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Spacetime geometry from causal structure and a measurement

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Abstract. The causal structure of spacetime defines a partial order on the events of spacetime. In an earlier paper, using techniques from domain theory, we showed that for globally hyperbolic spacetimes one could reconstruct the topology from the causal structure. However, the causal structure determines the metric only up to a local rescaling (a conformal transformation); in a four dimensional spacetime, the metric tensor has ten components, and thus effectively only nine are determined by the causal structure. After establishing the relationship between measurement in domain theory, the concept of global time function and the Lorentz distance, we are able to domain theoretically recover the final tenth component of the metric tensor, thereby obtaining causal reconstruction of not only the topology of spacetime, but also its geometry.

1. Introduction

The study of spacetime structure from an abstract viewpoint – i.e., not from the viewpoint of solving differential equations – was initiated by Penrose [**18**] in a dramatic paper in which he showed a fundamental inconsistency of gravity: all the spacetimes satisfying some general conditions develop singularities. Penrose's paper initiated a whole new way of studying general relativity: an abstract approach using ideas of differential topology and geometry rather than looking for solutions of Einstein's equations.

It was known since Chandrasekhar [**3**] that since gravity is universal and inherently attractive, a gravitating mass of sufficient size will eventually collapse. It was widely believed that the collapse phenomenon discovered by Chandrasekhar was an artifact of special symmetry assumptions and that in a realistic situation perturbations would prevent the appearance of singularities. Penrose dashed this hope by showing that singularities arise *generically.* What Penrose showed was that any such collapse eventually leads to a singularity where the mathematical description of spacetime as a continuum breaks down. This leads to the need to reformulate gravity. Part of the motivation for the search for a quantum theory of gravity is

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the hope that this elusive theory will resolve the problem of gravitational collapse. A good discussion of the history of these ideas is in a recent book by Hawking and Penrose [**9**].

Since the first singularity theorems [**18, 8**] causality has played a key role in understanding spacetime structure. The analysis of causal structure relies heavily on techniques of differential topology [**19**]. For the past decade Sorkin and others [**21**] have pursued a program for quantization of gravity based on causal structure. In this approach the causal relation is regarded as the fundamental ingredient and the topology and geometry are secondary.

In a paper that appeared in 2006 [**15**], we prove that the causality relation is much more than a relation – it turns a globally hyperbolic spacetime into what is known as a *bicontinuous poset*. The order on a bicontinuous poset allows one to define an intrinsic topology called *the interval topology*. On a globally hyperbolic spacetime, the interval topology is the manifold topology. Theorems that reconstruct the spacetime topology have been known [**19**] and Malament [**12**] has shown that the class of time-like curves determines the causal structure. We establish these results as well though in a purely order theoretic fashion: there is no need to know what "smooth curve" means.

Our more abstract stance also teaches us something *new*: a globally hyperbolic spacetime *itself* can be reconstructed in a purely order theoretic manner, beginning from only a countable dense set of events and the causality relation. The ultimate reason for this is that the category of globally hyperbolic posets, which contains the globally hyperbolic spacetimes, is *equivalent* to a very special category of posets called *interval domains*. This provides a profound connection between domain theory, first introduced for the purposes of assigning semantics to programming languages, and general relativity, a theory meant to explain gravity. Even from a purely mathematical perspective this equivalence is surprising, since globally hyperbolic spacetimes are usually not order theoretically complete, but interval domains always are.

While our previous work has focused on the role of domain theory in investigating qualitative aspects in relativity [**16**] – like the topology – in this paper, we investigate reconstructing quantitative aspects of spacetime structure: the metric. The theory of measurement was introduced by Martin in [**13**] as a way of incorporating quantitative information into domain theory. In this paper we will show how not only the topology, but the *geometry* of spacetime can be reconstructed order theoretically from the causal structure together with an appropriate measurement. The reason is that the Lorentz distance defines a Scott continuous function on the domain of spacetime intervals. What is even more interesting, though, is that our setting provides a way to *topologically* distinguish between Newtonian and relativistic notions of time. Every global time function defines a measurement on the domain of spacetime intervals, in particular, it is Scott continuous. The Lorentz distance is not only Scott continuous, but satisfies a stronger property, that it is interval continuous. An interval continuous function must assign zero to any element which approximates nothing. In all spacetimes there are non-empty intervals that correspond to a null line segment; these do not approximate anything (but they are not maximal either since they will contain other null sub-intervals) and indeed their "length" in the Lorentz metric is zero. Thus, no interval continuous function on the domain of spacetime intervals can ever be a measurement and the reason for this has entirely to do with relativity: a clock moving at the speed of light records no time as having elapsed, so an interval continuous function is incapable of distinguishing between a single event and a null interval. In Section 7 we discuss this point at length.

2. Domains, continuous posets and topology

We review some basic concepts which can be found, for example in the comprehensive book "Continuous Lattices and Domains" [**7**]. Occasionally our terminology differs; we will point out such occasions.

A *poset* is a partially ordered set, i.e., a set together with a reflexive, antisymmetric and transitive relation.

Definition 2.1. *Let* $(P, \sqsubseteq)$ *be a partially ordered set. A nonempty subset* $S \subseteq P$ *is* directed *if* $(\forall x, y \in S)(\exists z \in S)\, x, y \sqsubseteq z$. *The* supremum *of* $S \subseteq P$ *is the least of all its upper bounds provided it exists. This is written* $\bigsqcup S$.

These ideas have duals that will be important to us: a nonempty $S \subseteq P$ is *filtered* if $(\forall x, y \in S)(\exists z \in S)\, z \sqsubseteq x, y$. The *infimum* $\bigwedge S$ of $S \subseteq P$ is the greatest of all its lower bounds provided it exists.

Definition 2.2. *For a subset* X *of a poset* P, *set*

$$\uparrow\! X := \{y \in P : (\exists x \in X)\, x \sqsubseteq y\} \ \ \& \ \ \downarrow\! X := \{y \in P : (\exists x \in X)\, y \sqsubseteq x\}.$$

We write $\uparrow x = \uparrow \{x\}$ *and* $\downarrow x = \downarrow \{x\}$ *for elements* $x \in X$.

A partial order allows for the derivation of several intrinsically defined topologies. Here is our first example.

Definition 2.3. *A subset* U *of a poset* P *is* Scott open *if*

(i) *U is an upper set:* $x \in U \ \&\ x \sqsubseteq y \Rightarrow y \in U$, *and*
(ii) *U is inaccessible by directed suprema: For every directed* $S \subseteq P$ *with a supremum,*

$$\bigsqcup S \in U \Rightarrow S \cap U \neq \emptyset.$$

The collection of all Scott open sets on P *is called the* Scott topology.

Closely related to directed sets are ideals.

Definition 2.4. *An* ideal I *in a poset is a directed set such that if* $x \in I$ *and* $y \leq x$, *then* $y \in I$. *A set with the latter property is called a* lower set.

Posets can have a variety of completeness properties. The following completeness condition has turned out to be particularly useful in applications.

