An Algebraic Theory of Markov Processes

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Abstract

Markov processes are a fundamental model of probabilistic transition systems and are the underlying semantics of probabilistic programs. We give an algebraic axiomatisation of Markov processes using the framework of quantitative equational logic introduced in [13]. We present the theory in a structured way using work of Hyland et al. [9] on combining monads. We take the interpolative barycentric algebras of [13] which captures the Kantorovich metric and combine it with a theory of contractive operators to give the required axiomatisation of Markov processes both for discrete and continuous state spaces. This work apart from its intrinsic interest shows how one can extend the general notion of combining effects to the quantitative setting.

Keywords Markov processes, equational logic, quantitative reasoning, combining monads.

1 Introduction

The theory of effects began with the pioneering work of Moggi [15, 16] on an algebraic treatment of programming languages via the theory of monads. This allowed a compositional treatment of various semantic phenomena such as state, IO, exceptions, etc. This work was followed up by the program of Plotkin and Power [18, 19] on understanding the monads as arising from operations and equations; see also the survey of Hyland and Power [10]. A fundamental contribution, due to Hyland et al. [9], was a way of combining effects by taking the "sum" of theories.

In the present paper we use the framework of [13] which introduced the quantitative analogue of equational logic and the techniques of [9] to develop an algebraic theory of Markov processes. In [13] it was shown how a certain set of equations gave as free algebras the space of probability distributions with Kantorovich metric. A challenge at the time was to extend this to the theory of Markov processes, which are dynamically evolving probability distributions. Instead of developing an equational theory in an *ad-hoc* way, we use the ideas of [9] to obtain a very general theory of probability distributions equipped with additional operators. Markov processes (or labelled Markov processes [17]) are just a very special instance of these where there is a set of unary operators for the transitions.

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It is very pleasing that one can obtain the axiomatisation of Markov processes in this systematic way. Some effort is involved in showing that the techniques apply to the quantitative setting; in that sense our results go beyond the example of Markov processes as they can be seen as an example of a general paradigm of forming sums of quantitative theories. Overall, we see our work as a first step towards a full-blown theory of quantitative effects.

The main conceptual advance we have attained is unifying an algebraic presentation of Markov processes with their well-known coalgebraic presentation. On the one hand, one sees them as algebras arising from a very natural quantitative theory; on the other hand, they arise from the theory of quantitative bisimulation via final coalgebras of behaviour functors. This all comes about as we have a coincidence of initial and final coalgebras. Such a coincidence is known in domain theory [20, 21] but seems not to have been developed in the metric case. The algebraic and coalgebraic viewpoints in semantics have largely developed independently. We feel our work contributes to building a bridge between the two and we hope in future work to use this unified perspective in applications.

Technical summary

In [13] it is shown that any quantitative equational theory $\mathcal{U}$ induces a monad $T_{\mathcal{U}}$ on **Met** (the category of metric spaces), namely the monad assigning to an arbitrary space M the quantitative algebra freely generated over M and satisfying the conditional quantitative equations (conditional equations) of $\mathcal{U}$. One can readily show that if one considers a signature Σ and the empty theory, the induced monad is the free monad Σ^* over the signature endofunctor (also called Σ) in **Met**.

Similarly, suppose that with each operator $f \in \Sigma$ of arity n we associate a contractive factor $0 < c < 1$ (written $f : \langle n, c\rangle \in \Sigma$) and add, for each $\delta \geq c\varepsilon$, the axiom

$$\{x_1 =_\varepsilon y_1, \dots, x_n =_\varepsilon y_n\} \vdash f(x_1, \dots, x_n) \equiv_\delta f(y_1, \dots, y_n)$$

obtaining the *quantitative theory of contractive operators in* Σ, denoted by $\mathcal{O}(\Sigma)$. Then the induced monad is the free monad $\tilde{\Sigma}^*$ on the endofunctor $\tilde{\Sigma} = \coprod_{f:\langle n,c\rangle\in\Sigma} c \cdot Id^n$, where $c \cdot X$ is the space X with metric rescaled by a factor of c (see Section 6).

In [13], the monad induced by the quantitative equational theory $\mathcal{B}$ of *interpolative barycentric algebras* (recalled in Section 5) was shown to be the monad Π of finitely supported Borel probability measures with Kantorovich metric. By taking the (disjoint) union of the axioms of interpolative barycentric algebras and of the algebra of contractive operators for Σ, one obtains $\mathcal{B}+\mathcal{O}(\Sigma)$, the *quantitative theory of interpolative barycentric algebras with contractive operators* in Σ.

In Section 7, the free monad induced by $\mathcal{B} + \mathcal{O}(\Sigma)$ is proven to be isomorphic to the sum of monads $\tilde{\Sigma}^* + \Pi$. Because of this characterisation, by using results in [9], we can show that $T_{\mathcal{B}+\mathcal{O}(\Sigma)}$ assigns to an arbitrary metric space M the *initial* solution of the functorial equation

$$X_M \cong \Pi(\tilde{\Sigma} X_M + M)\,.$$

We obtain analogous results for complete separable metric spaces by taking the completion of the monad. In this case the monad assigns to any complete separable metric space M the *unique solution* of the functorial equation

$$Y_M \cong \Delta(\tilde{\Sigma} Y_M + M)\,.$$

where Δ is the Giry monad of Borel probability measures with Kantorovich metric.

By observing that the maps from left to right of the above isomorphisms are coalgebra structures, in Section 8 we algebraically recover Markov processes by using the signature Σ which has a constant symbol $\mathbf{0}$, representing termination, and a unary operator $\diamond(t)$, representing the capability of performing a transition to t.

The above findings fit into a more general pattern: we show that under certain assumptions on the quantitative theories $\mathcal{U}, \mathcal{U}'$, the free monad $T_{\mathcal{U}+\mathcal{U}'}$ that arises from the disjoint union $\mathcal{U} + \mathcal{U}'$ of the two theories is the categorical sum $T_{\mathcal{U}} + T_{\mathcal{U}'}$ of the free monads on $\mathcal{U}$ and $\mathcal{U}'$, respectively. The only requirement on the theories is that they can be axiomatised by a set of quantitative inferences involving only quantitative equations between variables as hypotheses. In [14] this type of theory is called *basic*.

For basic quantitative theories we have another main result: the quantitative algebras satisfying a basic theory $\mathcal{U}$ are in one-to-one correspondence with the Eilenberg-Moore algebras for the free monad $T_{\mathcal{U}}$. This result generalises the classical isomorphism between the algebras of a functor F and the Eilenberg-Moore algebras of the free monad F^* on F [3].

2 Preliminaries

The basic structures with which we work are metric spaces. A metric induces a topology, and different metrics can induce the same topology. In particular, for any metric d, the 1-bounded function $d'(x, y) = d(x, y)/(1 + d(x, y))$ is also a metric and yields the same topology as d. We henceforth restrict to 1-bounded metrics. If a metric space has a countable dense subset we say it is *separable*; this is equivalent to having a countable base for the topology. A sequence (x_i) in a metric space is said to be *Cauchy* if $\forall \epsilon > 0, \exists N, \forall i, j \geq N, d(x_i, x_j) \leq \epsilon$. If every Cauchy sequence converges, the space is said to be *complete*. If a space is not complete it can be completed by a well-known construction called *Cauchy completion*. If (X, d) is a metric space we write $(\overline{X}, d)$ or just $\overline{X}$ for its completion. If, for a given topology, there is *some* metric that is complete we say that the topology is completely metrizable. Topological spaces underlying complete separable metric space are called *Polish*.

