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On renormalisation of $\lambda\phi^4$ field theory in curved space–time: I

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Abstract. Renormalisation of $\lambda\phi^4$ theory in curved space–time is considered in the interaction picture. A generalisation of normal ordering to curved space–time is introduced, based on the construction of adiabatic particle states in Robertson–Walker space–time. Dimensional regularisation is used to define uniquely the divergent quantities which are removed by normal ordering. It is shown that this normal ordering is sufficient to make finite all physical processes including vacuum polarisation to first order in λ. An alternative and equivalent procedure is given which requires renormalisation of the mass and of the constant which couples the field to the Ricci scalar. The stress tensor is found to be finite to first order in λ and it is shown that if the free-field theory in a Robertson–Walker universe predicts that particles are created by the gravitational field with a black-body spectrum then this spectrum is maintained when first-order self-interactions are taken into account. Finally, some aspects of the renormalisation of second-order physical processes are discussed. In particular, it is shown that some second-order Feynman diagrams give rise to divergences which involve state-dependent quantities. However, it appears that these state-dependent divergences disappear when all Feynman diagrams corresponding to a given physical process are summed.

1. Introduction

Although a considerable amount of work has been carried out in recent years on the development of quantum field theory in curved space–time, relatively little of this work has been concerned with interacting field theories. In this paper we consider the renormalisation to first order in the coupling constant of the theory of a scalar field, ϕ, with self-coupling $\lambda\phi^4$, and we make some remarks about renormalisation to second order. This theory has recently been studied by Birrell and Ford (1980), who investigated some first-order particle creation processes, and by Birrell and Taylor (1980), who investigated the renormalisability of n-point Green's functions. One problem not discussed by these authors is the renormalisation of vacuum-to-vacuum processes. Such processes were considered earlier by Drummond (1975) and Drummond and Shore (1979), but their work was restricted to the special case of a massless conformally invariant scalar field with $\lambda\phi^4$ self-interaction in De Sitter space. Calculations of the modification to the Casimir effect for a scalar field when a self-interaction is included have been performed by Ford (1978) and Kay (1980). Work on self-interacting scalar field theories in curved space–times has also been performed by Freedman and Weinberg (1974) and Boulware (1979).

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After describing some basic formalism in § 2, we show in § 3 that the introduction of a class of adiabatic vacuum states (Parker and Fulling 1974) enables normal ordering of field operators to be defined in curved space–time and that all first-order S-matrix elements are consequently finite, including the first-order vacuum-to-vacuum process. Although the adiabatic vacuum states are only explicitly defined in Robertson–Walker space–times, we assume that this normal ordering procedure is valid in general space–times (since the adiabatic formalism gives the same divergences in the Feynman propagator as does the DeWitt–Schwinger formalism (DeWitt 1975)). In § 4, the stress tensor is investigated and shown to be finite to first order in λ. We find that a stress tensor which represents radiation by the gravitational field of non-interacting particles with a Planckian spectrum maintains the Planckian form to first order in λ. In § 5 a preliminary investigation of second-order renormalisation is carried out. The main result is the appearance of non-geometrical state-dependent divergences in some Feynman diagrams. It will be shown in a following paper (Bunch and Panangaden 1980) that these divergences disappear in conformally flat space–times when all diagrams corresponding to a particular physical process are summed, so that $\lambda\phi^4$ theory remains renormalisable in curved space–time, at least to second order in λ. That this second-order result is valid in general curved space–times is proved in Bunch and Parker (1979).

2. Basic formalism

The Lagrangian density is

$$\hat{\mathscr{L}} = \tfrac{1}{2}\sqrt{g}\,[\partial^\mu\hat{\phi}_0\partial_\mu\hat{\phi}_0 - (m_0^2+\xi_0 R)\hat{\phi}_0^2 - \tfrac{1}{2}\lambda_0\hat{\phi}_0^4] \tag{2.1}$$

where the subscript zero on any quantity indicates that it is a bare quantity and the hat on any operator means that it is a Heisenberg picture operator. The Heisenberg picture field equation is

$$\Box\,\hat{\phi}_0 + (m_0^2) + \xi_0 R)\hat{\phi}_0 + \lambda_0\hat{\phi}_0^3 = 0. \tag{2.2}$$

The momentum conjugate to $\hat{\phi}_0$ is

$$\hat{\pi}_0 = \frac{\partial\hat{\mathscr{L}}}{\partial(\partial_0\hat{\phi}_0)} = \sqrt{g}\,g^{0\mu}\partial_\mu\hat{\phi}_0 \tag{2.3}$$

and the canonical commutation relations are

$$\begin{aligned}&[\hat{\phi}_0(x_0,\boldsymbol{x}),\,\hat{\phi}_0(x_0,\boldsymbol{x}')] = 0 = [\hat{\pi}_0(x_0,\boldsymbol{x}),\,\hat{\pi}_0(x_0,\boldsymbol{x}')]\\ &[\hat{\phi}_0(x_0,\boldsymbol{x}),\,\hat{\pi}_0(x_0,\boldsymbol{x}')] = \mathrm{i}\delta(\boldsymbol{x}-\boldsymbol{x}').\end{aligned} \tag{2.4}$$

The Hamiltonian density is

$$\begin{aligned}\hat{\mathscr{H}}(x) &= \hat{\pi}_0\partial_0\hat{\phi}_0 - \hat{\mathscr{L}}\\ &= \tfrac{1}{2}\sqrt{g}\,[g^{00}(\partial_0\hat{\phi}_0)^2 - g^{ij}\partial_i\hat{\phi}_0\partial_j\hat{\phi}_0 + (m_0^2+\xi_0 R)\hat{\phi}_0^2 + \tfrac{1}{2}\lambda_0\hat{\phi}_0^4].\end{aligned} \tag{2.5}$$

This canonical formalism is well-known (see, for example, Fulling 1972) and requires only that the space–time be globally hyperbolic. Now introduce renormalised quantities

$$\hat{\phi} = Z_1^{-1/2}\hat{\phi}_0 \qquad m^2 = Z_2^{-1}m_0^2 \qquad \xi = Z_3^{-1}\xi_0 \qquad \lambda = Z_4^{-1}\lambda_0. \tag{2.6}$$

Then

$$\hat{\mathscr{L}} = \tfrac{1}{2}\sqrt{g}\, Z_1[\partial^\mu\hat{\phi}\partial_\mu\hat{\phi} - (Z_2 m^2 + Z_3\xi R)\hat{\phi}^2] - \frac{\lambda}{4}\sqrt{g}\, Z_4 Z_1^2\hat{\phi}^4 \tag{2.7}$$

$$\hat{\pi} = Z_1^{1/2}\pi_0 \tag{2.8}$$

$$[\hat{\phi}(x_0, \boldsymbol{x}), \hat{\pi}(x_0, \boldsymbol{x}')] = \mathrm{i}\delta(\boldsymbol{x} - \boldsymbol{x}') \tag{2.9}$$

and

$$\hat{\mathscr{H}} = \tfrac{1}{2}\sqrt{g}\, Z_1[g^{00}(\partial_0\hat{\phi})^2 - g^{ij}\partial_i\hat{\phi}\partial_j\hat{\phi} + (Z_2 m^2 + Z_3\xi R)\hat{\phi}^2] + \tfrac{1}{4}\sqrt{g}\,\lambda Z_4 Z_1^2\hat{\phi}^4. \tag{2.10}$$

The Hamiltonian density decomposes into free and interacting parts:

$$\hat{\mathscr{H}}(x) = \hat{\mathscr{H}}_0(x) + \hat{\mathscr{H}}'(x) \tag{2.11}$$

where

$$\hat{\mathscr{H}}_0(x) = \tfrac{1}{2}\sqrt{g}\,[g^{00}(\partial_0\hat{\phi}_0)^2 - g^{ij}\partial_i\hat{\phi}_0\partial_j\hat{\phi}_0 + (m^2 + \xi R)\hat{\phi}_0^2] \tag{2.12}$$

$$\hat{\mathscr{H}}'(x) = \tfrac{1}{2}\sqrt{g}\,[(Z_2 - 1)m^2 + (Z_3 - 1)\xi R]\hat{\phi}_0^2 + \tfrac{1}{4}\sqrt{g}\, Z_4\lambda\hat{\phi}_0^4. \tag{2.13}$$