Definition 2.5. *A* dcpo *is a poset in which every directed subset has a supremum. The* least element *in a poset, when it exists, is the unique element* $\perp$ *with* $\perp \sqsubseteq x$ *for all* x.

If one takes any poset the collection of ideals ordered by inclusion forms a dcpo. This means that the union of any directed family of ideals is an ideal. It is easy to check this explicitly from the definition.

The set of *maximal elements* in a dcpo D is

$$\max(D) := \{x \in D : \uparrow\! x = \{x\}\}.$$

Each element in a dcpo has a maximal element above it; this follows at once from Zorn's Lemma and indeed is equivalent to it and hence to the Axiom of Choice.

Definition 2.6. *For elements x, y of a poset, write $x \ll y$ iff for all directed sets S with a supremum,*

$$y \sqsubseteq \bigsqcup S \Rightarrow (\exists s \in S)\, x \sqsubseteq s.$$

We set $\mathord{\downarrow\!\!\downarrow} x = \{a \in D : a \ll x\}$ *and* $\mathord{\uparrow\!\!\uparrow} x = \{a \in D : x \ll a\}$.

For the symbol "$\ll$," read "approximates." A number of basic properties are immediate from the definition. For example, the fact that the relation is transitive and the following:

$$x \le y \ll z \le w \Rightarrow x \ll w.$$

Definition 2.7. *A* basis *for a poset D is a subset B such that $B \cap \mathord{\downarrow\!\!\downarrow} x$ contains a directed set with supremum x for all $x \in D$. A poset is* continuous *if it has a basis. A poset is* ω-continuous *if it has a countable basis.*

Continuous posets have an important property, they are *interpolative.*

Proposition 2.8. If $x \ll y$ in a continuous poset P, then there is $z \in P$ with $x \ll z \ll y$.

Proof: Consider the set $K = \{u \mid \exists v.u \ll v \ll y\}$. This set is clearly not empty since x is the supremum of elements that approximate it so for some w we have $w \ll x \ll y$. Let $u_1, u_2 \in K$ then we have $u_1 \ll v_1 \ll y$ and $u_2 \ll v_2 \ll y$ for some v_1, v_2. Since $\mathord{\downarrow\!\!\downarrow} y$ is directed there is some $v \ll y$ with $v_1, v_2 \le v$. Thus $u_1, u_2 \ll v$ and since $\mathord{\downarrow\!\!\downarrow} v$ is directed we have an element $u \ll v$ with $u_1, u_2 \le u$. Now since $u \ll v \ll y$, $u \in K$, hence K is a directed set. Now clearly $\bigsqcup K = z$ so by the definition of $x \ll z$ we have that there is some $w \in K$ with $x \le w$ which means that there is some z such that $x \le w \ll z \ll y$. $\square$

This proof is taken from [**10**]. A very short proof using ideals can be found in [**7**].

This enables a clear description of the Scott topology.

Theorem 2.9. *The collection $\{\mathord{\uparrow\!\!\uparrow} x : x \in D\}$ is a basis for the Scott topology on a continuous poset.*

Proof: From the interpolation property it easily follows that sets of the form $\mathord{\uparrow\!\!\uparrow} x$ are Scott open. If U is any Scott open set and $x \in U$ then the directed set $\mathord{\downarrow\!\!\downarrow} x$ must intersect U, since $\bigsqcup \mathord{\downarrow\!\!\downarrow} x = x \in U$. Let $y \in U \cap \mathord{\downarrow\!\!\downarrow} x$, then $\mathord{\uparrow\!\!\uparrow} y \subset U$, thus for any point x in U we can find a set of the form $\mathord{\uparrow\!\!\uparrow} y$ containing x and contained in U so these sets form a basis for the Scott topology. $\square$

Definition 2.10. *A* continuous dcpo *is a continuous poset which is also a dcpo. A* domain *is a continuous dcpo.*

The next example is due to Scott[**20**] and worth keeping in mind when we consider the analogous construction for globally hyperbolic spacetimes.

Example 2.11. *The collection of compact intervals of the real line*

$$\mathbf{I}\mathbb{R} = \{[a, b] : a, b \in \mathbb{R}\ \&\ a \le b\}$$

ordered under reverse inclusion

$$[a, b] \sqsubseteq [c, d] \Leftrightarrow [c, d] \subseteq [a, b]$$

is an ω-continuous dcpo:

- *For directed* $S \subseteq \mathbf{I}\mathbb{R}$, $\bigsqcup S = \bigcap S$,
- $I \ll J \Leftrightarrow J \subseteq int(I)$, *and*
- $\{[p,q] : p,q \in \mathbb{Q}\ \&\ p \leq q\}$ *is a countable basis for* $\mathbf{I}\mathbb{R}$.

In the above $\mathbb{Q}$ *stands for the rationals. The domain* $\mathbf{I}\mathbb{R}$ *is called the* interval domain.

We also have $\max(\mathbf{I}\mathbb{R}) \simeq \mathbb{R}$ in the Scott topology. More precisely the subspace topology that $\max(\mathbf{I}\mathbb{R})$ inherits from the domain equipped with the Scott topology is homeomorphic to the reals with its usual topology. Approximation can help explain why:

Example 2.12. *A basic Scott open set in* $\mathbf{I}\mathbb{R}$ *is*

$$\twoheaduparrow[a,b] = \{x \in \mathbf{I}\mathbb{R} : x \subseteq (a,b)\}.$$

One of the interesting things about $\mathbf{I}\mathbb{R}$ is that it is a domain that is derived from an underlying poset with an abundance of order theoretic structure. Part of this structure is that the real line is *bicontinuous*, a fundamental notion in the present work:

Definition 2.13. *A continuous poset* P *is* bicontinuous *if*

- *For all* $x, y \in P$, $x \ll y$ *iff for all filtered* $S \subseteq P$ *with an infimum,*
$$\bigwedge S \sqsubseteq x \Rightarrow (\exists s \in S)\, s \sqsubseteq y,$$
and
- *For each* $x \in P$, *the set* $\twoheaduparrow x$ *is filtered with infimum* x.

In order to clarify the above definition we deconstruct it as follows. Given a continuous poset with its approximation relation $\ll$ we define the dual relation $\ll^{op}$ by

$$x \ll^{op} y \text{ iff } \inf S \leq x \text{ implies } S \cap \downarrow y \neq \emptyset$$

for any filtered set S with an infimum. Of course, there is no prima facie reason why $\ll$ and $\ll^{op}$ should be related. We can say that a poset is "dually continuous" if for every x the set $\{y | x \ll^{op} y\}$ is filtered and has x as its infimum. Our definition then amounts to saying that the poset is continuous, dually continuous and the two relations $\ll$ and $\ll^{op}$ coincide. In other work [**7**] the term "bicontinuous" is used for the situation where the two approximation relations do not coincide; such authors use the term "strongly bicontinuous" for what we have called bicontinuous. For us the present terminology seems more natural and leads to the pleasing theory of the interval topology described below.