The categories of metric spaces that we consider are **Met**: metric spaces, **CMet**: complete metric spaces and **CSMet**: complete separable metric spaces (recall that all the spaces we consider are 1-bounded). The morphisms are the non-expansive maps, *i.e.* the $f : (X, d_X) \to (Y, d_Y)$ such that $d_Y(f(x), f(y)) \leq d_X(x, y)$. These categories have all countable products and coproducts. One can define products by taking the set theoretic product and defining the metric to be the sup of the pointwise metrics, *i.e.* given $\{(M_i, d_i) | i \in I\}$ the metric on $\Pi_i M_i$ is

$$d((x_1, \ldots, x_n, \ldots), (y_1, \ldots, y_n, \ldots)) = \sup_i d_i(x_i, y_i)\,.$$

We assume that the reader is familiar with the basic notions of σ-algebras, measurable functions, and measures. Given a topology the σ-algebra generated is called its *Borel σ-algebra* and its elements are called *Borel sets*. A probability measure defined on the Borel sets is a *Borel probability measure*. Given a topological space with its Borel σ-algebra, we define the *support* of a measure to be the complement of the union of all open sets with zero measure. A measure is said to be *finitely supported* if its support is a finite set. A finitely supported probability measure is just a convex sum of point measures; *i.e.* measures whose support is a single point.

We assume the reader is familiar with monads and with algebras of a monad (the Eilenberg-Moore algebras of a monad).

2.1 Kantorovich Metric

We review some well-known facts about metrics between spaces of probability distributions.

Let M be a metric space. The *Kantorovich metric*[1] between Borel probability measures μ, ν over M is defined as:

$$\mathcal{K}(d_M)(\mu, \nu) = \sup_{f \in \Phi_M} \left| \int f \, \mathrm{d}\mu - \int f \, \mathrm{d}\nu \right|\,.$$

with supremum ranging over the set Φ_M of positive 1-bounded non-expansive real-valued functions $f : M \longrightarrow [0, 1]$.

Under suitable restrictions on the type of measures, the above distance has a well-known dual characterization, based on the notion of *coupling*. A coupling for a pair of Borel probability measures (μ, ν) over M, is a Borel probability measure ω on the product space $M \times M$, such that, for all Borel sets $E \subseteq M$

$$\omega(E \times M) = \mu(E) \qquad \text{and} \qquad \omega(M \times E) = \nu(E)\,.$$

A Borel probability measure μ over M is *Radon* if for any Borel set $E \subseteq M$, $\mu(E)$ is the supremum of $\mu(K)$ over all compact subsets K of E. We write $C(\mu, \nu)$ for the set of Radon couplings for a pair of Borel probability measures (μ, ν).

Theorem 2.1 (Kantorovich-Rubinstein Duality [26, Thm. 5.10])**.** *Let M be a metric space. Then, for arbitrary Radon probability measures μ, ν over M*

$$\mathcal{K}(d_M)(\mu, \nu) = \min \left\{ \int d \, \mathrm{d}\omega \mid \omega \in C(\mu, \nu) \right\}\,.$$

Examples of Radon probability measures are finitely supported Borel probability measures on any metric space and generic Borel probability measures over complete separable metric spaces.

We write $\Delta(M)$ for the space of Borel probability measures over M with the Kantorovich metric and $\Pi(M)$ for the subspace of $\Delta(M)$ of the finitely supported Borel probability measures over M.

Lemma 2.2. *Let M be a separable metric space. Then, the Cauchy completion of $\Pi(M)$ is isomorphic to the set of Borel probability measures over the Cauchy completion of M, i.e., $\overline{\Pi(M)} \cong \Delta(\overline{M})$.*

3 Quantitative Equational Theories

Quantitative equations were introduced in [13]. In this framework equalities $t \equiv_\varepsilon s$ are indexed by a positive rational number, to capture the idea that t is "within ε" of s. This informal notion is formalised in a manner analogous to traditional equational logic and it is shown that one can axiomatise quantitative analogues of algebras. Analogues of Birkhoff's completeness theorem [13] and

[1] Sometimes called the Wasserstein-1 metric.

variety theorem [14] were established. The collection of equationally defined quantitative algebras form the algebras for monads on suitable categories of metric spaces. In this section we review this formalism.

Let Σ be an algebraic signature of function symbols $f : n \in \Sigma$ of arity $n \in \mathbb{N}$. Let X be a countable set of variables, ranged over by $x, y, z, \ldots$. We write $\mathbb{T}(\Sigma, X)$ for the set of Σ-terms freely generated over X, ranged over by $t, s, u, \ldots$.

A *substitution of type* Σ is a function $\sigma : X \to \mathbb{T}(\Sigma, X)$ that is homomorphically extended to terms as

$$\sigma(f(t_1, \ldots, t_n)) = f(\sigma(t_1), \ldots, \sigma(t_n))\,;$$

we write $\mathcal{S}(\Sigma)$ for the set of substitutions of type Σ.

A *quantitative equation of type* Σ over X is an expression of the form $t \equiv_\varepsilon s$, for $t, s \in \mathbb{T}(\Sigma, X)$ and $\varepsilon \in \mathbb{Q}_{\geq 0}$. We use $\mathcal{V}(\Sigma, X)$ to denote the set of quantitative equations of type Σ over X, and its subsets will be ranged over by $\Gamma, \Theta, \ldots$. Let $\mathcal{E}(\Sigma, X)$ be the set of *conditional quantitative equations* on $\mathbb{T}(\Sigma, X)$, which are expressions of the form

$$\{t_1 \equiv_{\varepsilon_1} s_1, \ldots, t_n \equiv_{\varepsilon_n} s_n\} \vdash t \equiv_\varepsilon s\,,$$

for arbitrary $s_i, t_i, s, t \in \mathbb{T}(\Sigma, X)$ and $\varepsilon_i, \varepsilon \in \mathbb{Q}_{\geq 0}$.

A *quantitative equational theory of type* Σ *over* X is a set $\mathcal{U}$ of conditional quantitative equations on $\mathbb{T}(\Sigma, X)$ closed under the relation $\vdash$ as axiomatised below, for arbitrary $x, y, z, x_i, y_i \in X$, terms $s, t \in \mathbb{T}(\Sigma, X)$, rationals $\varepsilon, \varepsilon' \in \mathbb{Q}_{\geq 0}$, and $\Gamma, \Theta \subseteq \mathcal{V}(\Sigma, X)$,

(Refl) $\vdash x \equiv_0 x\,,$

(Symm) $\{x \equiv_\varepsilon y\} \vdash y \equiv_\varepsilon x\,,$

(Triang) $\{x \equiv_\varepsilon z, z \equiv_{\varepsilon'} y\} \vdash x \equiv_{\varepsilon+\varepsilon'} y\,,$

(1-Bdd) $\emptyset \vdash x \equiv_1 y\,,$

(Max) $\{x \equiv_\varepsilon y\} \vdash x \equiv_{\varepsilon+\varepsilon'} y\,,$ for all $\varepsilon' > 0\,,$

(Inf) $\{x \equiv_{\varepsilon'} y \mid \varepsilon' > \varepsilon\} \vdash x \equiv_\varepsilon y\,,$

(f-NE) $\{x_1 \equiv_\varepsilon y_1, \ldots, x_n \equiv_\varepsilon y_n\} \vdash f(x_1, \ldots, x_n) \equiv_\varepsilon f(y_1, \ldots, y_n)\,,$

(Subst) If $\Gamma \vdash t \equiv_\varepsilon s$, then $\sigma(\Gamma) \vdash \sigma(t) \equiv_\varepsilon \sigma(s)$, for all $\sigma \in \mathcal{S}(\Sigma)\,,$

(Assum) If $t \equiv_\varepsilon s \in \Gamma$, then $\Gamma \vdash t \equiv_\varepsilon s\,,$

(Cut) If $\Gamma \vdash \Theta$ and $\Theta \vdash t \equiv_\varepsilon s$, then $\Gamma \vdash t \equiv_\varepsilon s\,,$

where we write $\Gamma \vdash \Theta$ to mean that $\Gamma \vdash t \equiv_\varepsilon s$ holds for all $t \equiv_\varepsilon s \in \Theta$ and $\sigma(\Gamma) = \{\sigma(t) \equiv_\varepsilon \sigma(s) \mid t \equiv_\varepsilon s \in \Gamma\}$.

