenthesis a \rrparenthesis) \geq r_i\} \\ &= \bigcap_i \llparenthesis F_{r_i} a \rrparenthesis \in \Sigma.\end{aligned}$$

□

Now we are ready to prove the algebraic version of a truth lemma for Aumann algebras.

Lemma 18 (Extended Truth Lemma). *Let $\mathcal{A}$ be a countable Aumann algebra and $[\![\cdot]\!]$ an interpretation of elements of $\mathcal{A}$ as measurable sets in $\mathcal{M}$ such that for any generator p of $\mathcal{A}$, $[\![p]\!] = \{u \in \mathcal{U} \mid p \in u\}$. Then, for arbitrary $a \in A$,*

$$[\![a]\!] = \llparenthesis a \rrparenthesis.$$

In particular, if we consider Markovian logic $\mathcal{L}$, we can construct the corresponding Stone Markov process $\mathcal{M}_{\mathcal{L}}$. The following Theorem is a consequence of the Extended Truth Lemma and Theorem 6.

Theorem 19. *Given an arbitrary Markov process $\mathcal{M}$ and an arbitrary $m \in \operatorname{supp}(\mathcal{M})$, $(\mathcal{M}, m) \approx (\mathcal{M}_{\mathcal{L}}, [\![m]\!])$, where $[\![m]\!] = \{\varphi \in \mathcal{L} \mid \mathcal{M}, m \vDash \varphi\}$.*

6.3. From SMPs to AAs

Let $\mathcal{M} = (M, \mathcal{B}, \theta)$ be a Stone Markov process with distinguished base $\mathcal{B}$. By definition, $\mathcal{B}$ is a field of clopen sets closed under the operations

$$F_r(A) = \{m \in M \mid \theta(m)(A) \geq r\}.$$

Theorem 20. *The structure $\mathcal{B}$ with the set-theoretic Boolean operations and the operations F_r, $r \in \mathbb{Q}_0$ is a countable Aumann algebra.*

We denote this algebra by $\mathbb{A}(\mathcal{M})$.

Proof: We need to verify all the axioms of Aumann algebra. The proof is routine and we omit it from this abstract. □

6.4. Duality

In this section we summarize the previous results in the form of the duality theorem.

Theorem 21 (Duality Theorem).

(i) *Any countable Aumann algebra $\mathcal{A}$ is isomorphic to $\mathbb{A}(\mathbb{M}(\mathcal{A}))$ via the map $\beta : \mathcal{A} \to \mathbb{A}(\mathbb{M}(\mathcal{A}))$ defined by*

$$\beta(a) = \{u \in \operatorname{supp}(\mathbb{M}(\mathcal{A})) \mid a \in u\} = \llparenthesis a \rrparenthesis.$$

(ii) *Any Stone Markov process $\mathcal{M} = (M, \mathcal{A}, \theta)$ is homeomorphic to $\mathbb{M}(\mathbb{A}(\mathcal{M}))$ via the map $\alpha : \mathcal{M} \to \mathbb{M}(\mathbb{A}(\mathcal{M}))$ defined by*

$$\alpha(m) = \{A \in \mathcal{A} \mid m \in A\}.$$

Proof: (i) The set $\beta(a)$ is the set of good ultrafilters of $\mathcal{A}$ that contain a; that is, $\beta(a) = \llparenthesis a \rrparenthesis$. By the classical Stone representation theorem, $\mathcal{A}$ and $\llparenthesis \mathcal{A} \rrparenthesis^*$ are isomorphic as Boolean algebras via the map $a \mapsto \llparenthesis a \rrparenthesis^*$. By the Rasiowa–Sikorski lemma (Theorem 2) and Lemma 12, the good ultrafilters are dense in $\llparenthesis \mathcal{A} \rrparenthesis^*$, and since the space $\mathcal{U}^*$ has a base of clopens, $\llparenthesis \mathcal{A} \rrparenthesis^*$ and $\llparenthesis \mathcal{A} \rrparenthesis$ are isomorphic as Boolean algebras via the map $\llparenthesis a \rrparenthesis^* \mapsto \llparenthesis a \rrparenthesis$. Indeed, consider two elements a, b of $\mathcal{A}$ such that $\llparenthesis a \rrparenthesis^* \subseteq \llparenthesis b \rrparenthesis^*$. This implies $\llparenthesis a \rrparenthesis \subseteq \llparenthesis b \rrparenthesis$ by definition. Reverse, if $\llparenthesis a \rrparenthesis \subseteq \llparenthesis b \rrparenthesis$ and $\llparenthesis a \rrparenthesis^* \not\subseteq \llparenthesis b \rrparenthesis^*$, then $\llparenthesis a \rrparenthesis^* \setminus \llparenthesis b \rrparenthesis^*$ is a non-empty clopen. By density, $(\llparenthesis a \rrparenthesis^* \setminus \llparenthesis b \rrparenthesis^*) \cap \mathcal{U} \neq \varnothing$ implying $\llparenthesis a \rrparenthesis \setminus \llparenthesis b \rrparenthesis \neq \varnothing$ - contradiction.