The Heisenberg equation of motion for any operator $\hat{\Omega} = \hat{\Omega}(\hat{\phi}_0, \hat{\pi}_0)$ is

$$\partial_0\hat{\Omega} = \mathrm{i}[\hat{H}, \hat{\Omega}] \tag{2.14}$$

where $\hat{H}$ is the Hamiltonian

$$\hat{H} = \hat{H}(\hat{\phi}_0, \hat{\pi}_0) = \int \hat{\mathscr{H}}(x)\, \mathrm{d}^3\boldsymbol{x}. \tag{2.15}$$

The Hamiltonian can be expressed as the sum of a free part and an interacting part:

$$\hat{H} = \hat{H}_0 + \hat{H}' \tag{2.16}$$

with

$$\hat{H}_0 = \int \hat{\mathscr{H}}_0(x)\mathrm{d}^3\boldsymbol{x} \tag{2.17}$$

$$\hat{H}' = \int \hat{\mathscr{H}}'(x)\mathrm{d}^3\boldsymbol{x}. \tag{2.18}$$

Taking $\hat{\Omega} = \hat{\phi}_0$ and $\hat{\Omega} = \hat{\pi}_0$ in (2.14) yields equations (2.3) and (2.2).

Given any Heisenberg picture operator $\hat{\Omega}$ and state $|\psi\rangle_H$, one can define an interaction picture operator Ω and state $|\psi\rangle_I$ by

$$\Omega = U(x^0, -\infty)\hat{\Omega}U(x^0, -\infty)^{-1} \tag{2.19}$$

and

$$|\psi(x^0)\rangle_I = U(x^0, -\infty)|\psi_H\rangle \tag{2.20}$$

where $U = U(x^0, -\infty)$ is a unitary operator which satisfies

$$\mathrm{i}\partial_0 U = H'U \equiv U\hat{H}' \tag{2.21}$$

and the boundary condition

$$\lim_{x^0\to-\infty} U(x^0, -\infty) = I \qquad \text{(weak operator convergence).} \tag{2.22}$$

The equation of motion for an interaction picture operator is easily derived from (2.14), (2.16), (2.19) and (2.21):

$$\partial_0 \Omega = \mathrm{i}[H_0, \Omega] \tag{2.23}$$

from which one obtains

$$\pi_0 = \sqrt{g}\, g^{0\mu} \partial_\mu \phi_0 \qquad \pi = \sqrt{g}\, g^{0\mu} Z_1 \partial_\mu \phi \tag{2.24}$$

and

$$\Box \phi_0 + (m^2 + \xi R)\phi_0 = 0 \qquad \Box \phi + (m^2 + \xi R)\phi = 0. \tag{2.25}$$

Thus the interaction picture field operator satisfies the free-field equation with renormalised mass and coupling to the scalar curvature.

The S matrix can now be constructed as in Minkowski space–time:

$$S = \lim_{x^0 \to \infty} U(x^0, -\infty) \qquad \text{(weak operator convergence).} \tag{2.26}$$

One obtains in the usual manner

$$S = \sum_{n=0}^{\infty} S^{(n)} \tag{2.27}$$

where

$$S^{(0)} = I \qquad S^{(n)} = \frac{(-\mathrm{i})^n}{n!} \int T(\mathscr{H}'(x_1) \ldots \mathscr{H}'(x_n))\, \mathrm{d}^4 x_1 \ldots \mathrm{d}^4 x_n. \tag{2.28}$$

In order to construct physical particle states we assume, as for interacting fields in Minkowski space–time, that the interaction is switched on at some early time and off again at some late time. The divergences which we will be dealing with are local so that switching the interaction on and off asymptotically is not expected to affect renormalisability. In the in-region, before the interaction is switched on, the physical renormalised field operator is assumed to be a linear combination of positive and negative frequency solutions of the free wave equation:

$$(\phi(x))_{x^0 \to -\infty} = \int \mathrm{d}\mu(\boldsymbol{k})[A_{\boldsymbol{k}}^{\mathrm{in}} \phi_{\boldsymbol{k}}^{\mathrm{in}} + A_{\boldsymbol{k}}^{\dagger} \phi_{\boldsymbol{k}}^{\mathrm{in}*}]. \tag{2.29}$$

The in-vacuum is then defined by

$$A_{\boldsymbol{k}} |\mathrm{in}\rangle = 0 \qquad \text{for all } \boldsymbol{k}. \tag{2.30}$$

If the space–time is flat at early times the functions $\phi_{\boldsymbol{k}}^{\mathrm{in}}$ will be positive frequency plane waves of momentum $\boldsymbol{k}$ and the vacuum state will correspond to the absence of Minkowski particles. However, it is not necessary to assume that the geometry is initially flat, although if it is not, some physically motivated criterion for defining positive frequency solutions of the wave equation must be found so that the construction of physical particle states can be carried out. Given a state $|\mathrm{in}\rangle$, an 'in' Fock space, $\mathscr{F}_{\mathrm{in}}$, can be constructed by applying creation operators $A_{\boldsymbol{k}}^{\mathrm{in}\dagger}$ to $|\mathrm{in}\rangle$. At late times, after the interaction has switched off, a similar decomposition can be made:

$$(\phi(x))_{x^0 \to \infty} = \int \mathrm{d}\mu(\boldsymbol{k})[A_{\boldsymbol{k}}^{\mathrm{out}} \phi_{\boldsymbol{k}}^{\mathrm{out}}(x) + A_{\boldsymbol{k}}^{\mathrm{out}\dagger} \phi_{\boldsymbol{k}}^{\mathrm{out}*}(x)] \tag{2.31}$$

where ϕ_k^{out} is positive frequency at late times, and an 'out' Fock space, $\mathscr{F}_{\text{out}}$, is constructed with vacuum $|\text{out}\rangle$ for which

$$A_k^{\text{out}}|\text{out}\rangle = 0 \qquad \text{for all } \boldsymbol{k}. \tag{2.32}$$

At intermediate times the renormalised interaction picture field operator satisfies the free-field equation and we take

$$\phi(x) = Z_1^{-1/2}\int \mathrm{d}\mu(\boldsymbol{k})[A_k\phi_k + A_k^\dagger\phi_k^*] \tag{2.33}$$

where the solutions ϕ_k reduce to ϕ_k^{in} at early times and we can make the identification

$$A_k \equiv A_k^{\text{in}}. \tag{2.34}$$

The factor $Z_1^{-1/2}$ must be included in (2.33) to ensure that A_k and $A_{k'}^\dagger$ have the correct commutation relations for annihilation and creation operators. These commutation relations are derived from (2.9) which contains a factor $Z_1^{1/2}$ through (2.8). Note that at early and late times the interaction switches off so that $Z_1 \to 1$ and $\phi(x)$ becomes

$$(\phi(x))_{x^0\to\infty} = \int \mathrm{d}\mu(\boldsymbol{k})[A_k(\phi_k(x))_{x^0\to\infty} + A_k^\dagger(\phi_k^*(x))_{x^0\to\infty}]. \tag{2.35}$$

The solutions ϕ_k do not necessarily reduce at late times to ϕ_k^{out}, since particles can be created by the gravitational field leading to a mixing of positive and negative frequency components of the field. In general we have

$$(\phi_k(x))_{x^0\to\infty} = \int \mathrm{d}\mu(\boldsymbol{k}')\,[\alpha_{kk'}\phi_{k'}^{\text{out}} + \beta_{kk}\phi_{k'}^{*\text{out}}]. \tag{2.36}$$

This defines the Bogolubov transformation relating $\mathscr{F}_{\text{in}}$ to $\mathscr{F}_{\text{out}}$:

$$A_{k'}^{\text{out}} = \int \mathrm{d}\mu(\boldsymbol{k})[\alpha_{kk'}A_k + \beta_{kk'}^*A_k]. \tag{2.37}$$

Let the initial state of the system in the interaction picture be $|\phi_i\rangle \in \mathscr{F}_{\text{in}}$. At time t, the state will be

$$|\psi(t)\rangle = U(t, -\infty)|\phi_i\rangle. \tag{2.38}$$

Thus the amplitude that at late times the system will be in a state $|\phi_f\rangle \in \mathscr{F}_{\text{out}}$ is

$$\lim_{t\to\infty}\langle\phi_f|\psi(t)\rangle = \langle\phi_f|S|\phi_i\rangle. \tag{2.39}$$

In this expression, all information about particle creation due to the self-interaction is contained in S and all information about gravitational particle creation is contained in the Bogolubov transformation relating $\mathscr{F}_{\text{in}}$ to $\mathscr{F}_{\text{out}}$.