Example 2.14. $\mathbb{R}$, $\mathbb{Q}$ *are bicontinuous.*

Definition 2.15. *On a bicontinuous poset* P, *sets of the form*

$$(a,b) := \{x \in P : a \ll x \ll b\}$$

form a basis for a topology called the interval topology.[1]

The proof that such sets form the base for a topology uses interpolation and bicontinuity and is given in our previous paper [**15**]. In contrast to a domain, a bicontinuous poset P has $\twoheaduparrow x \neq \emptyset$ for each x, so it is rarely a dcpo.

[1] The term "interval topology" means something different in lattice theory.

3. The mathematical structure of spacetime

The mathematical structure used to define spacetime in general relativity is very rich and can be described in a sequence of layers. We give a quick overview of this structure emphasizing particularly the causal structure. This is standard material and is explained well in a number of text books. Ones that we recommend particularly are: *The Large-Scale Structure of Spacetime* by Hawking and Ellis [**8**], *Techniques of differential topology in relativity* by Penrose [**19**], *General Relativity* by Wald [**23**] and *Global Lorentzian Geometry* by Beem, Ehrlich and Easley [**2**]. A beautiful account of how some of these structures are related to the physics of particles and light rays is given in an article appropriately named, "The geometry of free fall and light propagation" by Ehlers, Pirani and Schild [**4**] which we highly recommend for a reader interested in the physical significance of the mathematics.

The basic ingredient of general relativity is an *event* which we take to be an undefined primitive concept in the same way that a point is taken as a primitive concept in the geometry of space. Note that an event is not to be understood, as in ordinary language, as the occurrence of some action but rather as a *potential* occurrence. This is just as a point in space is not necessarily the location of a physical entity but the place where a material particle *could* be. The collection of events is a set called spacetime. A set is, of course, no structure at all. It is the canvas on which we paint the rest of the structure.

The next level of mathematical structure is to make the spacetime into a topological space. It is at this point that one incorporates the fact that it is a 4-dimensional topological manifold. Here is the precise definition

DEFINITION 3.1. *A topological n-dimensional* manifold $\mathcal{M}$ *is a topological space equipped with a family of open sets $\{O_i\}_{i\in I}$ together with a family of continuous functions $\phi_i : O_i \to \mathbb{R}^n$ such that each ϕ_i is a homeomorphism of O_i onto its image. We assume that as a topological space M is connected, Hausdorff and has a countable basis.*

It is often assumed that a manifold is paracompact: this means that every open cover has a locally-finite refinement. It is a very useful technical condition that lies at the heart of partition-of-unity arguments and is crucially used to prove the existence of metrics. We will not be discussing anything at that level of detail so we will never mention paracompactness again except to note that a connected Hausdorff manifold is paracompact iff it has a countable basis. Given a manifold as we have defined it above, a pair $(O_i, \phi_i)_{i\in I}$ is called a *chart*; we will use the word "chart" ambiguously for the pair, for the set O_i and for the function ϕ_i. The collection of charts is called an *atlas*.

The next structure that one defines on spacetime is the differential structure. This allows one to "do calculus" or at least to define the notion of derivative. A manifold could be something like the surface of a sphere: there would be no sense in "adding" the points of a sphere. The formula for the derivative of a function f has in it the expression $f(x+\epsilon)$: what does $x+\epsilon$ mean if one is not working in a vector space? The notion of manifold is precisely designed to allow one to think of a manifold as locally like a vector space; that is what the charts are for. However, we have to be sure that the charts agree on the notion of derivative. This brings us to the concept of a *differential manifold*.

Consider what happens when two charts intersect: $V \stackrel{\text{def}}{=} O_i \cap O_j \neq \emptyset$. Define $U_i = \phi_i(O_i), U_j = \phi_j(O_j), U = \phi_i(V)$ and $W = \phi_j(V)$. Now the function $\phi_j \circ \phi_i^{-1}$ is a well-defined continuous function, in fact a homeomorphism, from U to W. Since U and W are open subsets of $\mathbb{R}^n$ it is clear what one means by saying that they are differentiable. Such functions are conveniently called *transition functions* as they allow one to translate between charts.

Definition 3.2. *A manifold is said to be* smooth *if all the transition functions are infinitely differentiable.*

The charts allow one to endow patches of the topological space M with the structure of a vector space: exactly what one needs to define the notion of a derivative. The condition on the transition functions ensures that the notion of what is a differentiable function will not be chart dependent.

We will not review the entire apparatus of differential geometry here. However, the reader should be convinced that there is a clear strategy for developing the notions of the differential calculus on manifolds now. One uses the charts to move to $\mathbb{R}^n$ and uses the usual undergraduate calculus notions there. Thus, for example, it should be clear how one can define a smooth real-valued function on a manifold or a smooth function between two manifolds.

Once one has the notion of a smooth structure – another more snappy name for differential structure – one can define curves and tangent vectors. A *smooth curve* on M is a smooth function γ from some interval[2] $[a, b]$ to M. Note that the curve is not just the image of γ but γ itself: this is what one normally thinks of as a parametrized curve. Two different functions that happen to have the same image are different curves. One can now define the tangent vector to a curve in the usual way using the charts to move back and forth between M and $\mathbb{R}^n$. At each point there is a vector space now attached to the point: the collection of all tangent vectors at that point, it is called the *tangent space* at a point p. The whole apparatus of multi-linear algebra can now be brought to bear and one can define dual vectors at a point and indeed arbitrary tensors at every point. If V is the tangent space and V^* is the dual space one says that a tensor has type (p, q) if it belongs to $V \otimes V \otimes \ldots (p \text{ terms}) \ldots \otimes V \otimes V^* \otimes \ldots (q \text{ terms}) \ldots \otimes V^*$.

Now we come to the absolutely crucial part of the structure. Given a tangent space one can define a cone[3] which we call the forward or future light cone. Mathematically, any cone will do but, of course, the physics determines the forward light cone through the propagation of light rays emanating from a point. We similarly define another cone by taking the negatives of the vectors in the forward light cone: this is the backward or past light cone. We come to the first important restriction on the spacetimes that we consider.

Definition 3.3. *A manifold is* time orientable *if it is possible to choose globally a consistent definition of future and past light cones.*

It might seem *prima facie* that every manifold will be time orientable but [**8**] gives examples showing that this is not the case. Essentially the same type of construction that one uses to produce the Mobius strip can produce a non time

[2]It could also be an open or half-open interval.

[3]Technically a cone C is a subset of a real vector space V which is closed under addition and multiplication by *positive* scalars and such that if both x and $-x$ are in C then x is zero.

orientable manifold. Henceforth, we assume that all manifolds we consider are time orientable.

A choice of future and past light cones defines the causal structure in the following way. Given a smooth curve we can determine whether its tangent vector at a point p through which the curve passes lies inside the future light cone, or on its boundary, or in the past light cone, or on its boundary or outside both cones. The tangent vector is said to be future timelike, future null, past timelike, past null or spacelike, respectively. Of course the tangent to a curve may be at different places all of the above. However, we are interested in curves that have a timelike or null tangent vector as these are the curves along which causal effects propagate.