It remains to show that the operations F_r are preserved. Let $\mathcal{U} = \operatorname{supp}(\mathbb{M}(\mathcal{A}))$. For each $r \in \mathbb{Q}_0$,

$$\begin{aligned}\beta(F_r a) &= \{u \in \mathcal{U} \mid F_r a \in u\} \\ &= \{u \in \mathcal{U} \mid \theta(u)(\llparenthesis a \rrparenthesis) \geq r\} \\ &= \{u \in \mathcal{U} \mid \theta(u)(\beta(a)) \geq r\} \\ &= F_r(\beta(a)).\end{aligned}$$

(ii) The set $\alpha(m)$ is the set of all elements of $\mathcal{A}$ that contain m. We first prove that this is a good ultrafilter of $\mathbb{A}(\mathcal{M})$. It is clearly an ultrafilter, as it is a principal ultrafilter of a set-theoretic Boolean algebra. To show that it is good, we need to reason that if $a \in \mathcal{A}$ and $F_r a \in \alpha(m)$ for all $r < s$, then $F_s a \in \alpha(m)$. This follows immediately from the fact that $F_t a \in \alpha(m)$ iff $m \in F_t a$ iff $\theta(m)(a) \geq t$.

The map α is a a strict embedding, since the two distinguished bases $\mathcal{A}$ of $\mathcal{M}$ and $\llparenthesis \mathcal{A} \rrparenthesis$ of $\mathbb{M}(\mathbb{A}(\mathcal{M}))$ are isomorphic. This embedding must be a homeomorphism, since $\mathcal{M}$ is saturated.

□

6.5. Duality in Categorical Form

We present the previous results in a more categorical format. The categories of Aumann algebras ($\mathbf{AA}$) and Stone Markov processes ($\mathbf{SMP}$) were defined in §4 and §5, respectively.

We define contravariant functors $\mathbb{A} : \mathbf{SMP} \to \mathbf{AA}^{\mathrm{op}}$ and $\mathbb{M} : \mathbf{AA} \to \mathbf{SMP}^{\mathrm{op}}$. The functor $\mathbb{A}$ on an object $\mathcal{M}$ produces the Aumann algebra $\mathbb{A}(\mathcal{M})$ defined in Theorem 20. On arrows $f : \mathcal{M} \to \mathcal{N}$ we define $\mathbb{A}(f) = f^{-1} : \mathbb{A}(\mathcal{N}) \to \mathbb{A}(\mathcal{M})$. It is well known that this is a Boolean algebra homomorphism. It is also easy to verify from the definition of morphisms in the category $\mathbf{SMP}$ (Definition 10) that it is an Aumann algebra homomorphism.

To see this explicitly, let $A \in \mathcal{A}_{\mathcal{N}}$. We wish to show that

$$f^{-1}(F_r^{\mathcal{N}}(A)) = F_r^{\mathcal{M}}(f^{-1}(A)).$$

Using the fact that

$$\theta_{\mathcal{N}}(f(m))(A) = \theta_{\mathcal{M}}(m)(f^{-1}(A)),$$

we have

$$\begin{aligned} m \in f^{-1}(F_r^{\mathcal{N}}(A)) &\Leftrightarrow f(m) \in F_r^{\mathcal{N}}(A) \\ &\Leftrightarrow \theta_{\mathcal{N}}(f(m))(A) \geq r \\ &\Leftrightarrow \theta_{\mathcal{M}}(m)(f^{-1}(A)) \geq r \\ &\Leftrightarrow m \in F_r^{\mathcal{M}}(f^{-1}(A)). \end{aligned}$$