3. Renormalisation of S-matrix elements to first order in λ

Suppose that the initial state of the system is the vacuum state, $|\text{in}\rangle$. Then the amplitude that the final state of the system will be an n-particle out-state $|\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n, \text{out}\rangle$ where $\boldsymbol{k}_1, \ldots \boldsymbol{k}_n$ are the quantum numbers of the n final particles is

$$\text{amplitude} = \langle \boldsymbol{k}_1, \ldots \boldsymbol{k}_n, \text{out}\,|\,S\,|\,\text{in}\rangle. \tag{3.1}$$

It will be sufficient to consider only processes in which the initial state is $|\text{in}\rangle$ to discuss renormalisation. Once renormalisation has been carried out, the calculation of other amplitudes is straightforward. Inserting a complete set of in-states gives

$$\text{amplitude} = \langle \boldsymbol{k}_1, \ldots \boldsymbol{k}_n, \text{out} \mid \text{in}\rangle\langle \text{in} \mid S \mid \text{in}\rangle + \int \mathrm{d}\mu(\boldsymbol{p}_1)\mathrm{d}\mu(\boldsymbol{p}_2)\langle \boldsymbol{k}_1, \ldots \boldsymbol{k}_n, \text{out} \mid \boldsymbol{p}_1, \boldsymbol{p}_2, \text{in}\rangle$$
$$\times\langle \boldsymbol{p}_1, \boldsymbol{p}_2, \text{in} | S | \text{in}\rangle + \int \mathrm{d}\mu(\boldsymbol{p}_1) \ldots \mathrm{d}\mu(\boldsymbol{p}_4)\langle \boldsymbol{k}_1, \ldots \boldsymbol{k}_n, \text{out} \mid \boldsymbol{p}_1, \ldots \boldsymbol{p}_4, \text{in}\rangle$$
$$\times\langle \boldsymbol{p}_1, \ldots \boldsymbol{p}_4, \text{in} | S | \text{in}\rangle + \ldots \tag{3.2}$$

where we have used the fact that the $\lambda\phi^4$ interaction can create particles only in pairs. Expression (3.2) can contain only a finite number of terms when the S matrix is calculated to finite order in perturbation theory. To first order in λ it contains only the three terms explicitly shown. Each of the quantities $\langle \boldsymbol{k}_1, \ldots, \boldsymbol{k}_n, \text{out} | \boldsymbol{p}_1, \ldots, \boldsymbol{p}_m, \text{in}\rangle$ is finite and independent of λ and can be obtained in terms of the Bogolubov coefficients (Parker 1977). For $\lambda\phi^4$ theory, power counting arguments show that the only divergent S-matrix elements are those with zero, two or four external lines. Thus the S-matrix elements will all be finite provided that we renormalise $\langle \text{in} | S | \text{in}\rangle$, $\langle \boldsymbol{p}_1\boldsymbol{p}_2, \text{in} | S | \text{in}\rangle$ and $\langle \boldsymbol{p}_1, \ldots, \boldsymbol{p}_4 | S | \text{in}\rangle$. To first order in λ, these correspond respectively to figures 1–3. The third of these is in fact finite so only the first two need to be renormalised.

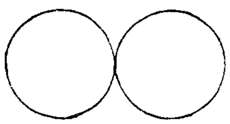

Figure 1.

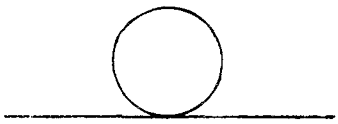

Figure 2.

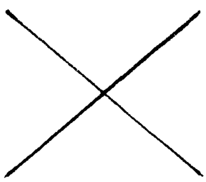

Figure 3.

The interaction Hamiltonian density is given by (2.13). The renormalisation constants $Z_i (i = 1, 2, 3, 4)$ are power series in λ:

$$Z_i = 1 + \sum_{n=1}^{\infty} \lambda^n Z_i^{(n)}. \tag{3.3}$$

Thus to first order in λ, the interaction Hamiltonian density is

$$\mathscr{H}'(x) = \tfrac{1}{2}\lambda\sqrt{g}[Z_2^{(1)} m^2 + Z_3^{(1)}\xi R]\phi_0^2 + \tfrac{1}{4}\lambda\sqrt{g}\,\phi_0^4. \tag{3.4}$$

In order to obtain finite S-matrix elements from this interaction Hamiltonian density we will introduce a generalisation to curved space–time of normal ordering of field operators. Let the space–time be an arbitrary Robertson–Walker universe with metric

$$ds^2 = C(\eta)[d\eta^2 - (1-\epsilon r^2)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)] \tag{3.5}$$

where the universe is spatially open, flat or closed for $\epsilon = -1, 0$ or $+1$. We will indicate how this restriction on the metric of the space–time is to be removed later. Since normal ordering of field operators corresponds to the removal of divergent vacuum polarisation contributions, the most important step in defining normal ordering in curved space–time is to define what is meant by a physical vacuum state. In Minkowski space–time this is straightforward since there exists a unique Poincaré invariant vacuum. In curved space–time no such unique vacuum exists. However, for metrics of the form (3.5) and for the spatially flat anisotropic Kasner metrics it is possible to define a class of approximate physical vacuum states called adiabatic vacuum states (Parker and Fulling 1974, Fulling *et al* 1974). This is achieved by decomposing the field $\phi(x)$ according to

$$\phi_0(x) = C^{-1/2}(\eta)\int d\tilde{\mu}(\boldsymbol{k})[a_k Y_k(\boldsymbol{x})\chi_k(\eta) + a_k^\dagger Y_k^*(\boldsymbol{x})\chi_k^*(\eta)] \tag{3.6}$$

where $Y_k(\boldsymbol{x})$ are eigenfunctions of the three-dimensional Laplacian operator on an $\eta =$ constant hypersurface, $d\tilde{\mu}(\boldsymbol{k})$ is a measure on the space of quantum numbers $\boldsymbol{k}$ and the functions $\chi_k(\eta)$ satisfy

$$\frac{d^2\chi_k}{d\eta^2} + [\omega_k^2 + (\xi - \tfrac{1}{6})CR]\chi_k = 0 \tag{3.7}$$

$$\omega_k^2 = k^2 + Cm^2 \tag{3.8}$$

and the normalisation condition

$$\frac{d\chi_k^*}{d\eta}\chi_k - \chi_k^*\frac{d\chi_k}{d\eta} = \mathrm{i}. \tag{3.9}$$

The range of k is restricted to be $0 < k < \infty$ for $\epsilon = 0, -1$ and $k = 1, 2, 3, \ldots$ for $\epsilon = 1$. Approximate solutions $\chi_k(\eta)$ to (3.7) are now sought having the form of a positive frequency WKB solution:

$$\chi_k(\eta) = \frac{\exp[-\mathrm{i}\int^\eta W_k(\eta')d\eta']}{[2W_k(\eta)]^{1/2}} \tag{3.10}$$

where $W_k(\eta)$ satisfies (from (3.7) and (3.10)):

$$W_k^2 = \omega_k^2 + (\xi - \tfrac{1}{6})\,CR - \frac{1}{2}\left[\frac{W_k''}{W_k} - \frac{3}{2}\left(\frac{W_k'}{W_k}\right)^2\right]. \tag{3.11}$$