The rules (Subst), (Cut), (Assum) are the usual rules of equational logic. The axioms (Refl), (Symm), (Triang) correspond, respectively, to reflexivity, symmetry, and the triangle inequality; (Max) represents inclusion of neighborhoods of increasing diameter; (Inf) is the limiting property of a decreasing chain of neighborhoods with converging diameters; and (f-NE) expresses nonexpansivness of the $f \in \Sigma$. We have added the axiom (1-Bdd) to ensure that the algebras we get also have 1-bounded metrics. This is a minor variation of the theory presented in [13]. The results that we use from that paper all hold with this change.

A set A of conditional quantitative equations *axiomatises* a quantitative equational theory $\mathcal{U}$, if $\mathcal{U}$ is the smallest quantitative equational theory containing A.

The models of quantitative equational theories, called *quantitative algebras*, are universal Σ-algebras equipped with a metric.

Definition 3.1. A *quantitative* Σ*-algebra* is a tuple $\mathcal{A} = (A, \Sigma^{\mathcal{A}})$, where A is a metric space and $\Sigma^{\mathcal{A}} = \{f^{\mathcal{A}} : A^n \to A \mid f : n \in \Sigma\}$ is a set of non-expansive *interpretations* for the algebraic operators in Σ, *i.e.*, satisfying the following, for all $0 \leq i \leq n$ and $a_i, b_i \in A$,

$$\max_i d_A(a_i, b_i) \geq d_A(f^{\mathcal{A}}(a_1, \ldots, a_n), f^{\mathcal{A}}(b_1, \ldots, b_n))\,.$$

The morphisms between quantitative Σ-algebras are non-expansive Σ-homomorphisms. Quantitative Σ-algebras and their morphism form a category $\mathbf{QA}(\Sigma)$.

A quantitative algebra $\mathcal{A} = (A, \Sigma^{\mathcal{A}})$ *satisfies* the conditional quantitative equation $\Gamma \vdash t \equiv_\varepsilon s$ over $\mathcal{E}(\Sigma, X)$, written $\Gamma \models_{\mathcal{A}} t \equiv_\varepsilon s$, if for any assignment $\iota : X \to A$,

$$\big(\text{for all } t' \equiv_{\varepsilon'} s' \in \Gamma,\ d_A(\iota(t'), \iota(s')) \leq \varepsilon'\big) \text{ implies } d_A(\iota(t), \iota(s)) \leq \varepsilon\,,$$

where $\iota(t)$ is the homomorphic interpretation of $t \in \mathbb{T}(\Sigma, X)$ in $\mathcal{A}$.

A quantitative algebra $\mathcal{A}$ is said to *satisfy* (or be a *model* for) the quantitative theory $\mathcal{U}$, if $\Gamma \models_{\mathcal{A}} t \equiv_\varepsilon s$ whenever $\Gamma \vdash t \equiv_\varepsilon s \in \mathcal{U}$. We write $\mathbb{K}(\Sigma, \mathcal{U})$ for the collection of models of a theory $\mathcal{U}$ of type Σ.

Sometimes it is convenient to consider the quantitative Σ-algebras whose carrier is a complete metric space. This class of algebras forms a full subcategory of $\mathbf{QA}(\Sigma)$, written $\mathbf{CQA}(\Sigma)$. Similarly, we write $\mathbb{CK}(\Sigma, \mathcal{U})$ for the full subcategory of quantitative Σ-algebras in $\mathbf{CQA}(\Sigma)$ which are models of $\mathcal{U}$.

The following definition lifts the Cauchy completion of metric spaces to quantitative algebras.

Definition 3.2. The *Cauchy completion of a quantitative* Σ*-algebra* $\mathcal{A} = (A, \Sigma^{\mathcal{A}})$, is the quantitative Σ-algebra $\overline{\mathcal{A}} = (\overline{A}, \Sigma^{\overline{\mathcal{A}}})$, where $\overline{A}$ is the Cauchy completion of A and $\Sigma^{\overline{\mathcal{A}}} = \{f^{\overline{\mathcal{A}}} : \overline{A}^n \to \overline{A} \mid f : n \in \Sigma\}$ is such that for Cauchy sequences $(b^i_j)_j$ converging to $b^i \in \overline{A}$, for $1 \leq i \leq n$, we have:

$$f^{\overline{\mathcal{A}}}(b^1, \ldots, b^n) = \lim_j f^{\mathcal{A}}(b^1_j, \ldots, b^n_j)\,.$$

The above extends to a functor $\mathbb{C} : \mathbf{QA}(\Sigma) \to \mathbf{CQA}(\Sigma)$ which is the left adjoint to the functor embedding $\mathbf{CQA}(\Sigma)$ into $\mathbf{QA}(\Sigma)$.

The completion of quantitative Σ-algebras extends also to a functor from $\mathbb{K}(\Sigma, \mathcal{U})$ to $\mathbb{CK}(\Sigma, \mathcal{U})$, whenever $\mathcal{U}$ can be axiomatised by a collection of *continuous schemata*, which are conditional quantitative equations of the form

$$\{x_1 \equiv_{\varepsilon_1} x_1, \ldots, x_n \equiv_{\varepsilon_n} y_n\} \vdash t \equiv_\varepsilon s\,, \quad \text{for all } \varepsilon \geq f(\varepsilon_1, \ldots, \varepsilon_n),$$

where $f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a continuous real-valued function, and $x_i, y_i \in X$. We call such a theory *continuous*.

Free Monads on Quantitative Theories. Recall that to every signature Σ, one can associate a *signature endofunctor* (also called Σ) on $\mathbf{Met}$ by:

$$\Sigma = \coprod_{f : n \in \Sigma} Id^n\,.$$

It is easy to see that, by couniversality of the coproduct, quantitative Σ-algebras correspond to Σ-algebras for the functor Σ in $\mathbf{Met}$, and the morphisms between them to non-expansive homomorphisms of Σ-algebras. Below we pass between the two points of view as convenient.

In [13] it is shown that any quantitative theory $\mathcal{U}$ of type Σ induces a monad $T_{\mathcal{U}}$ on $\mathbf{Met}$, called the *free monad on* $\mathcal{U}$. The relevant results leading to its definition are summarized in the following theorem.

Theorem 3.3 (Free Algebra)**.** *Let $\mathcal{U}$ be a quantitative theory of type Σ. Then, for any $X \in$ **Met** there exists a metric space $T_X \in$ **Met**, a non-expansive map $\eta^{\mathcal{U}}_X : X \to T_X$, and a quantitative Σ-algebra $(T_X, \psi^{\mathcal{U}}_X)$ satisfying $\mathcal{U}$, such that, for any quantitative Σ-algebra (A, α) in $\mathbb{K}(\Sigma, \mathcal{U})$ and non-expansive map $\beta : X \to A$, there exists a unique homomorphism $h : T_X \to A$ of quantitative Σ-algebras making the following diagram commute*

$$\begin{tikzcd} X \arrow[r, "\eta^{\mathcal{U}}_X"] \arrow[dr, "\beta"'] & T_X \arrow[d, dashed, "h"] & \Sigma T_X \arrow[l, "\psi^{\mathcal{U}}_X"'] \arrow[d, "\Sigma h"] \\ & A & \Sigma A \arrow[l, "\alpha"] \end{tikzcd}$$

The map h is also called the homomorphic extension of α along β.