DEFINITION 3.4. *A curve is said to be a* future-directed causal *curve if its tangent vector everywhere lies inside or on the boundary of the future light cone. A curve is said to be a* future-directed timelike curve *if its tangent vector is everywhere strictly inside the future light cone. A curve is said to be a* future-directed null *curve if its tangent vector is everywhere on the boundary of the future light cone.*

We usually work with future directed curves; there are analogous definitions for past directed curves.

The next structure that one usually defines is the *affine* structure. This defines what it means to "move a vector parallel to itself" along a curve: this is called parallel transport and the mathematical gadget that describes this is called the affine connection. We will not discuss the affine connection here. We remark in passing that it is used to define what it means to be a "straight line" or a geodesic on a manifold.

Finally we get to measure the length of a curve. This is done by a symmetric tensor. A *Lorentz metric* on a manifold is a symmetric, nondegenerate tensor field of type $(0, 2)$ whose signature is $(- + ++)$; it is traditionally denoted by g. The fact that it is of type $(0, 2)$ means that given a vector v it assigns a number $g(v, v)$ which is quadratic in v: just what we expect for length squared. What is unusual is that some vectors have positive length and some non-zero vectors have zero length. Vectors that are timelike have negative length and null vectors have zero length. While those brought up on metric spaces may be disturbed by the indefiniteness of this kind of metric it is worth getting used to and accepting it as a reasonable definition. The physical fact that forces this is the *experimental* observation that the real world is covariant with respect to a particular group, the Lorentz group, and that the invariant for this group is indeed of the given signature.

Our presentation of the layers of spacetime structure is not how most textbooks present it. They tend to take the metric as fundamental and present all aspects of the structure in one shot, but this is conceptually confusing. As we have presented it each layer requires the previous layer for a proper definition. Here is how a spacetime is usually defined.

DEFINITION 3.5. *A* spacetime *is a real four-dimensional*[4] *smooth manifold* $\mathcal{M}$ *with a Lorentz metric* g_{ab}.

Of course, once one has a metric it encodes the light cones by telling you which vectors are timelike, null and spacelike through their "length" squared. But in order to even define smooth tensor fields, the differential structure must be in place and before that the topology.

[4]The results in the present paper work for any dimension $n \geq 2$ [**11**].

Let $(\mathcal{M}, g_{ab})$ be a time-orientable spacetime. Let $\Pi^+_\leq$ denote the future directed causal curves, and $\Pi^+_<$ denote the future directed time-like curves. These curves can be used to define order relations on spacetime. The following definitions are standard in the relativity literature.

Definition 3.6. *For $p \in \mathcal{M}$,*

$$I^+(p) := \{q \in \mathcal{M} : (\exists \pi \in \Pi^+_<)\, \pi(0) = p, \pi(1) = q\}$$

and

$$J^+(p) := \{q \in \mathcal{M} : (\exists \pi \in \Pi^+_\leq)\, \pi(0) = p, \pi(1) = q\}$$

Similarly, we define $I^-(p)$ and $J^-(p)$.

We write the relation J^+ as

$$p \sqsubseteq q \equiv q \in J^+(p).$$

The following properties from [**8**] are very useful:

Proposition 3.7. Let $p, q, r \in \mathcal{M}$. Then

(i) The sets $I^+(p)$ and $I^-(p)$ are open.
(ii) $p \sqsubseteq q$ and $r \in I^+(q) \Rightarrow r \in I^+(p)$
(iii) $q \in I^+(p)$ and $q \sqsubseteq r \Rightarrow r \in I^+(p)$
(iv) $\mathrm{Cl}(I^+(p)) = \mathrm{Cl}(J^+(p))$ and $\mathrm{Cl}(I^-(p)) = \mathrm{Cl}(J^-(p))$, where Cl stands for topological closure.

From the physical point of view there are a number of causality conditions that one can imagine imposing on a spacetime. Time orientability is a precondition for even discussing causality in any global sense. The basic causality condition is that there are no closed causal curves. In other words, we always assume the chronology conditions that ensure $(\mathcal{M}, \sqsubseteq)$ is a partially ordered set. There are a number of other, stronger conditions that one can impose; they are discussed at length in [**8**]. We will not mention all of them here. Two that we will mention are strong causality and global hyperbolicity. Intuitively, strong causality says that one cannot even come close to violating causality in the sense that for every point there is an open neighbourhood such that causal curves that leave it cannot reenter it. Thus not only do causal curves not come back to their starting point they do not come arbitrarily close to it.

We will not give the formal version of the above definition referring instead to [**8**] or [**19**]. There is a convenient topological characterization of strong causality. First we define the Alexandroff topology on a spacetime. It is the topology which has $\{I^+(p) \cap I^-(q) : p, q \in \mathcal{M}\}$ as a basis [**19**][5]. Penrose [**19**] proved the following important theorem.

Theorem 3.8. *A spacetime $\mathcal{M}$ is strongly causal iff its Alexandroff topology is Hausdorff iff its Alexandroff topology is the manifold topology.*

This shows the topological significance of strong causality. The number of topologies proliferate when the spacetime lacks strong causality.

[5]This terminology is common among relativists but order theorists use the phrase "Alexandrov topology" to mean something else: the topology generated by the upper sets.

4. Global hyperbolicity

Beyond strong causality the most discussed conditions are *stable causality*, *causal simplicity* and *global hyperbolicity*. Stable causality is intuitive but requires some technical machinery to formalize properly. Roughly speaking, it says that the spacetime is still causal even if the light cones are opened out a little. Causal simplicity says that the sets $J^{\pm}(p)$ are always closed; effectively it means that spacetime does not have holes cut out of it. Global hyperbolicity is the strongest and the name does not give many clues to the uninitiated.

One of the most important things that a mathematical physicist wants to do is to solve the following problem. One is told the complete configuration of a system at some time and one wants to determine the complete future evolution. In relativity this means one is given a spacelike surface which is complete in some sense and one wants to be able to predict the evolution for all future times. If there is such a surface the spacetime is called globally hyperbolic. The use of the term "global" should now be clear but why "hyperbolic"? The equations one is most interested in are wave equations which are hyperbolic partial differential equations. What does one mean by a "complete" spacelike surface? A reasonable way of thinking of this is a surface such that every timelike curve extended indefinitely in both directions must hit this this surface exactly once. If there is such a surface it is called a Cauchy surface. The problem of determining the evolution from data on such a surface (also called an "initial value surface") is called the Cauchy problem (also called the "initial value problem").

Penrose has called *globally hyperbolic* spacetimes "the physically reasonable spacetimes" [**23**].

Definition 4.1. *A spacetime $\mathcal{M}$ is* globally hyperbolic *if it is strongly causal and if $\uparrow a \cap \downarrow b$ is compact in the manifold topology, for all $a, b \in \mathcal{M}$.*

This is the most useful version of the definition but it gives none of the intuitions about solving the Cauchy problem.