The solution $W_k(\eta)$ to (3.11) is obtained iteratively and it is not difficult to see that as the iteration proceeds the terms obtained contain an increasing number of derivatives with respect to η and at the same time involve increasingly large inverse powers of ω_k so that the WKB solutions provide a good approximation in the limit of slowly and smoothly varying geometry or of high frequency ω_k. If, following Fulling *et al* (1974), we say that a term involving n derivatives with respect to η is a term of adiabatic order

T^{-n}, then we find that to adiabatic order T^0

$$W_k = \omega_k \tag{3.12}$$

and to adiabatic order T^{-2}

$$W_k = \omega_k + \frac{(\xi - \frac{1}{6})CR}{2\omega_k^3} - \frac{1}{4\omega_k^3}\left[\frac{\omega_k''}{\omega_k} - \frac{3}{2}\left(\frac{\omega_k'}{\omega_k}\right)^2\right]. \tag{3.13}$$

The solution to adiabatic order T^{-4} is given in Bunch (1980). Because the solution for $\chi_k(\eta)$ is only approximate, the annihilation and creation operators a_k and $a_k^\dagger$ are not completely specified and so there is not a unique vacuum: there is instead a class of vacuum states which can be made smaller by carrying out the WKB approximation to increasingly high order. The important physical properties of an adiabatic vacuum are that it reduces to the usual particle concept in the static limit, since the lowest order solution (3.12) becomes exact in this limit, and that for a time-varying geometry the adiabatic vacuum corresponds to the absence of particles in sufficiently high frequency modes. Actually, the inherent indeterminacy of an adiabatic vacuum is a reflection of the fact that particles are created by the gravitational field in all modes including high frequency modes. However, the energy density of created particles at any time in mode k falls off as some high inverse power of k as $k \to \infty$, the precise behaviour depending on the adiabatic order to which the vacuum is defined (for more details, see Fulling (1979)). It is this property that makes an adiabatic vacuum a suitable state for investigating the renormalisation of the stress tensor and for defining normal ordering since the divergences to be removed by renormalisation or normal ordering come from the high frequency modes which are accurately given by the WKB approximation. To renormalise the stress tensor the WKB approximation must be carried out to adiabatic order T^{-4} but we shall find that for normal ordering of S-matrix elements the order T^{-2} solution (3.13) is sufficient.

Suppose that the field operator $\phi(x)$ is decomposed:

$$\phi_0(x) = \phi_0^{(+)}(x) + \phi_0^{(-)}(x) \tag{3.14}$$

where $\phi_0^{(+)}(x)$ is a positive frequency WKB solution defined to adiabatic order T^{-2} and $\phi_0^{(-)}(x)$ the corresponding negative frequency solution. To each decomposition (3.14) there corresponds an adiabatic vacuum $|0\rangle_A$. The operators $\phi_0^2(x)$ and $\phi_0^4(x)$ are normal ordered with respect to $|0\rangle_A$ by making all positive frequency parts (annihilation operators) stand to the right. The normal ordered operators are thus

$$N(\phi_0^2(x)) = \phi_0^2(x) - [\phi_0^{(+)}(x), \phi_0^{(-)}(x)] \tag{3.15}$$

$$N(\phi_0^4(x)) = \phi_0^4(x) - 6\phi_0^2(x)[\phi_0^{(+)}(x), \phi_0^{(-)}(x)] + 3[\phi_0^{(+)}(x), \phi_0^{(-)}(x)]^2 \tag{3.16}$$

where the commutator of the positive and negative frequency parts is just

$$[\phi_0^{(+)}(x), \phi_0^{(-)}(x)] = {}_A\langle 0|\phi_0^2(x)|0\rangle_A. \tag{3.17}$$

Thus a different normal ordering procedure is obtained for each $|0\rangle_A$ in the class of adiabatic vacuum states of order T^{-2}. To make the normal ordering procedure unique we will evaluate (3.17) by dimensional regularisation and retain only the pole term, which is independent of which vacuum $|0\rangle_A$ is chosen. This procedure must ultimately be justified by renormalisation of coupling constants and we will show at the end of this section that normal ordering of the two-particle creation amplitude $\langle \boldsymbol{p}_1, \boldsymbol{p}_2, \text{in}|S|\text{in}\rangle$ is equivalent to renormalisations of m and ξ. In a following paper (Bunch and

Panangaden 1980) it will be shown that normal ordering of vacuum-to-vacuum processes can be justified by renormalisation of coupling constants in the Einstein action.

To evaluate (3.17) we substitute (3.6) and use equation (5.21) of Fulling *et al* (1974) to obtain

$$_{\mathrm{A}}\langle 0|\phi_0^2(x)|0\rangle_{\mathrm{A}} = \frac{1}{4\pi^2 C}\int_0^\infty \frac{k^{n-2}\,\mathrm{d}k}{W_k} \tag{3.18}$$

where we have regularised by replacing k^2 by k^{n-2} where n is the dimension of space–time. When $\epsilon = 1$ the integral should be replaced by a sum, but we will use (3.18) for all values of ϵ since we are only intending to keep the divergent pole terms. Using (3.13) to obtain W_k^{-1} to order T^{-2} and evaluating (3.18) we find that the pole terms are

$$_{\mathrm{A}}\langle 0|\phi_0^2(x)|0\rangle_{\mathrm{A}} \approx \frac{m^2 + (\xi - \frac{1}{6})R}{8\pi^2(n-4)}. \tag{3.19}$$

Denoting this quantity by $G_{\mathrm{D}}(x)$, the normal ordered interaction Hamiltonian density is

$$\begin{aligned}\mathcal{H}'(x) = {} & \tfrac{1}{2}\lambda\sqrt{g}\,[Z_2^{(1)}m^2 + Z_3^{(1)}\xi R][\phi_0^2(x) - G_{\mathrm{D}}(x)] \\ & + \tfrac{1}{4}\lambda\sqrt{g}[\phi_0^4(x) - 6\phi_0^2(x)G_{\mathrm{D}}(x) + 3G_{\mathrm{D}}^2(x)].\end{aligned} \tag{3.20}$$

Finally we notice that the expression (3.19) is exactly the same as the expression obtained for the divergences in the coincidence limit of the Feynman Green's function, $\lim_{x\to y} G(x, y)$, by the DeWitt–Schwinger method (DeWitt 1975). This method does not rely on the introduction of an adiabatic vacuum but it has the advantage of being valid for an arbitrary space–time. We therefore assume that if adiabatic vacuum states were constructed in an arbitrary space–time, equation (3.19) would continue to hold so that the normal ordering we have defined can be taken to be valid for a space–time with arbitrary metric.

We are now in a position to consider the evaluation of S-matrix elements and we will start with the S-matrix element which represents the creation of a pair of in-particles from the vacuum. This is

$$\begin{aligned}\langle \boldsymbol{p}_1, \boldsymbol{p}_2, \mathrm{in}|S|\mathrm{in}\rangle \\ = -\frac{\mathrm{i}\lambda}{2}\int \sqrt{g(x)}\,\mathrm{d}^n x\,\phi_{\boldsymbol{p}_1}^*(x)\phi_{\boldsymbol{p}_2}^*(x)[2(m^2 Z_2^{(1)} + \xi R Z_3^{(1)} \\ + 6\int \mathrm{d}\boldsymbol{k}\,|\phi_{\boldsymbol{k}}(x)|^2 - 6G_{\mathrm{D}}(x)].\end{aligned} \tag{3.21}$$

In this expression, $\phi_{\boldsymbol{p}_1}^*(x)$ and $\phi_{\boldsymbol{p}_2}^*(x)$ are the wavefunctions of the created particles. The contribution from figure 2 is the term involving

$$\int \mathrm{d}\boldsymbol{k}\,|\phi_{\boldsymbol{k}}(x)|^2 \equiv \langle \mathrm{in}|\phi_0^2(x)|\mathrm{in}\rangle \equiv G(x). \tag{3.22}$$