The universal property above says that $(T_X, \psi^{\mathcal{U}}_X)$ is the free quantitative Σ-algebra for X in $\mathbb{K}(\Sigma, \mathcal{U})$. From Theorem 3.3, one defines the monad $(T_{\mathcal{U}}, \eta^{\mathcal{U}}, \mu^{\mathcal{U}})$ as follows: the functor $T_{\mathcal{U}} :$ **Met** $\to$ **Met** associates to $X \in$ **Met** the carrier T_X of the free quantitative Σ-algebra for X in $\mathbb{K}(\Sigma, \mathcal{U})$; the maps $\eta^{\mathcal{U}}_X$ form the components of the unit $\eta^{\mathcal{U}} : Id \Rightarrow T_{\mathcal{U}}$; and the multiplication $\mu^{\mathcal{U}} : T_{\mathcal{U}} T_{\mathcal{U}} \Rightarrow T_{\mathcal{U}}$ is defined at X as the unique map that, by Theorem 3.3, satisfies $\mu^{\mathcal{U}}_X \circ \eta^{\mathcal{U}}_X = id$ and $\mu^{\mathcal{U}}_X \circ \psi^{\mathcal{U}}_{T_{\mathcal{U}} X} = \psi^{\mathcal{U}}_X \circ \Sigma \mu^{\mathcal{U}}_X$.

A similar free construction also holds for quantitative algebras in **CQA**(Σ) for continuous quantitative equational theories:

Theorem 3.4 (Free Complete Algebra)**.** *Let $\mathcal{U}$ be a continuous quantitative theory of type Σ. Then, for any $X \in$ **CMet**, quantitative Σ-algebra (A, α) in $\mathbb{CK}(\Sigma, \mathcal{U})$ and non-expansive map $\beta : X \to A$, there exists a unique homomorphism $h : \mathbb{C}T_{\mathcal{U}} \to A$ of quantitative Σ-algebras making the following diagram commute*

$$\begin{tikzcd} X \arrow[r, "\mathbb{C}\eta^{\mathcal{U}}_X"] \arrow[dr, "\beta"'] & \mathbb{C}T_{\mathcal{U}} X \arrow[d, dashed, "h"] & \Sigma \mathbb{C}T_{\mathcal{U}} X \arrow[l, "\mathbb{C}\psi^{\mathcal{U}}_X"'] \arrow[d, "\Sigma h"] \\ & A & \Sigma A \arrow[l, "\alpha"] \end{tikzcd}$$

The above is equivalent to saying that the forgetful functor from $\mathbb{CK}(\Sigma, \mathcal{U})$ to **CMet** has a left adjoint. In particular, $\mathbb{C}T_{\mathcal{U}}$ is the free monad on $\mathcal{U}$ in **CMet**, provided that the quantitative equational theory is continuous.

4 Disjoint Union of Quantitative Theories

One of the advantages of the approach followed in [9] is that it allows one to combine different computational phenomena in a smooth way. In our setting we need to combine quantitative theories. The major example which will be discussed in this paper is the combination of interpolative barycentric algebras (which we had shown in [13] to axiomatise probability distributions with the Kantorovich metric) and the algebras that give a transition structure. This combination gives us the usual theory of Markov processes but now enriched with metric reasoning principles for the underlying probability distributions.

In this section we develop the theory of the *disjoint union* of quantitative equational theories. The proofs given here are mostly, but not completely, categorical. We use the fact that the quantitative equational theories are *basic* (*cf.* [14]). This appears in the proof of Theorem. 4.2 and is remarked on there.

Let Σ, Σ' be two disjoint signatures. The disjoint union of two quantitative equational theories $\mathcal{U}, \mathcal{U}'$ of respective types Σ and Σ', written $\mathcal{U} + \mathcal{U}'$, is the smallest quantitative theory containing $\mathcal{U}$ and $\mathcal{U}'$. Following Kelly [12], we show that any model for $\mathcal{U} + \mathcal{U}'$ is a $\langle \mathcal{U}, \mathcal{U}' \rangle$*-bialgebra*: a metric space A with both a Σ-algebra structure $\alpha : \Sigma A \to A$ satisfying $\mathcal{U}$ and a Σ'-algebra structure $\beta : \Sigma' A \to A$ satisfying $\mathcal{U}'$. Formally, let $\mathbb{K}((\Sigma, \mathcal{U}) \oplus (\Sigma', \mathcal{U}'))$ be the category of $\langle \mathcal{U}, \mathcal{U}' \rangle$-bialgebras with non-expansive maps preserving the two algebraic structures. Then, the following isomorphism of categories holds.

Proposition 4.1. $\mathbb{K}(\Sigma + \Sigma', \mathcal{U} + \mathcal{U}') \cong \mathbb{K}((\Sigma, \mathcal{U}) \oplus (\Sigma', \mathcal{U}'))$.

Next, we show that under certain assumptions on the theories, the free monad $T_{\mathcal{U}+\mathcal{U}'}$ on the disjoint union $\mathcal{U} + \mathcal{U}'$ corresponds to the categorical sum $T_{\mathcal{U}} + T_{\mathcal{U}'}$ of the free monads on $\mathcal{U}$ and $\mathcal{U}'$, respectively. The only requirement we ask is that the theories can be axiomatised by a set of *basic* conditional equations, *i.e.*, conditional equations of the form $\{x_1 \equiv_{\varepsilon_1} y_1, \dots, x_n \equiv_{\varepsilon_n} y_n\} \vdash t \equiv_{\varepsilon} s$, where x_i, y_i are variables in X; as in [14], we call these theories *basic*.

Let T-**Alg** be the category of Eilenberg-Moore T-algebras of a monad T. Then, we have:

Theorem 4.2. *For any basic quantitative equational theory $\mathcal{U}$ of type Σ, $T_{\mathcal{U}}$-**Alg** $\cong \mathbb{K}(\Sigma, \mathcal{U})$.*

Proof. The isomorphism is given by the following pair of functors

$$\begin{tikzcd} T_{\mathcal{U}}\text{-}\mathbf{Alg} \arrow[r, bend left, "H"] & \mathbb{K}(\Sigma, \mathcal{U}) \arrow[l, bend left, "K"] \end{tikzcd}$$

both mapping morphisms essentially to themselves and on objects acting as follows: for $(A, \alpha) \in T_{\mathcal{U}}$-**Alg** and $(B, \beta) \in \mathbb{K}(\Sigma, \mathcal{U})$,

$$H(A, \alpha) = (A, \alpha \circ \psi^{\mathcal{U}}_A \circ \Sigma \eta^{\mathcal{U}}_A), \qquad K(B, \beta) = (B, \beta^{\flat}),$$

where $\beta^{\flat} : T_{\mathcal{U}} B \to B$ is the unique map that, by Theorem 3.3, satisfies the equations $\beta^{\flat} \circ \eta^{\mathcal{U}}_B = id_B$ and $\beta^{\flat} \circ \psi^{\mathcal{U}}_B = \beta \circ \Sigma \beta^{\flat}$.

To show that K is well defined, we need to prove that the unit and the associativity laws for the $T_{\mathcal{U}}$-algebra hold. The unit law follows directly by definition of $\beta^{\flat}$. The associativity law follows by Theorem 3.3, since both $\beta^{\flat} \circ \mu^{\mathcal{U}}_B$ and $\beta^{\flat} \circ T_{\mathcal{U}} \beta^{\flat}$ fit as the unique homomorphic extension of β along $\eta^{\mathcal{U}}_B$. Given any morphism $h : (B, \beta) \to (B', \beta')$ of quantitative Σ-algebras in $\mathbb{K}(\Sigma, \mathcal{U})$, $K(h) = h$ is proved to be a $T_{\mathcal{U}}$-homomorphism by showing that both $(\beta')^{\flat} \circ T_{\mathcal{U}} h$ and $h \circ \beta^{\flat}$ fit as the unique homomorphic extension of $\psi^{\mathcal{U}}_{B'}$ along h. Functoriality of K follows similarly, using the universal property in Theorem 3.3.