Given the sets $I^{\pm}(p)$ we can define an irreflexive transitive relation $\prec$ by $p \prec q$ if $q \in I^{+}(p)$ (or $p \in I^{-}(q)$); in other words, there is a future directed *timelike* curve from p to q. In our previous paper we proved the following.

Theorem 4.2 ([**16**]). *If $\mathcal{M}$ is globally hyperbolic, then $(\mathcal{M}, \sqsubseteq)$ is a bicontinuous poset with $\ll \,=\, \prec$ whose interval topology is the manifold topology.*

This gives a striking connection between the approximation relation in domain theory and the notion of timelike order.

This result motivates the following definition:

Definition 4.3. *A poset $(X, \leq)$ is* globally hyperbolic *if it is bicontinuous and each interval $[a, b] = \{x : a \leq x \leq b\}$ is compact in the interval topology.*

This abstracts the spacetime concept of global hyperbolicity to posets.

Globally hyperbolic posets are very much like the real line. In fact, a well-known domain theoretic construction pertaining to the real line extends in perfect form to the globally hyperbolic posets:

Theorem 4.4 ([**16**]). *The closed intervals of a globally hyperbolic poset X*

$$\mathbf{I}X := \{[a, b] : a \leq b \;\&\; a, b \in X\}$$

ordered by reverse inclusion

$$[a, b] \sqsubseteq [c, d] \equiv [c, d] \subseteq [a, b]$$

form a continuous domain with

$$[a, b] \ll [c, d] \equiv a \ll c \;\&\; d \ll b.$$

The poset X has a countable basis iff $\mathbf{I}X$ is ω-continuous. Finally,

$$\max(\mathbf{I}X) \simeq X$$

where the set of maximal elements has the relative Scott topology from $\mathbf{I}X$ and X has the interval topology.

Globally hyperbolic posets also have rich enough structure that we can deduce many properties of spacetime from them *without* appealing to differentiable structure or geometry. Here is one such example:

Definition 4.5. *Let $(X, \leq)$ be a globally hyperbolic poset. A subset $\pi \subseteq X$ is a* causal curve *if it is compact, connected and linearly ordered. We define*

$$\pi(0) := \perp \quad \textit{and} \quad \pi(1) := \top$$

where $\perp$ and $\top$ are the least and greatest elements of π. For $P, Q \subseteq X$,

$$C(P, Q) := \{\pi : \pi \ \textit{causal curve}, \pi(0) \in P, \pi(1) \in Q\}$$

and call this the space of causal curves *between P and Q.*

Here we are adapting the definitions from spacetime geometry to arbitrary posets. A causal curve as just defined is not a function from $[0, 1]$ to X as in the case of curves on manifolds but we are mimicking that definition. The $\top$ and $\perp$ refer to the top and bottom elements of the subset π viewed as a poset. When π is embedded into X these are points of X (not necessarily the top or bottom of X, of course); we are writing $\pi(0)$ and $\pi(1)$ to be suggestive of a curve in the geometric setting.

This definition is motivated by the fact that a subset of a globally hyperbolic spacetime $\mathcal{M}$ is the image of a causal curve iff it is the image of a continuous monotone increasing $\pi : [0, 1] \to \mathcal{M}$ iff it is a compact connected linearly ordered subset of $(\mathcal{M}, \sqsubseteq)$.

Theorem 4.6 ([**15**]). *If $(X, \leq)$ is a separable globally hyperbolic poset, then the space of causal curves $C(P, Q)$ is compact in the Vietoris topology and hence in the upper topology.*

In addition, while events in spacetime are maximal elements of $\mathbf{I}\mathcal{M}$, causal curves are maximal elements in a higher order domain $\mathrm{C}(\mathbf{I}\mathcal{M})$, called the *convex powerdomain* of $\mathbf{I}\mathcal{M}$ [**15**]. In addition, the fact that spacetime has a canonical domain theoretic model teaches us something new: from only a countable set of events and the causality relation, one can reconstruct spacetime in a purely order theoretic manner. Explaining how requires domain theory.

5. Spacetime from a discrete causal set

Given a set with an order relation $(X, \leq)$ we will use the notation $F \leq x$ where F is a finite subset of X and $x \in X$ to mean that every $y \in F$ satisfies $y \leq x$. We use this to define an abstract basis.

An *abstract basis* is a set $(C, \ll)$ with a *transitive* relation that is *interpolative* from the $-$ *direction*:

$$F \ll x \Rightarrow (\exists y \in C)\, F \ll y \ll x,$$

for all finite subsets $F \subseteq C$ and all $x \in F$. Suppose, though, that it is also interpolative from the $+$ *direction*:

$$x \ll F \Rightarrow (\exists y \in C)\, x \ll y \ll F.$$

Then we can define a new abstract basis of *intervals*

$$\mathrm{int}(C) = \{(a, b) : a \ll b\} = \ll \subseteq C^2$$

whose relation is

$$(a, b) \ll (c, d) \equiv a \ll c \;\&\; d \ll b.$$

We recall some basic facts about ideal completions. Given an abstract basis $(C, \ll)$ as above, recall that an *ideal* is a subset of C that is a lower set and also a directed set. The collection of ideals ordered by inclusion is a directed complete poset called the ideal completion of $(C, \ll)$.

Let $\mathbf{I}C$ denote the ideal completion of the abstract basis $\mathrm{int}(C)$.

THEOREM 5.1 ([**16**]). *Let C be a countable dense subset of a globally hyperbolic spacetime $\mathcal{M}$ and $\ll = I^+$ be timelike causality. Then*

$$\max(\mathbf{I}C) \simeq \mathcal{M}$$

where the set of maximal elements has the relative Scott topology and $\mathcal{M}$ has the manifold topology.

In "ordering the order" I^+, taking its completion, and then the set of maximal elements, we recover spacetime by reasoning only about the causal relationships between a countable dense set of events. One objection to this might be that we begin from a *dense* set C, and then order theoretically recover the space $\mathcal{M}$ – but *dense* is a topological idea so we need to know the topology of $\mathcal{M}$ before we can recover it! But the denseness of C can be expressed in purely causal terms:

$$C \text{ dense} \equiv (\forall x, y \in \mathcal{M})(\exists z \in C)\, x \ll z \ll y.$$

Now the objection might be that we still have to reference $\mathcal{M}$. We too would like to not reference $\mathcal{M}$ at all. However, some global property needs to be assumed, either directly or indirectly, in order to reconstruct $\mathcal{M}$.

Theorem 5.1 is very different from results like "Let $\mathcal{M}$ be a certain spacetime with relation $\leq$. Then the interval topology is the manifold topology." Here we identify, in abstract terms, a process by which a countable set with a causality relation determines a space. The process is entirely order theoretic in nature, spacetime is not required to understand or execute it (i.e., if we put $C = \mathbb{Q}$ and $\ll = <$, then $\max(\mathbf{I}C) \simeq \mathbb{R}$). In this sense, our understanding of the relation between causality and the topology of spacetime is now explainable independently of geometry. Ideally, one would now like to know what constraints on C in general imply that

$\max(\mathbf{I}C)$ is a manifold. However, that is only to clarify the relationship with standard relativity; the process above may offer a more flexible definition of spacetime in general that is applicable to different physical situations – we have not ruled that possibility out.