But $G(x)$ has the same divergence structure as $G_{\mathrm{D}}(x)$ so that we may write

$$G(x) = G_{\mathrm{D}}(x) + G_{\mathrm{R}}(x) \tag{3.23}$$

where $G_{\mathrm{R}}(x)$ is the renormalised $G(x)$ which is finite in four dimensions. It is now clear that (3.21) is finite if we take

$$Z_2^{(1)} = Z_3^{(1)} = 0 \tag{3.24}$$

and the amplitude to create two particles from the vacuum is

$$\langle p_1p_2, \mathrm{in}|S|\mathrm{in}\rangle = -3\mathrm{i}\lambda \int \sqrt{g(x)}\, \mathrm{d}^4x\phi^*_{p_1}(x)\phi^*_{p_2}(x)G_{\mathrm{R}}(x). \tag{3.25}$$

Because of (3.20) we can write, to first order in λ,

$$\mathscr{H}'(x) = \tfrac{1}{4}\lambda\sqrt{g}\,[\phi_0^4(x) - 6\phi_0^2G_{\mathrm{D}}(x) + 3G_{\mathrm{D}}^2(x)]. \tag{3.26}$$

The vacuum-to-vacuum amplitude is now finite and using the fact that

$$\langle \mathrm{in}|\phi_0^4(x)|\mathrm{in}\rangle = 3\langle \mathrm{in}|\phi_0^2(x)|\mathrm{in}\rangle^2 \equiv 3G(x)^2 \tag{3.27}$$

one sees that

$$\langle \mathrm{in}|S^{(1)}|\mathrm{in}\rangle = -\frac{3\mathrm{i}\lambda}{4}\int \sqrt{g(x)}\, \mathrm{d}^4x\, G_{\mathrm{R}}^2(x). \tag{3.28}$$

Thus normal ordering with respect to an adiabatic vacuum makes all first-order processes finite without any renormalisations of the physical parameters of the theory being required. This is exactly as in Minkowski space–time except that in curved space–time the renormalised amplitudes (3.25) and (3.28) are in general non-zero to first order. We will next consider briefly an alternative to normal ordering, namely subtracting from the Lagrangian operator its expectation value in an adiabatic vacuum state. This leads to a first-order interaction Hamiltonian:

$$\mathscr{H}'(x) = \tfrac{1}{2}\lambda\sqrt{g}\,[Z_2^{(1)}m^2 + Z_3^{(1)}\xi R][\phi_0^2(x) - G_{\mathrm{D}}(x)] + \tfrac{1}{4}\lambda\sqrt{g}\,[\phi_0^4(x) - 3G_{\mathrm{D}}^2(x)] \tag{3.29}$$

since

$$_{\mathrm{A}}\langle 0|\phi_0^4(x)|0\rangle_{\mathrm{A}} = 3(_{\mathrm{A}}\langle 0|\phi_0^2(x)|0\rangle_{\mathrm{A}})^2. \tag{3.30}$$

Then

$$\langle p_1p_2, \mathrm{in}|S|\mathrm{in}\rangle = -\frac{\mathrm{i}\lambda}{2}\int \sqrt{g(x)}\, \mathrm{d}^nx\phi^*_{p_1}(x)\phi^*_{p_2}(x)[2(m^2Z_2^{(1)} + \xi RZ_3^{(1)}) + 6G(x)] \tag{3.31}$$

but

$$G(x) = \frac{m^2 + (\xi - \frac{1}{6})R}{8\pi^2(n-4)} + G_{\mathrm{R}}(x). \tag{3.32}$$

Thus, taking

$$Z_2^{(1)} = -\frac{3}{8\pi^2(n-4)} \tag{3.33}$$

$$\xi Z_3^{(1)} = (\xi - \tfrac{1}{6})Z_2^{(1)} \tag{3.34}$$

leads to the same expression as before, namely (3.25). Similarly, the vacuum-to-vacuum amplitude is given by (3.28). Thus this alternative procedure gives the same results as normal ordering but requires renormalisations of m and ξ. It is not difficult to see that the two procedures will be equivalent to all orders in perturbation theory.

4. The energy–momentum tensor

The expectation value of the stress tensor at late times, if the initial state is the in-vacuum, is

$$\langle T_{\mu\nu}^{(\mathrm{out})}\rangle \equiv \langle \mathrm{in}|\hat{T}_{\mu\nu}|\mathrm{in}\rangle = \langle \mathrm{in}|S^\dagger T_{\mu\nu}S|\mathrm{in}\rangle \tag{4.1}$$

where the stress tensor operator in the Heisenberg or interaction picture is evaluated at late times, after the $\lambda\phi^4$ interaction has effectively been switched off. This means that the stress tensor at late times must have the form of the free-field stress tensor:

$$T_{\mu\nu} \equiv T_{\mu\nu}^{(\text{out})} = \tfrac{1}{2}(1-2\xi)\{\partial_\mu\phi, \partial_\nu\phi\} + \tfrac{1}{2}(2\xi - \tfrac{1}{2})g_{\mu\nu}\{\partial_\rho\phi, \partial^\rho\phi\} - \xi\{\phi, \nabla_\mu\partial_\nu\phi\}$$
$$+ \xi g_{\mu\nu}\{\phi, \Box\phi\} - \xi G_{\mu\nu}\phi^2 + \tfrac{1}{2} m^2 g_{\mu\nu}\phi^2. \tag{4.2}$$

We have written $T_{\mu\nu}^{(\text{out})}$ entirely in terms of renormalised quantities: at late times, when the system is dispersed, the only effect of the self-interaction is to renormalise the physical parameters. Since $T_{\mu\nu}^{(\text{out})}$ is bilinear in the field and its derivatives, normal ordering of $T_{\mu\nu}^{(\text{out})}$ is straightforward:

$$:T_{\mu\nu}^{(\text{out})}: = T_{\mu\nu}^{(\text{out})} - {}_{\text{A}}\langle 0|T_{\mu\nu}^{(\text{out})}|0\rangle_{\text{A}}. \tag{4.3}$$

To first order in λ

$$\langle :T_{\mu\nu}^{(\text{out})}:\rangle = \langle \text{in}|:T_{\mu\nu}^{(\text{out})}:|\text{in}\rangle + \langle \text{in}|[T_{\mu\nu}^{(\text{out})}, S^{(1)}]|\text{in}\rangle \tag{4.4}$$

where $S^{(1)}$ is defined by equation (2.28) and we used $S^{(1)+} = -S^{(1)}$. The first term is the same as in the absence of the $\lambda\phi^4$ interaction and is known to be finite provided that the normal ordering of $T_{\mu\nu}^{(\text{out})}$ is carried out to adiabatic order T^{-4}.

The final term in equation (4.4) can be evaluated by inserting a complete set of states of $\mathscr{F}_{\text{in}}$ between $S^{(1)}$ and $T_{\mu\nu}^{(\text{out})}$. Only the two-particle states contribute and one obtains

$$\langle \text{in}|[T_{\mu\nu}^{(\text{out})}, S^{(1)}]|\text{in}\rangle = 2\,\text{Re}\int d\mu(\boldsymbol{p}_1)d\mu(\boldsymbol{p}_2)[\langle \text{in}|T_{\mu\nu}^{(\text{out})}|\boldsymbol{p}_1, \boldsymbol{p}_2, \text{in}\rangle\langle \text{in}, \boldsymbol{p}_1\boldsymbol{p}_2|S^{(1)}|\text{in}\rangle]. \tag{4.5}$$

The renormalised matrix element of $S^{(1)}$ appearing in (4.5) is given by (3.25). The contribution of a typical term from the stress tensor, say ϕ^2, to (4.5) is:

$$\langle \text{in}|[\phi^2(x), S^{(1)}]|\text{in}\rangle$$
$$= 2\,\text{Re}\int d\mu(\boldsymbol{p}_1)d\mu(\boldsymbol{p}_2)\langle \text{in}\,|\phi^2(x)|\boldsymbol{p}_1\boldsymbol{p}_2, \text{in}\rangle\langle \text{in}, \boldsymbol{p}_1\boldsymbol{p}_2|S^{(1)}|\text{in}\rangle \tag{4.6}$$
$$= -12\lambda\,\text{Re}\,\text{i}\int d\mu(\boldsymbol{p}_1)d\mu(\boldsymbol{p}_2)\int \sqrt{g(x')}\,d^4x'\phi_{\boldsymbol{p}_1}(x)\phi_{\boldsymbol{p}_2}(x)$$
$$\times G_{\text{R}}(x')\phi^*_{\boldsymbol{p}_1}(x')\phi^*_{\boldsymbol{p}_2}(x'). \tag{4.7}$$