We show that H is well defined, *i.e.*, for any $(A, \alpha) \in T_{\mathcal{U}}$-**Alg**, $H(A, \alpha)$ satisfies $\mathcal{U}$. Since, by hypothesis, $\mathcal{U}$ is a basic quantitative equational theory, it is axiomatised by a set $\mathcal{A} \subseteq \mathcal{U}$ of basic conditional quantitative equations. Thus, $H(A, \alpha)$ is a model for $\mathcal{U}$ iff it satisfies all the conditional equations in $\mathcal{A}$. To this end, note that for any assignment $\iota : X \to A$ of the variables, the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "\eta^{\Sigma}_X"] \arrow[d, "\iota"] & \mathbb{T}(\Sigma, X) \arrow[d, "T_{\Sigma}\iota"] & & & \Sigma\mathbb{T}(\Sigma, X) \arrow[lll, "\psi^{\Sigma}_X"'] \arrow[d, "\Sigma T_{\Sigma}\iota"] \\ A \arrow[r, "\eta^{\mathcal{U}}_A"] \arrow[dr, "id"'] & T_{\mathcal{U}} A \arrow[d, "\alpha"] & & & \Sigma T_{\mathcal{U}} A \arrow[lll, "\psi^{\mathcal{U}}_A"'] \arrow[d, "\Sigma\alpha"] \\ & A & T_{\mathcal{U}} A \arrow[l, "\alpha"'] & \Sigma T_{\mathcal{U}} A \arrow[l, "\psi^{\mathcal{U}}_A"'] & \Sigma A \arrow[l, "\Sigma\eta^{\mathcal{U}}_A"'] \end{tikzcd} \tag{1}$$

where the commutativity of the bottom-right square is proven as follows, by naturality of the maps, the monad laws, the unit, and the associativity laws of $(A, \alpha) \in T_{\mathcal{U}}$-**Alg**:

$$\begin{tikzcd}
 & T_{\mathcal{U}}A \arrow[dl, "id"'] & \Sigma T_{\mathcal{U}}A \arrow[l, "\psi^{\mathcal{U}}_A"'] & \Sigma T_{\mathcal{U}}A \arrow[l, "id"'] \arrow[dl, "\Sigma T_{\mathcal{U}}\eta^{\mathcal{U}}_A"'] \arrow[d, "id"] \\
T_{\mathcal{U}}A \arrow[d, "\alpha"'] & T_{\mathcal{U}}T_{\mathcal{U}}A \arrow[l, "T_{\mathcal{U}}\alpha"] \arrow[u, "\mu^{\mathcal{U}}_A"'] \arrow[d, "\mu^{\mathcal{U}}_A"] & \Sigma T_{\mathcal{U}}T_{\mathcal{U}}A \arrow[l, "(\psi^{\mathcal{U}}T^{\mathcal{U}})_A"] \arrow[u, "\Sigma\mu^{\mathcal{U}}_A"] \arrow[r, "\Sigma T_{\mathcal{U}}\alpha"'] \arrow[d, "\Sigma\mu^{\mathcal{U}}_A"] & \Sigma T_{\mathcal{U}}A \arrow[d, "\Sigma\alpha"] \\
A & T_{\mathcal{U}}A \arrow[l, "\alpha"] & \Sigma T_{\mathcal{U}}A \arrow[l, "\psi^{\mathcal{U}}_A"] \arrow[r, "\Sigma\alpha"] & \Sigma A \arrow[dl, "id"] \\
 & & \Sigma A \arrow[u, "\Sigma\eta^{\mathcal{U}}_A"] &
\end{tikzcd}$$

By commutativity of (1) and uniqueness of homomorphic extension, we have that the homomorphic extension $\iota^\sharp \colon \mathbb{T}(\Sigma, X) \to A$ of ι on (A, α) can be factorised as $\iota^\sharp = \alpha \circ T_{\mathcal{U}}\iota$. Moreover $T_\Sigma\iota$, is the homomorphic extension of $\eta^{\mathcal{U}}_A \circ \iota \colon X \to T_{\mathcal{U}}A$ on $(T_{\mathcal{U}}A, \psi^{\mathcal{U}}_A)$.

Let $\Gamma \vdash t \equiv_\varepsilon s \in \mathcal{A}$ and $\iota \colon X \to A$ be an assignment. Assume that, for all $x \equiv_\delta y \in \Gamma$, $d_A(\iota^\sharp(x), \iota^\sharp(y)) \le \delta$. Since $x, y \in X$, then $d_A(\iota(x), \iota(y)) \le \delta$; here we have used the fact that $\mathcal{U}$ is basic.

By Theorem 3.3, $(T_{\mathcal{U}}A, \psi^{\mathcal{U}}_A)$ satisfies $\mathcal{U}$, and because $T_\Sigma\iota$ is the homomorphic extension of $\eta^{\mathcal{U}}_A \circ \iota \colon X \to T_{\mathcal{U}}A$ on $(T_{\mathcal{U}}A, \psi^{\mathcal{U}}_A)$, we have that

$$\begin{gathered}\big(\text{for all } x \equiv_\delta y \in \Gamma,\ d_{T_{\mathcal{U}}A}(\eta^{\mathcal{U}}_A \circ \iota(x), \eta^{\mathcal{U}}_A \circ \iota(y)) \le \delta\big) \\ \text{implies} \\ d_{T_{\mathcal{U}}A}(T_\Sigma\iota(t), T_\Sigma\iota(s)) \le \varepsilon\,. \end{gathered} \tag{2}$$

By definition of $d_{T_{\mathcal{U}}A}$, we have that $d_{T_{\mathcal{U}}A}(\eta^{\mathcal{U}}_A \circ \iota(x), \eta^{\mathcal{U}}_A \circ \iota(y)) \le d_A(\iota(x), \iota(y))$, hence, by (2), we get $d_{T_{\mathcal{U}}A}(T_\Sigma\iota(t), T_\Sigma\iota(s)) \le \varepsilon$. Thus,

$$\begin{aligned}
\varepsilon &\ge d_{T_{\mathcal{U}}A}(T_\Sigma\iota(t), T_\Sigma\iota(s)) \\
&\ge d_A(\alpha \circ T_\Sigma\iota(t), \alpha \circ T_\Sigma\iota(s)) && (\alpha \text{ non-expansive}) \\
&\ge d_A(\iota^\sharp(t), \iota^\sharp(s))\,. && (\alpha \circ T_\Sigma\iota = \iota^\sharp)
\end{aligned}$$

Therefore $H(A, \alpha)$ satisfies $\mathcal{U}$. Note the crucial role played by the requirement of having basic conditional equations in using (2).