Finally, let us mention that the category of globally hyperbolic posets[6] is in fact *naturally isomorphic* to a special category of domains called interval domains [**16**]. Thus, questions about spacetime can be converted to domain theoretic form, where we can use domain theory to answer them, and then translate the answers back to the language of physics (and vice-versa). It also implies that causality between events is equivalent to an order on *regions* of spacetime. Most importantly, it means that a globally hyperbolic spacetime with causality is equivalent to a structure $\mathbf{I}X$ whose origins are "discrete." This can be taken as the formal explanation for why spacetime can be reconstructed from a countable dense set of events in a purely order theoretic manner.

6. Time and measurement

A *domain* is a partially ordered set with intrinsic notions of completeness and approximation defined by the order. A *measurement* is a function μ that to each informative object x assigns a "measure" μx of the information content in x. In many cases, μx will be a number, but it need not be. Let us now define measurement precisely before discussing it further. We use the notation σ_D and σ_E to mean the Scott topology of D and E viewed as a collection of open sets.

Definition 6.1. *A function $f : D \to E$ between posets is* Scott continuous *if the inverse image of a Scott open set in E is Scott open in D.*

Scott continuity can be characterized order theoretically: a function $f : D \to E$ between posets is Scott continuous iff f is monotone,

$$(\forall x, y \in D)\, x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y),$$

and preserves directed suprema:

$$f(\bigsqcup S) = \bigsqcup f(S),$$

for all directed $S \subseteq D$ with a supremum. In particular, for the domain $[0,\infty)^*$ of non-negative reals in their opposite order, a Scott continuous function $\mu : D \to [0,\infty)^*$ will satisfy

(1) For all $x, y \in D$, $x \sqsubseteq y \Rightarrow \mu x \geq \mu y$, and
(2) If (x_n) is an increasing sequence in D, then

$$\mu\left(\bigsqcup_{n\geq 1} x_n\right) = \lim_{n\to\infty} \mu x_n$$

provided (x_n) has a supremum.

Definition 6.2. *A Scott continuous map $\mu : D \to E$ between posets is said to* measure the content of $x \in D$ *if*

$$x \in U \Rightarrow (\exists \varepsilon \in \sigma_E)\, x \in \mu_\varepsilon(x) \subseteq U,$$

[6] The morphisms are monotone functions that are continuous in the interval topology.

whenever $U \in \sigma_D$ is Scott open and

$$\mu_\varepsilon(x) := \mu^{-1}(\varepsilon) \cap \downarrow x$$

are the elements ε close to x in content. The map μ measures X *if it measures the content of each $x \in X$.*

Definition 6.3. *A* measurement *is a Scott continuous map $\mu : D \to E$ between posets that measures* $\ker \mu := \{x \in D : \mu x \in \max(E)\}$.

We often refer to μ as simply "measuring" $x \in D$ or as measuring $X \subseteq D$ when it measures each element of X. The case $E = [0, \infty)^*$, the set of non-negative reals in their dual order, is of particular interest in this paper: in this case, for $\varepsilon > 0$ and assuming $\mu x = 0$, we define

$$\mu_\varepsilon(x) := \mu_{[0,\varepsilon)}(x) = \{y \in D : y \sqsubseteq x \;\&\; \mu y < \varepsilon\}$$

and see that a Scott continuous $\mu : D \to [0, \infty)^*$ measures the content of $x \in D$ when

$$x \in U \Rightarrow (\exists \varepsilon > 0)\; x \in \mu_\varepsilon(x) \subseteq U$$

for all Scott open $U \subseteq D$. The map μ is then a measurement when it measures the content of its kernel $\ker(\mu) = \{x \in D : \mu x = 0\}$, the elements with no uncertainty. All such elements are maximal in the information order $\sqsubseteq$ on D. Let us now explain the intuition behind this idea on a continuous poset D.

The order on D defines a clear sense in which one object has 'more information' than another: a *qualitative* view of information content. The definition of measurement attempts to identify those monotone mappings μ which offer a *quantitative* measure of information content in the sense specified by the order. The essential point in the definition of measurement is that μ measure content in a manner that is consistent with the particular view offered by the order. There are plenty of monotone mappings that are not measurements – and while some of them may measure information content in *some other sense*, each sense must first be specified by a different information order. The definition of measurement is then a minimal test that a function μ must pass if we are to regard it as providing a measure of information content.

We now consider a few properties that measures of information content have which arbitrary monotone mappings in general need not have: qualities that make them 'different' from maps that are simply monotone. Other such properties may be found in [**13**].

Theorem 6.4 ([**13**]). *Let $\mu : D \to [0, \infty)^*$ be a measurement on a continuous poset.*

(i) *If $x \in \ker(\mu)$, then $x \in \max(D) = \{x \in D : \uparrow x = \{x\}\}$.*

(ii) *If μ measures the content of $y \in D$, then*

$$(\forall x \in D)\; x \sqsubseteq y \;\&\; \mu x = \mu y \Rightarrow x = y.$$

(iii) *If μ measures $X \subseteq D$, then*

$$\{\uparrow \mu_\varepsilon(x) \cap X : x \in X, \varepsilon > 0\}$$

is a basis for the Scott topology on X.

Unfortunately, within the realm of physics, it is normally far from trivial to prove that a function is actually a measurement. Let us remedy this now by considering an easy lemma but one that has striking applications to measurements.

LEMMA 6.5. *For a sequence (x_n) in a compact Hausdorff space X, the following are equivalent:*

(i) *The sequence (x_n) converges to x,*
(ii) *For any convergent subsequence (x_{n_k}) of (x_n), we have $x_{n_k} \to x$.*

Proof. (ii) $\Rightarrow$ (i): if (x_n) does not converge to x, then there is an open set $U \subseteq X$ with $x \in U$ such that for all k there is $n_k \geq k$ with $x_{n_k} \notin U$. By compactness of X, (x_{n_k}) has a convergent subsequence (y_n). Because (y_n) is a subsequence of (x_n), we have $y_n \to x$ by (ii), so eventually $y_n \in U$, in contrast to $x_{n_k} \notin U$. □

It is difficult to believe that such an easy lemma could be useful. But in fact:

THEOREM 6.6. *Let $\mu : D \to [0,\infty)^*$ be a strictly monotone, Scott continuous function defined on a poset D. If τ is a Hausdorff topology on D such that*

(i) *every Scott open set is τ open,*
(ii) *every sequence (x_n) in $\downarrow\!x$ with $\mu x_n \to \mu x$ is contained in some τ-compact $K \subseteq \downarrow\!x$,*
(iii) *the function μ is continuous from (D,τ) to $[0,\infty)$ with the Euclidean topology,*

then μ measures all of D.