The functor $\mathbb{M} : \mathbf{AA} \to \mathbf{SMP}^{\mathrm{op}}$ on an object $\mathcal{A}$ gives the Stone–Markov process $\mathbb{M}(\mathcal{A})$ defined in Theorem 17. On morphisms $h : \mathcal{A} \to \mathcal{B}$, it maps ultrafilters to ultrafilters by $\mathbb{M}(h) = h^{-1} : \mathbb{M}(\mathcal{B}) \to \mathbb{M}(\mathcal{A})$; that is,

$$\mathbb{M}(h)(u) = h^{-1}(u) = \{A \in \mathcal{A}_{\mathcal{N}} \mid h(A) \in u\}.$$

Another way to view $\mathbb{M}(h)$ is by composition, recalling that an ultrafilter can be identified with a homomorphism $\overline{u} : \mathcal{A} \to \mathbb{2}$ by $u = \{a \mid \overline{u}(a) = 1\}$. In this view,

$$\mathbb{M}(h)(\overline{u}) = \overline{u} \circ h,$$

where $\circ$ denotes function composition.

We know from classical Stone duality that this is continuous. We need to verify that it is a morphism. It suffices to verify it on sets of the form $(\!|a|\!)$ as these generate the σ-algebra. Because h is a homomorphism, we calculate as follows:

$$\begin{aligned} \theta_{\mathcal{B}}(u)(\mathbb{M}(h)^{-1}((\!|a|\!))) &= \sup\{r \mid F_r(h(a)) \in u\} \\ &= \sup\{r \mid h(F_r(a)) \in u\} \\ &= \{r \mid \overline{u}(h(F_r(a))) = 1\} \\ &= \{r \mid F_r(a) \in \mathbb{M}(h)(u)\} \\ &= \theta_{\mathcal{A}}(\mathbb{M}(h)(u)((\!|a|\!))). \end{aligned}$$

Theorem 22. *The functors $\mathbb{M}$ and $\mathbb{A}$ define a dual equivalence of categories.*

$$\mathbf{SMP} \underset{\mathbb{M}}{\overset{\mathbb{A}}{\rightleftarrows}} \mathbf{AA}^{\mathrm{op}}$$

The proof is given in the full version.

7. Related Work

Stone duality in semantics originates with the pioneering work of Plotkin [19] and Smyth [20] who discovered a Stone-type duality between Dijkstra's predicate-transformer semantics and state-transformer semantics. Kozen [21] developed the probabilistic analogue of this duality.

The theory of Stone-type dualities for transition systems has been investigated at length by Bonsangue and Kurz [22]. There have been many recent investigations of Stone-type dualities in logic and computation. Recent very interesting work by Jacobs [23] has explored convex dualities for probability and quantum mechanics.

Duality theory for LMPs was discussed by Mislove et al. [24] which is based on Gelfand duality for C^*-algebras. This is very interesting work but in rather a different direction from the present work which is very much in the spirit of logics for Markov processes and is related to bisimulation and its logical characterization [9]. By contrast, the work of Mislove et al. [24] is related to testing.

The most closely related work to ours is the work by Goldblatt [3] on the role of the Baire cateogry theorem on completeness proofs and even more closely his work on deduction systems [6] for coalgebras. The main difference between his work and ours is that we have eliminated some of the infinitary axioms that he uses, though we still retain one and, of course, we have developed a duality rather than just a completeness theorem. He uses one of his infinitary axioms in order to show countable additivity of the measures that he defines; this is what we have been able to eliminate by our use of the Rasiowa–Sikorski lemma to eliminate the bad ultrafilters; as far as we know this is a new idea.

8. Conclusions

As promised we have proved a duality theorem between Stone-Markov processes and Aumann algebras which subsumes and extends the completeness theorems in the literature. Our treatment improves on the existing axiomatizations as well.

The following novel features appear in our proof:

1) We have to remove ultrafilters that fail to satisfy a key infinitary axiom and we have to show that this does not change anything which we do by showing that these are "rare" in a topological sense (meager).
2) As a result, the usual compactness for Stone spaces fails and we need a new concept, which

we call saturation, instead.

3) We have to establish the relevant measure theoretic properties of the Markov kernels that we construct from the algebras, which again uses the Baire category theorem in a crucial way.
4) We define our Markov processes with a distinguished base and use this to constrain the morphisms in order to get the duality.

There are many variations one can imagine exploring. Perhaps the most interesting one is to consider more general measure spaces and work with different bisimulation notions [25] that apply more generally. Our treatment is not fully localic and perhaps some of the topological subtleties of the present proof would disappear once we adopted a more localic point of view.

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