It remains to show that K and H are inverses of each other. On morphisms this is clear. As for objects, for $(A, \alpha) \in T_{\mathcal{U}}$-**Alg** and $(B, \beta) \in \mathbb{K}(\Sigma, \mathcal{U})$, we have

$$\begin{aligned}
KH(A, \alpha) &= (A, (\alpha \circ \psi^{\mathcal{U}}_A \circ \Sigma\eta^{\mathcal{U}}_A)^\flat)\,, \\
HK(B, \beta) &= (B, \beta^\flat \circ \psi^{\mathcal{U}}_B \circ \Sigma\eta^{\mathcal{U}}_B)\,,
\end{aligned}$$

thus, we need to show $\beta^\flat \circ \psi^{\mathcal{U}}_B \circ \Sigma\eta^{\mathcal{U}}_B = \beta$ and $(\alpha \circ \psi^{\mathcal{U}}_A \circ \Sigma\eta^{\mathcal{U}}_A)^\flat = \alpha$. These are proved by the commutativity of the following diagrams

$$\begin{tikzcd}
T_{\mathcal{U}}B \arrow[d, "\beta^\flat"'] & \Sigma T_{\mathcal{U}}B \arrow[l, "\psi^{\mathcal{U}}_B"'] \arrow[d, "\Sigma\beta^\flat"'] & \Sigma B \arrow[l, "\Sigma\eta^{\mathcal{U}}_B"'] \arrow[dl, "id"] \\
B & \Sigma B \arrow[l, "\beta"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\eta^{\mathcal{U}}_A"] \arrow[ddr, "id"'] & T_{\mathcal{U}}A \arrow[dd, "\alpha"] & & \Sigma T_{\mathcal{U}}A \arrow[ll, "\psi^{\mathcal{U}}_A"'] \arrow[dl, "(\theta T^{\mathcal{U}})_A"] \arrow[dd, "\Sigma\alpha"] \\
 & & T_{\mathcal{U}}T_{\mathcal{U}}A \arrow[ul, "\mu^{\mathcal{U}}_A"'] \arrow[d, "T_{\mathcal{U}}\alpha"] & \\
 & A & T_{\mathcal{U}}A \arrow[l, "\alpha"] & \Sigma A \arrow[l, "\theta_A"]
\end{tikzcd}$$

where $\theta = \psi^{\mathcal{U}} \circ \Sigma\eta^{\mathcal{U}}$. In particular, the second one proves that α satisfies the equalities $\alpha \circ \eta^{\mathcal{U}}_A = id_A$ and $\alpha \circ \psi^{\mathcal{U}}_A = \alpha \circ \psi^{\mathcal{U}}_A \circ \Sigma\eta^{\mathcal{U}}_A \circ \Sigma\alpha$ which, by Theorem 3.3, are uniquely satisfied by $(\alpha \circ \psi^{\mathcal{U}}_A \circ \Sigma\eta^{\mathcal{U}}_A)^\flat$. Hence, $(\alpha \circ \psi^{\mathcal{U}}_A \circ \Sigma\eta^{\mathcal{U}}_A)^\flat = \alpha$. □

When the quantitative equational theories $\mathcal{U}$ and $\mathcal{U}'$ are basic, by Theorem 4.2, we get a refinement of Proposition 4.1 as follows.

Let T, G be two monads on a category $\mathbf{C}$. A $\langle T, G\rangle$-*bialgebra* is an object $A \in \mathbf{C}$ with Eilenberg-Moore algebra structures $\alpha \colon TA \to A$ and $\beta \colon GA \to A$. We write $\langle T, G\rangle$-**biAlg** for the category of $\langle T, G\rangle$-bialgebras with morphisms those in $\mathbf{C}$ preserving the two algebraic structures.

Corollary 4.3. *Let $\mathcal{U}, \mathcal{U}'$ be basic quantitative equational theories. Then $\mathbb{K}(\Sigma + \Sigma', \mathcal{U} + \mathcal{U}') \cong \langle T_{\mathcal{U}}, T_{\mathcal{U}'}\rangle$-**biAlg**.*

Proof. Immediate from Theorem 4.2 and Proposition 4.1. □

Now we are ready to state the main theorem of the section.

Theorem 4.4. *Let $\mathcal{U}, \mathcal{U}'$ be basic quantitative theories. Then, the monad $T_{\mathcal{U}+\mathcal{U}'}$ in* **Met** *is the sum of monads $T_{\mathcal{U}} + T_{\mathcal{U}'}$.*

Proof. By Corollary 4.3 and Theorem 3.3 the obvious forgetful functor from $\langle T_{\mathcal{U}}, T_{\mathcal{U}'}\rangle$-**biAlg** to **Met** has a left adjoint. The monad generated by this adjunction is isomorphic to $T_{\mathcal{U}+\mathcal{U}'}$. Thus, by [12] (cf. also [1, Proposition 2.8]), the monad $T_{\mathcal{U}+\mathcal{U}'}$ is also isomorphic to $T_{\mathcal{U}} + T_{\mathcal{U}'}$. □

The above constructions do not use any specific property of the category **Met**, apart from requiring its morphisms to be non-expansive. Thus, under mild restrictions on the type of theories and conditions on the free monad induced by them, we can reformulate alternative versions of Theorem 4.4 which are valid on specific full subcategories of **Met**.

The first one applies to **CMet**, provided that the quantitative equational theories are continuous. Recall that, continuous theories are basic. Moreover, the disjoint union $\mathcal{U} + \mathcal{U}'$ of two continuous quantitative theories $\mathcal{U}, \mathcal{U}'$ is also continuous, so that, by Theorem 3.4, the free monad on it in **CMet** is $\mathbb{C}T_{\mathcal{U}+\mathcal{U}'}$. Thus:

Theorem 4.5. *Let $\mathcal{U}, \mathcal{U}'$ be continuous quantitative theories. Then, the monad $\mathbb{C}T_{\mathcal{U}+\mathcal{U}'}$ in* **CMet** *is the sum of $\mathbb{C}T_{\mathcal{U}}$ and $\mathbb{C}T_{\mathcal{U}'}$.*

Note that, for a continuous quantitative theory $\mathcal{U}$, if the functor $T_{\mathcal{U}}$ preserves separability of the metric spaces, then the free monad on $\mathcal{U}$ in **CSMet** is given by $\mathbb{C}T_{\mathcal{U}}$. This is the case, for example, for countable signatures. Thus, under an additional condition on the free monads, the theorem above can also be stated for the case of complete separable metric spaces.

Corollary 4.6. *Let $\mathcal{U}, \mathcal{U}'$ be continuous quantitative theories and assume $T_{\mathcal{U}+\mathcal{U}'}$, $T_{\mathcal{U}}$, and $T_{\mathcal{U}'}$ preserve separability of metric spaces. Then, the monad $\mathbb{C}T_{\mathcal{U}+\mathcal{U}'}$ in* **CSMet** *is the sum of $\mathbb{C}T_{\mathcal{U}}$ and $\mathbb{C}T_{\mathcal{U}'}$.*

5 Interpolative Barycentric Algebras

Interpolative barycentric algebras [13] are the quantitative algebras for the signature

$$\Sigma_{\mathcal{B}} = \{+_e : 2 \mid e \in [0, 1]\}$$

having a binary operator $+_e$, for each $e \in [0, 1]$ (a.k.a. *barycentric signature*), and satisfying the following axioms

(B1) $\vdash x +_1 y \equiv_0 x\,,$

(B2) $\vdash x +_e x \equiv_0 x\,,$

(SC) $\vdash x +_e y \equiv_0 y +_{1-e} x\,,$

(SA) $\vdash (x +_e y) +_{e'} z \equiv_0 x +_{ee'} (y +_{\frac{e'-ee'}{1-ee'}} z)\,,$ for $e, e' \in [0, 1)\,,$

(IB) $\{x \equiv_\varepsilon y, x' \equiv_{\varepsilon'} y'\} \vdash x +_e x' \equiv_\delta y +_e y'$, for $\delta \ge e\varepsilon + (1 - e)\varepsilon'$.

The quantitative theory induced by the axioms above, written $\mathcal{B}$, is called *interpolative barycentric quantitative equational theory*. The axioms (B1), (B2), (SC), (SA) are those of *barycentric algebras* (a.k.a. *abstract convex sets*) due to M. H. Stone [23] where (SC) stands for *skew commutativity* and (SA) for *skew associativity*; (IB) is the *interpolative barycentric axiom* introduced in [13].