Proof. Let $x_n \sqsubseteq x$ with $\mu x_n \to \mu x$. Take a compact set K with $x_n \in K \subseteq \downarrow\!x$. Let (x_{n_k}) be any convergent subsequence of (x_n). Let us write $x_{n_k} \to y$. Then since K is closed, $y \in K$ and hence $y \sqsubseteq x$. However, since the sequence $\mu x_n \to \mu x$, we know that $\mu x_{n_k} \to \mu x$. Since μ is continuous with respect to τ, we get

$$\mu y = \mu\left(\lim_{k\to\infty} x_{n_k}\right) = \lim_{k\to\infty} \mu x_{n_k} = \mu x$$

and thus by strict monotonicity, $x = y$. Then every convergent subsequence of (x_n) converges to x and all of this happens in the compact Hausdorff space K. Thus, $x_n \to x$ in (D,τ).

If μ does not measure the content of $x \in D$, then there is a Scott open set $U \subseteq D$ and a sequence $x_n \sqsubseteq x$ with $\mu x_n \to \mu x$ and $x_n \notin U$. By our above remarks, $x_n \to x$ in (D,τ), and since U is τ open, we have $x_n \in U$ for all but a finite number of n, which is a contradiction. □

Notice that the proof above also shows that the previous result holds for maps of the form $\mu : D \to E$, where $E = \mathbb{R}$ or $E = \mathbb{R}^*$.

DEFINITION 6.7. *A* global time function $t : \mathcal{M} \to \mathbb{R}$ *on a globally hyperbolic spacetime $\mathcal{M}$ is a continuous function such that $x < y \Rightarrow t(x) < t(y)$ and $t^{-1}(r) = \Sigma$ is a Cauchy surface for $\mathcal{M}$, for each $r \in \mathbb{R}$.*

Because global time functions always exist on a globally hyperbolic spacetime, each such spacetime admits a natural measurement on the domain of spacetime intervals:

THEOREM 6.8. *For any global time function $t : \mathcal{M} \to \mathbb{R}$ on a globally hyperbolic spacetime, the function $\Delta t : \mathbf{I}(\mathcal{M}) \to [0,\infty)^*$ given by $\Delta t[a,b] = t(b) - t(a)$ measures all of $\mathbf{I}(\mathcal{M})$. It is a measurement with $\ker(\Delta t) = \max(\mathbf{I}(\mathcal{M}))$.*

Proof. The function Δt inherits its monotonicity from that of t; it is Scott continuous because t is continuous with respect to the manifold topology and directed suprema in $\mathbf{I}(\mathcal{M})$ are calculated using limits in the manifold topology. To prove that Δt measures $\mathbf{I}(\mathcal{M})$, we will show that t measures the continuous poset $(\mathcal{M}, \leq)$ and that it also measures $(\mathcal{M}, \leq^*)$, whose order $\leq^*$ is given by $x \leq^* y \equiv y \leq x$.

We apply the remark following Theorem 6.6 to $t : \mathcal{M} \to \mathbb{R}$ as follows. (i) The Scott topology is contained in the manifold topology. (ii) Given any sequence $x_n \leq x$ with $t(x_n) \to t(x)$, we have $x_n \in J^+(\Sigma) \cap J^-(x) \subseteq \downarrow x$ for some $\Sigma = t^{-1}(r)$, where r exists because $t(x_n)$ has a limit and the set $J^+(\Sigma) \cap J^-(x)$ is compact [**23**]. By the remark after Theorem 6.6, t measures $(\mathcal{M}, \leq)$. Because $(\mathcal{M}, \leq)$ is *bicontinuous*, $t : (\mathcal{M}, \leq^*) \to \mathbb{R}^*$ measures the continuous poset $(\mathcal{M}, \leq^*)$, again by the remark after Theorem 6.6. $\square$

What is so interesting about this proof is that in order to apply Theorem 6.6, we not only need continuity, strict monotonicity and the connection between causal structure and topology, we also make use of the Cauchy surface Σ, the latter of which implies that spacetime has an initial value formulation. Another point of interest is that the same technique used here to prove Δt is a measurement, Theorem 6.6, has also been used to show the same about capacity on the domain of binary channels and entropy on the domains of classical and quantum states [**17**].

7. The Lorentz distance

The *Lorentz distance* on a globally hyperbolic spacetime $\mathcal{M}$ is the function $d : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ given by

$$d(a, b) = \sup_{\pi_{ab}} \mathrm{len}(\pi_{ab})$$

where the sup is taken over all causal curves π_{ab} that join a to b; when $a \not\leq b$, $d(a, b) := 0$. By global hyperbolicity [**2**], the supremum in the definition of d is finite, yields the length of the maximum geodesic joining causally related events and the function d is continuous as a map from the manifold topology to the usual topology on $[0, \infty)$. Physically, $d(a, b)$ measures the amount of time recorded by a clock that travels from a to b when $a \leq b$. Thus, the Lorentz distance is determined by a map between domains of the form

$$d : \mathbf{I}(\mathcal{M}) \to [0, \infty)^* :: [a, b] \mapsto d(a, b)$$

and for the remainder of this paper we shall regard it as such. Crucially, $d[a, b] > 0$ iff $a \ll b$.

LEMMA 7.1. *The function $d : \mathbf{I}(\mathcal{M}) \to [0, \infty)^*$ is Scott continuous.*

Proof. Monotonicity of d: First, the "reverse triangle inequality"

$$a \leq b \leq c \Rightarrow d[a, c] \geq d[a, b] + d[b, c]$$

holds since d is defined as a sup and $\mathrm{len}(\pi_{ab} + \pi_{bc}) = \mathrm{len}(\pi_{ab}) + \mathrm{len}(\pi_{bc})$; here the notation $\pi_{ab} + \pi_{bc}$ means the causal curve obtained by joining π_{ab} and π_{bc} at b. Of course, such a joined curve may not be smooth but Penrose has shown [**19**] that one can always "smooth out this curve" with an arbitrarily small change in length.

We now apply it twice as follows: given $[a, b] \sqsubseteq [c, d]$, we have $a \leq c \leq d \leq b$, so

$$d[a, b] \geq d[a, c] + d[c, b] \geq d[a, c] + (d[c, d] + d[d, b]) \geq d[c, d]$$

which proves monotonicity.

Scott continuity of d: By the separability of $\mathcal{M}$, $\mathbf{I}(\mathcal{M})$ is ω-continuous, so it is enough to consider an increasing sequence (x_n) of intervals $x_n = [a_n, b_n]$. Then we have

$$d\left(\bigsqcup x_n\right) = d[\lim a_n, \lim b_n] = \lim d[a_n, b_n] = \lim d(x_n)$$

where the second to last equality uses the continuity of d as a map $\mathcal{M}\times\mathcal{M} \to [0,\infty)$ given in [**2**]. $\Box$

We now give a "new" definition of the interval topology which applies to any continuous poset. For bicontinuous posets it is equivalent to the old definition but this new version applies more generally.

Definition 7.2. *The* interval topology *on a continuous poset P exists when sets of the form*

$$(a,b) = \{x \in P : a \ll x \ll b\} \quad \& \quad \mathord{\Uparrow}x = \{y \in P : x \ll y\}$$

form a basis for a topology on P.