In this equation, x is taken to be at late times so that $\phi_{\boldsymbol{p}}(x)$ may be expressed according to (2.36). The momentum integrals in (4.7) do not give rise to divergences since the integration over $x^{0'}$ extends over the finite range, say from $-T$ to T, during which the interaction is switched on, so that $x^{0'}$ is always earlier than x^0. Thus the momentum integrals are effectively cut off for large momenta by the oscillating terms of the form

$$\phi_{\boldsymbol{p}}(x)\phi^*_{\boldsymbol{p}}(x') \sim \exp[\text{i}p(x^0 - x^{0'})]. \tag{4.8}$$

Thus one should carry out the integrations over $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ in (4.7) before taking the limit $T \to \infty$. The other terms in $T_{\mu\nu}^{(\text{out})}$ make similar contributions to (4.5) with appropriate derivatives acting on the $\phi_{\boldsymbol{p}}(x)$. Thus, the expression for $\langle T_{\mu\nu}^{(\text{out})}\rangle$ is finite to first order in λ.

We will now specialise to a spatially flat Robertson–Walker metric

$$ds^2 = dt^2 - a^2(t)d\boldsymbol{x}\,.\,d\boldsymbol{x} \tag{4.9}$$

which we assume reduces to Minkowski space–time at late and early times, so that we can discuss how the presence of a self-interaction affects the spectrum of created particles. For simplicity we will assume that, after renormalisation, $\xi = 0$, and we will start by showing that the energy density of the created particles is dominated at late times by a term which can be expressed as

$$\langle T_{00}\rangle = \int d^3\boldsymbol{p}\mathcal{N}(\boldsymbol{p})\Omega_p \tag{4.10}$$

where $\mathcal{N}(\boldsymbol{p})$ is the number density of created particles in mode $\boldsymbol{p}$ per unit proper volume and Ω_p is the energy of a particle in mode $\boldsymbol{p}$:

$$\Omega_p = \left(\frac{p^2}{a^2} + m^2\right)^{\frac{1}{2}}. \tag{4.11}$$

For the metric (4.9) the modes $\phi_{\boldsymbol{p}}(x)$ separate into

$$\phi_{\boldsymbol{p}}(x) = \exp(i\boldsymbol{p}\cdot\boldsymbol{x})\psi_p(t) \tag{4.12}$$

and since $G_R(x)$ is defined in terms of absolute values of modes, $G_R(x)$ depends only on t, so we will write it $G_R(t)$. Then, taking $\sqrt{g} = a^3(t)$, (4.7) becomes

$$\begin{aligned}\langle \mathrm{in}|[\phi^2(x), S^{(1)}]|\mathrm{in}\rangle \\ &= -12\lambda \,\mathrm{Re}\, i\int d\mu(\boldsymbol{p}_1)d\mu(\boldsymbol{p}_2)\int d^3\boldsymbol{x}'\exp[-i(\boldsymbol{p}_1+\boldsymbol{p}_2)\cdot\boldsymbol{x}']\int a^3(t')G_R(t')dt' \\ &\quad\times\phi_{\boldsymbol{p}_1}(x)\phi_{\boldsymbol{p}_2}(x)\psi^*_{p_1}(t')\psi^*_{p_2}(t') \qquad (4.13) \\ &= -96\pi^3\lambda\,\mathrm{Re}\, i\int d^3\boldsymbol{p}\int a^3(t')G_R(t')\psi_p(t)^2\psi_p^*(t')^2dt'. \qquad (4.14)\end{aligned}$$

For the spatially flat Robertson–Walker universe, the Bogolubov transformation (2.37) simplifies to

$$A_{\boldsymbol{k}}^{\mathrm{out}} = \alpha_k A_{\boldsymbol{k}} + \beta_k^* A_{-\boldsymbol{k}}^{\dagger} \tag{4.15}$$

so that we can write, for late times t,

$$\psi_p(t) = \alpha_p\psi_p^{\mathrm{out}}(t) + \beta_p\psi_p^{\mathrm{out}*}(t) \qquad (t > T). \tag{4.16}$$

Thus we obtain

$$\begin{aligned}\langle \mathrm{in}|[\phi^2(x), S^{(1)}]|\mathrm{in}\rangle \\ &= -96\pi^3\lambda\,\mathrm{Re}\, i\int d^3\boldsymbol{p}\int a^3(t')G_R(t')\psi_p^*(t')^2dt'[\alpha_p^2\psi_p^{\mathrm{out}}(t)^2 \\ &\quad+\beta_p^2\psi_p^{\mathrm{out}*}(t)^2 + 2\alpha_p\beta_p|\psi_p^{\mathrm{out}}(t)|^2]. \qquad (4.17)\end{aligned}$$

For large t the terms $\psi_p^{\mathrm{out}}(t)^2$ and $\psi_p^{\mathrm{out}*}(t)^2$ oscillate rapidly as functions of p, whereas $|\psi_p^{\mathrm{out}}(t)|^2$, which is independent of t, does not. Thus the dominant term in (4.16) at late times is

$$\langle \mathrm{in}|[\phi^2(x), S^{(1)}]|\mathrm{in}\rangle \approx -192\pi^3\lambda\,\mathrm{Re}\, i\int d^3\boldsymbol{p}\alpha_p\beta_p|\psi_p(t)|^2\int a^3(t')G_R(t')\psi_p^*(t')^2\,dt'. \tag{4.18}$$

The contribution of this term and similar terms coming from $\langle\partial_\mu\phi\partial_\nu\phi\rangle$, etc, to the energy density is

$$\rho_1 = -96\pi^3\lambda\,\mathrm{Re}\, i\int d^3\boldsymbol{p}\alpha_p\beta_p[|\partial_t\psi_p^{\mathrm{out}}|^2 + \Omega_p^2|\psi_p^{\mathrm{out}}|^2]\int a^3(t')G_R(t')\psi_p^*(t')^2dt'. \tag{4.19}$$

Similarly, the free-field contribution to the energy density has the dominant contribution

$$\rho_0 = \int d^3\boldsymbol{p}|\beta_p|^2[|\partial_t\psi_p^{\text{out}}|^2 + \Omega_p^2|\psi_p^{\text{out}}|^2]. \tag{4.20}$$

But at late times the space–time becomes flat and

$$\psi_p^{\text{out}}(t) = \frac{\exp(-\mathrm{i}\Omega_p t)}{(2\pi a_0)^{3/2}(2\Omega_p)^{1/2}} \tag{4.21}$$

where the quantity a_0 appearing in (4.21) is the constant limit of $a(t)$ for large t, so that

$$|\psi_p^{\text{out}}|^2 = \frac{1}{16\pi^3 a_0^3\Omega_p} \qquad |\partial_t\psi_p^{\text{out}}|^2 = \frac{\Omega_p}{16\pi^3 a_0^3}. \tag{4.22}$$

Hence

$$\langle T_{00}\rangle \equiv \rho_0 + \rho_1 = \frac{1}{8\pi^3 a_0^3}\int d^3\,\boldsymbol{p}\Omega_p|\beta_p|^2 - \frac{12\lambda}{a_0^3}\operatorname{Re}\mathrm{i}\int d^3\boldsymbol{p}\Omega_p\alpha_p\beta_p\int a^3(t')G_{\mathrm{R}}(t')\psi_p^*(t')^2 dt'. \tag{4.23}$$