5.1 On Metric Spaces

Let $\Pi\colon \mathbf{Met} \to \mathbf{Met}$ be the functor assigning to each $X \in \mathbf{Met}$ the metric space $\Pi(X)$ of finitely supported Borel probability measures with Kantorovich metric and acting on morphisms $f \in \mathbf{Met}(X, Y)$ as $\Pi(f)(\mu) = \mu \circ f^{-1}$, for $\mu \in \Pi(X)$.

This functor has a monad structure, with unit $\delta\colon Id \Rightarrow \Pi$ and multiplication $m\colon \Pi^2 \Rightarrow \Pi$, given as follows, for $x \in X$, $\Phi \in \Pi^2(X)$, and Borel subset $E \subseteq X$

$$\delta_X(x) = \delta_x\,, \qquad\qquad m(\Phi)(E) = \int ev_E \,\mathrm{d}\Phi\,,$$

where δ_x is the Dirac distribution at $x \in X$, and $ev_E\colon \Pi(X) \to [0,1]$ is the evaluation function, taking $\mu \in \Pi(X)$ to $\mu(E) \in [0,1]$. This monad is also known as the *finite distribution monad*.

For any $X \in \mathbf{Met}$, one can define a quantitative $\Sigma_{\mathcal{B}}$-algebra $(\Pi(X), \phi_X)$ as follows, for arbitrary $\mu, \nu \in \Pi X$

$$\phi_X\colon \Sigma_{\mathcal{B}}\Pi X \to \Pi X \qquad \phi_X(in_{+_e}(\mu,\nu)) = e\mu + (1-e)\nu\,.$$

This quantitative algebra satisfies the interpolative barycentric theory $\mathcal{B}$ [13, Theorem 10.4]) and is universal in the following sense.

Theorem 5.1 ([13, Th. 10.5])**.** *For any $\Sigma_{\mathcal{B}}$-algebra (A, α) satisfying $\mathcal{B}$ and non-expansive map $\beta\colon X \to A$, there exists a unique homomorphism $h\colon \Pi X \to A$ of quantitative $\Sigma_{\mathcal{B}}$-algebras making the diagram below commute*

$$\begin{tikzcd} X \arrow[r, "\delta_X"] \arrow[dr, "\beta"'] & \Pi X \arrow[d, dashed, "h"] & \Sigma_{\mathcal{B}}\Pi X \arrow[l, "\phi_X"'] \arrow[d, "\Sigma_{\mathcal{B}} h"] \\ & A & \Sigma_{\mathcal{B}} A \arrow[l, "\alpha"] \end{tikzcd}$$

From this we obtain that Π is isomorphic to the monad $T_{\mathcal{B}}$ on the quantitative theory $\mathcal{B}$ of interpolative barycentric algebras.

Theorem 5.2. *The monads $T_{\mathcal{B}}$ and Π in* **Met** *are isomorphic.*

5.2 On Complete Separable Metric Spaces

Define the functor $\Delta\colon \mathbf{CSMet} \to \mathbf{CSMet}$ assigning to each $X \in \mathbf{CSMet}$ the complete separable metric space $\Delta(X)$ of Borel probability measures with Kantorovich metric and acting on morphisms $f \in \mathbf{CSMet}(X, Y)$ as $\Delta(f)(\mu) = \mu \circ f^{-1}$, for $\mu \in \Delta(X)$. This functor has a monad structure, defined similarly to the one for Π. It is known as the *metric Giry monad*.

Note that Cauchy completion preserves separability. Thus the Cauchy completion functor $\mathbb{C}\colon \mathbf{Met} \to \mathbf{CMet}$ restricts to separable spaces. By Lemma 2.2 and the fact that $\mathbb{C}\mathbb{C} \cong \mathbb{C}$, one can verify that the canonical monad structure on $\mathbb{C}\Pi$ is isomorphic to the one on Δ in **CSMet**. In [13], it has been proven that $T_{\mathcal{B}}$ preserves separability. Hence, by Theorem 5.2, we obtain the following.

Theorem 5.3. *The monads $\mathbb{C}T_{\mathcal{B}}$ and Δ in* **CSMet** *are isomorphic.*

Note that, since $\mathcal{B}$ is axiomatised by a continuous schema of conditional quantitative equations, the free monad on $\mathcal{B}$ in **CSMet** is exactly given by $\mathbb{C}T_{\mathcal{B}}$. Therefore, Theorem 5.3 provides an algebraic characterisation of the metric Giry monad.

6 Algebras of Contractive Operators

A *signature of contractive operators* Σ is a signature of function symbols $f\colon n \in \Sigma$ with associated *contractive factor* $0 < c < 1$. We write this as $f\colon \langle n, c\rangle \in \Sigma$.

The *quantitative equational theory of contractive operators* associated to a signature Σ, written $\mathcal{O}(\Sigma)$, is the quantitative equational theory satisfying, for each $f\colon \langle n, c\rangle \in \Sigma$, the axioms

$$(f\text{-Lip})\ \{x_1 \equiv_\varepsilon y_1, \dots, x_n \equiv_\varepsilon y_n\} \vdash f(x_1,\dots,x_n) \equiv_\delta f(y_1,\dots,y_n)\,,$$

for all rationals $\delta \geq c\varepsilon$. The axiom (f-Lip) requires the interpretation of f to be c-Lipschitz continuous.

6.1 On Metric Spaces

For a contractive signature Σ, we define a modification of the signature endofunctor on **Met** by:

$$\tilde{\Sigma} = \coprod_{f\colon \langle n,c\rangle \in \Sigma} c \cdot Id^n$$

where $c \cdot Id$ is the *rescaling functor*, mapping a metric space (X, d_X) to $(X, c \cdot d_X)$.

Next we show that quantitative Σ-algebras satisfying $\mathcal{O}(\Sigma)$ are in one-to-one correspondence with universal $\tilde{\Sigma}$-algebras; moreover, this correspondence lifts the identity functor on **Met** (cf. [11]).

Lemma 6.1. *There exists an isomorphism of categories between $\tilde{\Sigma}$-**Alg** and $\mathbb{K}(\Sigma, \mathcal{O}(\Sigma))$ making the following diagram commute*

$$\begin{tikzcd} \tilde{\Sigma}\text{-}\mathbf{Alg} \arrow[rr, bend left, ""] \arrow[dr, "U_{\tilde{\Sigma}}"'] \arrow[rr, phantom, "\cong"] & & \mathbb{K}(\Sigma, \mathcal{O}(\Sigma)) \arrow[ll, bend left, ""] \arrow[dl, "U_{\Sigma}"] \\ & \mathbf{Met} & \end{tikzcd}$$

In virtue of Lemma 6.1, by an abuse of notation, we will denote by the same name the algebras in $\tilde{\Sigma}$-**Alg** and $\mathbb{K}(\Sigma, \mathcal{O}(\Sigma))$.

Next we show that the free monad $T_{\mathcal{O}(\Sigma)}$ is isomorphic to the free monad on $\tilde{\Sigma}$. For this result, we first need some discussion of free and initial algebras and free monads.

Given any endofunctor H on a category **C**, we write $(\mu y.Hy, \alpha_H)$ for the initial H-algebra, if it exists. If **C** has binary coproducts, the free H-algebra on $X \in \mathbf{C}$ with its unit can be identified with $(\mu y.(Hy + X), \alpha_{H+X})$, and the one exists if and only if the other does. These free algebras exist if, for example, **C** is locally countably presentable and H has countable rank.