A function between continuous posets is *interval continuous* when each poset has an interval topology and the inverse image of an interval open set is interval open. By the bicontinuity of $\mathcal{M}$, the interval topology on $\mathbf{I}(\mathcal{M})$ exists, so we can consider interval continuity for functions $\mathbf{I}(\mathcal{M}) \to [0,\infty)^*$.

Lemma 7.3.

(i) *For all $x, y \in \mathbf{I}(\mathcal{M})$, if $x \ll y$, then $dx > dy$, and*
(ii) *For all $x \in \mathbf{I}(\mathcal{M})$, if $dx > r \geq 0$, then there is $y \in \mathbf{I}(\mathcal{M})$ with $x \ll y$ and $dy = r$.*

Proof. (i) Given $x = [a,b] \ll [c,d] = y$, we know $a \ll b \ll c \ll d$, so as in the proof of Lemma 7.1, we again apply the reverse triangle inequality twice, this time noting that all distances involved are positive since $d[s,t] > 0$ iff $s \ll t$.

(ii) First, each interval $x = [a,b]$ is a path connected subset of $\mathcal{M}$, since any pair of points $s, t \in [a,b]$ can be joined by a continuous curve that first moves forward in time from s to a and then backward in time from a to t. In particular, $x = [a,b]$ is connected.

Because $dx > 0$, $a \ll b$, and interpolation gives $p \in x$ with $a \ll p \ll b$. The set $\uparrow p \cap \mathord{\Downarrow} b$ is directed with sup b, while $\downarrow p \cap \mathord{\Uparrow} a$ is filtered with inf a, so $x = [a,b]$ contains increasing and decreasing sequences of approximations with limits b and a respectively. By the continuity of $d : \mathcal{M}\times\mathcal{M} \to [0,\infty)$, there is an interval w with $p \in w$ such that

$$x \ll w \quad \& \quad dx > dw > r$$

where we make use of (i) to ensure $dx > dw$.

If $r = 0$, then set $y = [p,p]$ and the proof is finished. If $r > 0$, then the restriction of $d : \mathcal{M}\times\mathcal{M} \to [0,\infty)$ to the *connected set* $w \times w$ yields a continuous function that assumes the value dw and the value $0 = d[p,p]$. Thus it must also assume $r \in [0, dw]$, which gives $(c,d) \in w \times w$ with $d(c,d) = r$. Since $r > 0$, $c \ll d$, so let $y = [c,d] \in \mathbf{I}(\mathcal{M})$. Then y is the desired interval: we have $dy = r$, while $x \ll w \sqsubseteq y$ gives $x \ll y$. $\Box$

Notice that (i) above says that d preserves the way-below relation between domains, while (ii) is a kind of converse to (i). Together with Scott continuity they yield the following:

THEOREM 7.4. *The Lorentz distance* $d : \mathbf{I}(\mathcal{M}) \to [0,\infty)^*$ *is not only Scott continuous, it is also interval continuous. Thus, it does not measure* $\mathbf{I}(\mathcal{M})$ *at any point of* $\ker(d)$.

Proof. Let U be a basic interval open set in $[0,\infty)^*$. If U is Scott open, then by Lemma 7.1, $d^{-1}(U)$ is Scott open and the proof is finished. If $U = (s,t)$, then consider an interval $x \in d^{-1}(U)$, for which we have $s < dx < t$. By Scott continuity of d, there is $a \ll x$ with $da < t$. By Lemma 7.3(ii), there is b with $x \ll b$ and $db = s$. We claim $x \in (a,b) \subseteq d^{-1}(s,t)$.

If $y \in (a,b)$, then $a \ll y \ll b$, so by Lemma 7.3(i), $da > dy > db = s$, which gives $t > da > dy > s$, and finally $y \in d^{-1}(s,t)$, finishing the proof. $\Box$

That the Lorentz distance is not a measurement is a direct consequence of the fact that a clock travelling at the speed of light records no time as having elapsed i.e. the set of null intervals is

$$\ker(d) \setminus \max(\mathbf{I}(\mathcal{M})) \neq \emptyset$$

but measurements always have the property that $\mu x = 0$ implies $x \in \max(D)$ (Theorem 6.4).

In fact, no interval continuous function $\mu : \mathbf{I}(\mathcal{M}) \to [0,\infty)^*$ can be a measurement: by interval continuity, $\mu x = 0$ for any x with $\Uparrow x = \emptyset$. Just like the Lorentz distance, an interval continuous μ will also assign 0 to "null intervals." In this way, we see that interval continuity captures an essential aspect of the Lorentz distance. In addition, since Δt is a measurement, it cannot be interval continuous. This provides a surprising *topological* distinction between the Newtonian and relativistic concepts of time: d is interval continuous, Δt is not. Put another way, Δt can be used to reconstruct the *topology* of spacetime (Theorem 6.4(iii)), while d is used to reconstruct its *geometry*.

8. Spacetime geometry from a countable causal set

Let us return now to the reconstruction of spacetime (Section 5) from a countable dense set $(C, \ll)$. Specifically, we take the rounded ideal completion[7] $\mathbf{I}(C)$ of the abstract basis of *intervals*

$$\mathrm{int}(C) = \{(a,b) : a \ll b\} = \ll \subseteq C^2$$

whose relation is

$$(a,b) \ll (c,d) \equiv a \ll c \;\&\; d \ll b.$$

We are then able to recover spacetime as

$$\max(\mathbf{I}C) \simeq \mathcal{M}$$

where the set of maximal elements have the Scott topology. Let us now suppose that in addition to $\mathrm{int}(C)$ that we also begin with a countable collection of numbers l_{ab} chosen one for each $(a,b) \in \mathrm{int}(C)$ in such a way that the map

$$\mathrm{int}(C) \to [0,\infty)^* :: (a,b) \mapsto l_{ab}$$

[7]An ideal I is a directed downward-closed set, it is *rounded* if for any $x \in I$ there is a $y \in I$ with $x \ll y$.

is monotone. Then in the process of reconstructing spacetime, we can also construct the Scott continuous function $d : \mathbf{I}C \to [0, \infty)^*$ given by

$$d(x) = \inf\{l_{ab} : (a, b) \ll x\}.$$

In the event that the countable number of l_{ab} chosen are the Lorentz distances $l_{ab} = d[a, b]$, then the function d constructed above yields the Lorentz distance for any spacetime interval, the reason being that both are Scott continuous and are equal on a basis of the domain.

Thus, from a countable dense set of events and a countable set of distances, we can reconstruct the spacetime manifold together with its geometry in a purely order theoretic manner.

9. Conclusions

We have seen the following ideas in this paper:

(1) how to reconstruct the spacetime topology from the causal structure using purely order-theoretic ideas,
(2) an abstract order-theoretic definition of global hyperbolicity,
(3) that one can reconstruct spacetime, meaning its topology and *geometry*, from a countable dense subset,
(4) an equivalence of categories between a new category of interval domains and the category of globally hyperbolic posets.
(5) a topological distinction between Newtonian and relativistic notions of time.

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