This may be written

$$\rho = \int d^3\boldsymbol{p}\Omega_p\mathcal{N}(\boldsymbol{p}) \tag{4.24}$$

where

$$\mathcal{N}(\boldsymbol{p}) = \frac{|\beta_p|^2}{(2\pi a_0)^3} - \frac{12\lambda}{a_0^3}\operatorname{Re}\mathrm{i}\,\alpha_p\beta_p\int a^3(t')G_{\mathrm{R}}(t')\psi_p^*(t')^2\,dt' \tag{4.25}$$

is the expected number density of particles per unit proper volume in mode $\boldsymbol{p}$ at late times. To verify this final assertion for the order λ term (the free-field result is well-known to be given by (4.24) with $\lambda = 0$; see, for example, Parker (1977)) we must evaluate the order λ term in

$$\langle\text{in}|S^\dagger N_{\boldsymbol{p}}^{\text{out}}S|\text{in}\rangle \equiv \langle\text{in}|S^\dagger A_{\boldsymbol{p}}^{\dagger\text{out}}A_{\boldsymbol{p}}^{\text{out}}S|\text{in}\rangle \tag{4.26}$$

and then divide by the proper volume at late times which is a_0^3V where V is the coordinate volume:

$$V = \int d^3\boldsymbol{x}. \tag{4.27}$$

Strictly, the number density is defined as the limit of the number density of particles in a finite volume, V, of space–time as $V \to \infty$. The order λ term in (4.26) is

$$\langle\text{in}|[N_{\boldsymbol{p}}^{\text{out}}, S^{(1)}]|\text{in}\rangle = 2\operatorname{Re}\int d\boldsymbol{p}_1 d\boldsymbol{p}_2\langle\text{in}|N_{\boldsymbol{p}}^{\text{out}}|\boldsymbol{p}_1, \boldsymbol{p}_2, \text{in}\rangle\langle\boldsymbol{p}_1, \boldsymbol{p}_2, \text{in}|S^{(1)}|\text{in}\rangle. \tag{4.28}$$

But

$$N_{\boldsymbol{p}}^{\text{out}} = A_{\boldsymbol{p}}^{\text{out}\dagger}A_{\boldsymbol{p}} = |\alpha_p|^2A_{\boldsymbol{p}}^\dagger A_{\boldsymbol{p}} + |\beta_p|^2A_{-\boldsymbol{p}}A_{-\boldsymbol{p}}^\dagger + \alpha_p^*\beta_p^*A_{\boldsymbol{p}}^\dagger A_{-\boldsymbol{p}}^\dagger + \alpha_p\beta_pA_{\boldsymbol{p}}A_{-\boldsymbol{p}} \tag{4.29}$$

so that

$$\langle \text{in}|[N_p^{\text{out}}, S^{(1)}]|\text{in}\rangle$$

$$= 4 \operatorname{Re} \alpha_p\beta_p\langle \boldsymbol{p}, -\boldsymbol{p}, \text{in} \mid S^{(1)}|\text{in}\rangle \tag{4.30}$$

$$= -12\lambda \operatorname{Re} \mathrm{i}\, \alpha_p\beta_p \int \sqrt{g(x')}\, \mathrm{d}^4x'\phi_p^*(x')\phi_{-p}^*(x')G_{\mathrm{R}}(x') \tag{4.31}$$

$$= -12\lambda \operatorname{Re} \mathrm{i}\, \alpha_p\beta_p\, V \int a^3(t')\psi_p^*(t')^2 G_{\mathrm{R}}(t')\mathrm{d}t'. \tag{4.32}$$

Dividing by $a_0^3 V$ leads to the order λ term of (4.24) as claimed.

One can use the results of the preceding discussion to study to first order in λ the effect of a $\lambda\phi^4$ interaction on particle creation in a Robertson–Walker universe. For example, when $\xi = \frac{1}{6}$ and $m = 0$ then $G_{\mathrm{R}}(x) = 0$ and the $\lambda\phi^4$ interaction has no effect to first order in λ (see equations (4.7) and (4.25)). This result has been proved to first order by Birrell and Ford (1980). In Parker (1973) it was shown to all orders in λ that no massless conformally invariant scalar particles with $\lambda\phi^4$ interaction are created by the gravitational field in a conformally flat space–time. However, there may be higher order modifications to the stress tensor resulting from vacuum polarisation effects, similar to the trace anomaly of a free field. Another question one can study is the following one, involving consistency between the second law of thermodynamics and the predictions of quantum field theory.

Under the circumstances (i.e. special form of $a(t)$ and loss of correlations among created pairs) that the expansion of the universe gives rise directly to black-body radiation for the free field, does the inclusion of a $\lambda\phi^4$ interaction destroy the black-body spectrum? Free-field examples have been discussed by Parker (1977) and Audretsch and Schäfer (1978) and it is probable that almost any form of $a(t)$ will give rise to a spectrum of created particles which has the black-body form at sufficiently high frequencies. One would expect that interactions should not influence the black-body nature of the spectrum because the entropy is already a maximum and cannot decrease as the interaction is turned on.

The black-body radiation is in a mixed state rather than a pure state (otherwise it would not possess such a large entropy). In the free-field calculations which give rise to black-body radiation, it was therefore necessary to assume that the correlations among the created pairs are effectively destroyed (for example, through scattering, decays or as a result of large spatial separation). A formal method frequently employed to destroy such correlations in a superposition of states (for example, in discussions of the measurement process) is to introduce arbitrary phase factors in the coefficients of the states making up the superposition and to average over these phases. In the present context, we will use this method to destroy correlations among the created particles at late times.

Consider the wavefunction $\phi_p(x)$. At early times it reduces to $\phi_p^{\text{in}}(x)$ and describes a particle in the in-region. At late times it has the form

$$\phi_p(x) = \alpha_p\phi_p^{\text{out}}(x) + \beta_p\phi_{-p}^{\text{out}*}(x) \tag{4.33}$$

which describes a superposition of a particle and an antiparticle in the out-region, with α_p and β_p being the coefficients in the superposition. To destroy correlations between members of the created pairs we introduce arbitrary phase factors in α and β:

$$\alpha \to \exp(\mathrm{i}\gamma_\alpha)\alpha, \qquad \beta \to \exp(\mathrm{i}\gamma_\beta)\beta \tag{4.34}$$

where γ_α and γ_β are arbitrary real variables. To obtain the results for a statistical mixture, we average over these variables. From equations (4.18) and (4.25) it is then clear that when this averaging over phases is performed, the terms of order λ in $\mathcal{N}(\boldsymbol{p})$ and $\langle T_{\mu\nu}\rangle$ vanish. Thus, to order λ the spectrum at late times is not altered by the self-interaction.

5. Preliminary discussion of renormalisation in second order

To second order in λ, the normal ordered interaction Hamiltonian density is

$$\begin{aligned}\mathcal{H}'(x) = \tfrac{1}{4}\lambda\sqrt{g}\,[\phi_0^4(x) - 6\phi_0^2(x)G_{\mathrm{D}}(x) + 3G_{\mathrm{D}}^2(x)]\\ +\tfrac{1}{2}\lambda^2\sqrt{g}\,[Z_2^{(2)}m^2 + Z_3^{(2)}\xi R][\phi_0^2(x) - G_{\mathrm{D}}(x)]\\ +\tfrac{1}{4}\lambda^2\sqrt{g}\,Z_4^{(1)}[\phi_0^4(x) - 6\phi_0^2(x)G_{\mathrm{D}}(x) + 3G_{\mathrm{D}}^2(x)].\end{aligned} \tag{5.1}$$

Second-order S-matrix elements can be evaluated by using Wick's first theorem to express the time-ordered product of two field operators [for example, $T(\phi_0^4(x)\phi_0^4(y))$] in terms of contractions (free Feynman Green's functions) and normal ordered operators. It is most convenient to perform this decomposition of time-ordered products with respect to the 'in' Fock space, $\mathcal{F}_{\mathrm{in}}$. Then the Feynman Green's function is

$$G(x, y) = \langle \mathrm{in}|T(\phi_0(x)\phi_0(y))|\mathrm{in}\rangle \tag{5.2}$$

and the normal-ordering in the Wick expansion is with respect to the interaction picture annihilation and creation operators.

The simplest second-order S-matrix element to renormalise is that representing the creation of four in-particles from the vacuum, $\langle \boldsymbol{p}_1, \ldots, \boldsymbol{p}_4, \mathrm{in}|S|\mathrm{in}\rangle$. In terms of Feynman diagrams, this is the sum of figure 4 and 5.

It is clear that a suitable choice of $Z_4^{(1)}$ makes $\langle \boldsymbol{p}_1, \ldots, \boldsymbol{p}_4, \mathrm{in}|S|\mathrm{in}\rangle$ finite. We find that $Z_4^{(1)}$ has the same value as in Minkowski space–time, namely

$$Z_4^{(1)} = -\frac{9}{8\pi^2(n-4)}. \tag{5.3}$$

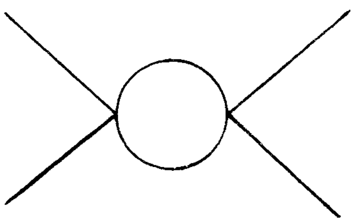

Figure 4.

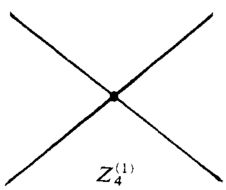

Figure 5.

In curved space–time, the essence of the calculation is the evaluation of the divergences in

$$\int G^2(x, y)\sqrt{g(y)}\,\mathrm{d}^n y \tag{5.4}$$

or, equivalently, in

$$\int G(x, y)\sqrt{g(y)}\,\mathrm{d}^n y\, G(y, x) \tag{5.5}$$

which has the same divergences as (5.4). But (5.5) is just

$$\zeta(2, x) \approx \frac{\mathrm{i}}{8\pi^2(n-4)} + \text{finite terms} \tag{5.6}$$

where $\zeta(\nu, x)$ is the zeta function for the manifold, sometimes written $\langle x|G^\nu|x\rangle$.

Consider now the creation of two in-particles from the vacuum. The second-order diagrams are figures 6–9.

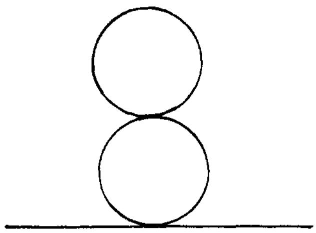

Figure 6.

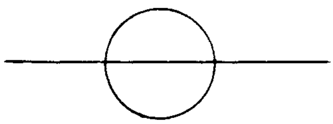

Figure 7.

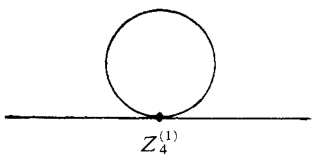

Figure 8.

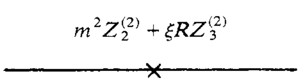

Figure 9.

The loops in figures 6 and 8 which begin and end at the same point are, in fact, made finite by the normal ordering we have performed. Thus the S-matrix element corresponding to figure 6 is proportional to

$$\int \phi^*_{p_1}(x)\phi^*_{p_2}(x)G^2(x, y)G_R(y)\sqrt{g(x)g(y)}\, d^nx\, d^ny. \tag{5.7}$$

The action of $G^2(x, y)$ as a distribution in y is, from (5.5) and (5.6),

$$G^2(x, y) = \frac{i}{8\pi^2(n-4)} g^{-1/2}(x)\delta(x, y) + \text{finite term} \tag{5.8}$$

where by 'finite term' is meant a distribution which maps test functions in n dimensions to quantities which are finite as $n \to 4$. Thus the divergence in (5.7) is proportional to

$$\frac{i}{8\pi^2(n-4)} \int \phi^*_{p_1}(x)\phi^*_{p_2}(x)G_R(x)\sqrt{g(x)}\, d^nx. \tag{5.9}$$

The divergence in figure 8 has the same structure. These two divergent quantities do not cancel out when added together, so it is to be hoped that they will be cancelled by a similar contribution from figure 7. This calculation has recently been performed by two of us (Bunch and Panangaden 1980) who found that there is indeed a contribution having the same form as (5.9) which achieves the required cancellation so that no divergences involving $G_R(x)$ remain when all diagrams representing the creation of two in-particles from the vacuum are summed.

The divergences in figure 7 are known once the structure of $G^3(x, y)$ as a distribution in y is known. Birrell and Taylor (1980) have shown that the divergences arising from the evaluation of $G^3(x, y)$ can be expressed by writing $G^3(x, y)$ as a distribution having the following form:

$$G^3(x, y) = f(x)\tilde{\delta}(x, y) + f^\mu(x)\partial_\mu\tilde{\delta}(x, y) + f^{\mu\nu}(x)\nabla_\mu\partial_\nu\tilde{\delta}(x, y) + \text{finite term} \tag{5.10}$$

where $f(x)$, $f^\mu(x)$ and $f^{\mu\nu}(x)$ are a scalar, a vector and a tensor field having poles at $n = 4$ and $\tilde{\delta}(x, y)$ is the invariant delta function, $g^{-1/2}(x)\delta(x, y)$. (Birrell and Taylor actually discussed the structure of $G^2(x, y)$ as a distribution in six dimensions (so that poles appear at $n = 6$), but their arguments apply equally well for $G^3(x, y)$ in four dimensions.) On the basis of dimensional arguments and by considering an expansion of the metric $g_{\mu\nu}$ about Minkowski space–time, Birrell and Taylor deduced that $f(x)$ and $f^\mu(x)$ must have dimensions (length)$^{-2}$ and (length)$^{-1}$ and that $f^{\mu\nu}(x)$ is dimensionless and, moreover, that they can all be expressed in terms of quantities which involve no more than two derivatives of the metric. This implies that $f(x)$ is a linear combination of m^2, R and $G_R(x)$, $f^\mu(x)$ is zero and $f^{\mu\nu}(x)$ is proportional to the metric, $g^{\mu\nu}$. Thus

$$G^3(x, y) = [c_0m^2 + c_1R + c_2G_R(x)]\tilde{\delta}(x, y) + c_3\Box\tilde{\delta}(x, y) \tag{5.11}$$

where c_0, c_1, c_2 and c_3 are constants having poles at $n = 4$. Birrell and Taylor assumed that $G^3(x, y)$ could only contain information about the mass of the field and the geometry of space–time and so did not consider the possibility of $G_R(x)$ appearing in (5.11). However, $G^3(x, y)$ contains information about the quantum state and this shows up in the appearance of $G_R(x)$ in (5.11).

The S-matrix element corresponding to figure 7 is proportional to

$$\int \phi^*_{p_1}(x)\phi^*_{p_2}(y)G^3(x,y)\sqrt{g(x)g(y)}\,\mathrm{d}^n x\,\mathrm{d}^n y$$
$$\approx \int \phi^*_{p_1}(x)\phi^*_{p_2}(x)[c_0 m^2 + c_1 R + c_2 G_{\mathrm{R}}(x)]\sqrt{g(x)}\,\mathrm{d}^n x$$
$$+ c_3 \int \Box\, \phi^*_{p_1}(x)\phi^*_{p_2}(x)\sqrt{g(x)}\,\mathrm{d}^n x. \tag{5.12}$$

The term involving $G_{\mathrm{R}}(x)$ has the same structure as (5.9) and, as mentioned above, we have verified that the sum of all such terms is zero. The remaining terms in (5.12) are, using the wave equation,

$$\int \phi^*_{p_1}(x)\phi^*_{p_2}(x)[(c_0 - c_3)m^2 + (c_1 - \xi c_3)R]\sqrt{g(x)}\,\mathrm{d}^n x. \tag{5.13}$$

Obviously these divergences are removed by suitable choices of $Z_2^{(2)}$ and $Z_3^{(2)}$ in the S-matrix element corresponding to figure 9.

In this section we have given a brief discussion of the renormalisation of second-order particle creation processes and we have indicated that the appearance of divergences involving $G_{\mathrm{R}}(x)$ does not affect renormalisability. Similar divergences appear in vacuum-to-vacuum processes and we have also verified that they cancel when all such diagrams are summed. A more complete account of the renormalisation of second-order processes, including vacuum-to-vacuum processes, appears in Bunch and Panangaden (1980).

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