A *free monad* on H is a monad H^* on **C** and a natural transformation $\gamma\colon H \Rightarrow H^*$ that is initial among all such pairs $(S, \lambda\colon H \Rightarrow S)$. If the forgetful functor from H-**Alg** to **C** has a left adjoint (equivalently, any **C**-object has a free H-algebra with unit), then the resulting monad on **C** is free on H and is said to be algebraic [3]. If H^* exists and is algebraic, the category H^*-**Alg** of Eilenberg-Moore algebras for the monad H^* is isomorphic to the category H-**Alg** of universal algebras for the endofunctor H.

We see from the above that, if **C** has binary sums, then H^* can be identified with $\mu y.(Hy + -)$ and the former exists if and only if the other does. We further see that if **C** is locally countably presentable and H has countable rank, then H^* exists and has countable rank [12].

Therefore, since **Met** is locally countably presentable [2] and $\tilde{\Sigma}$ has countable rank, the free algebra for $\tilde{\Sigma}$ exists and so does the free monad $\tilde{\Sigma}^*$. Let $\kappa_X : \tilde{\Sigma}\tilde{\Sigma}^*X \to \tilde{\Sigma}^*X$ be the free $\tilde{\Sigma}$-algebra on $X \in \mathbf{Met}$ with unit $\chi : Id \Rightarrow \tilde{\Sigma}^*$.

Then, the next result follows by Lemma 6.1 and freeness of $\tilde{\Sigma}^*$.

Corollary 6.2. *Let Σ be a signature of contractive operators. Then, for any quantitative Σ-algebra (A, α) satisfying $O(\Sigma)$ and non-expansive map $\beta : X \to A$, there exists a unique homomorphism $h : \tilde{\Sigma}^*X \to A$ of quantitative Σ-algebras making the diagram below commute*

$$\begin{tikzcd} X \arrow[r, "\chi_X"] \arrow[dr, "\beta"'] & \tilde{\Sigma}^*X \arrow[d, dashed, "h"] & \Sigma\tilde{\Sigma}^*X \arrow[l, "\kappa_X"'] \arrow[d, "\Sigma h"] \\ & A & \Sigma A \arrow[l, "\alpha"'] \end{tikzcd}$$

By freeness of $T_{O(\Sigma)}$ and Corollary 6.2, the following holds:

Theorem 6.3. *The monads $T_{O(\Sigma)}$ and $\tilde{\Sigma}^*$ in* **Met** *are isomorphic.*

6.2 On Complete Metric Spaces

The category **CMet** has coproducts and finite products. Moreover, since rescaling a metric by a factor $0 < c < 1$ preserves Cauchy completeness, the rescaling functor $c \cdot Id$ can be restricted to an endofunctor on **CMet**. Hence, for any contractive signature Σ, the endofunctor $\tilde{\Sigma} = \coprod_{f : \langle n,c \rangle \in \Sigma} c \cdot Id^n$ is well defined in **CMet**.

By repeating the construction in Lemma 6.1 we get the following.

Lemma 6.4. *There exists an isomorphism of categories between $\tilde{\Sigma}$-**Alg** and $\mathbb{K}(\Sigma, O(\Sigma))$ making the following diagram commute*

$$\begin{tikzcd} \tilde{\Sigma}\text{-}\mathbf{Alg} \arrow[rr, bend left, ""] \arrow[dr, "U_{\tilde{\Sigma}}"'] & \cong & \mathbb{CK}(\Sigma, O(\Sigma)) \arrow[ll, bend left, ""] \arrow[dl, "U_\Sigma"] \\ & \mathbf{CMet} & \end{tikzcd}$$

Because $O(\Sigma)$ is a continuous quantitative theory, by Theorem 3.4, the free monad on $O(\Sigma)$ in **CMet** is given by $\mathbb{C}T_{O(\Sigma)}$.

Note that, also **CMet** is locally countably presentable [2], and since $\tilde{\Sigma}$ is of countable rank, we have that the free monad on $\tilde{\Sigma}$ exists in **CMet** too. Therefore, by repeating the same argument we used before, by Lemma 6.4 and freeness of $\tilde{\Sigma}$ and $\mathbb{C}T_{O(\Sigma)}$ we obtain:

Theorem 6.5. *The monads $\mathbb{C}T_{O(\Sigma)}$ and $\tilde{\Sigma}^*$ in* **CMet** *are isomorphic.*

6.3 On Complete Separable Metric Spaces

The category **CSMet** has countable coproducts and finite products. Moreover, since the operation of rescaling a metric by a constant factor $0 < c < 1$, preserves both Cauchy completeness and separability, the endofunctor $c \cdot Id$ can be restricted to **CSMet**.

Hence, provided that the contractive signature Σ consists of only a *countable* set of operators, $\tilde{\Sigma} = \coprod_{f : \langle n,c \rangle \in \Sigma} c \cdot Id^n$ is a well defined endofunctor on **CSMet**. Hereafter, we assume the signature Σ to be countable.

Unlike **CMet**, the category **CSMet** is not locally countably presentable, because is not cocomplete (it does not have uncountable coproducts). However, **CSMet** has all $\aleph_1$-filtered colimits, and since every separable space in **CMet** is a countably presentable (or $\aleph_1$-presentable) object [2, Corollary 2.9], **CSMet** is $\aleph_1$-accessible. Since $\tilde{\Sigma}$ is of countable rank (*i.e.*, $\aleph_1$-accessible), by [8, Lemma 3.4] the free monad $\tilde{\Sigma}^*$ on $\tilde{\Sigma}$ exists in **CSMet** and is algebraic.

The functor Σ^* in **Met** preserves separability of the metric spaces. Thus, by Theorem 6.3, so does $T_{O(\Sigma)}$. Moreover, since the quantitative equational theory $O(\Sigma)$ is continuous, by Theorem 3.4, the free monad on $O(\Sigma)$ in **CSMet** is given by $\mathbb{C}T_{O(\Sigma)}$. Thus, by freeness of $\tilde{\Sigma}^*$ and Lemma 6.1, the following holds:

Theorem 6.6. *The monads $\mathbb{C}T_{O(\Sigma)}$ and $\tilde{\Sigma}^*$ in* **CSMet** *are isomorphic.*

7 Interpolative Barycentric Algebras with Contractive Operators

In this section we study a variation of interpolative barycentric quantitative algebras where we add operations from a contractive signature Σ assumed to be disjoint from $\Sigma_{\mathcal{B}}$.

These are quantitative algebras for the signature

$$\Sigma_{\mathcal{B}} + \Sigma = \{+_e : 2 \mid e \in [0, 1]\} \cup \Sigma\,,$$

satisfying the disjoint union of the axioms of interpolative barycentric quantitative theory, namely (B1), (B2), (SC), (SA), and (IB), and, for each $f \in \Sigma$, the axiom (f-Lip) from the quantitative theory of contractive operators.

The quantitative equational theory induced by these axioms will be called *interpolative barycentric theory with contractive operators in* Σ, and it coincides with the disjoint union $\mathcal{B} + O(\Sigma)$ of theories.

7.1 On Metric Spaces

We have already noted that the quantitative theories $\mathcal{B}$ and $O(\Sigma)$ are basic. Thus, by Theorem 4.4, the free monad $T_{\mathcal{B}+O(\Sigma)}$ on **Met** induced by $\mathcal{B} + O(\Sigma)$ is the sum of $T_{\mathcal{B}}$ and $T_{O(\Sigma)}$. Moreover, by Theorems 5.2 and 6.3, it is also isomorphic to $\Pi + \tilde{\Sigma}^*$.

In the following we will prove an alternative characterisation of $T_{\mathcal{B}+O(\Sigma)}$, that will eventually reveal the connection between interpolative barycentric algebras with operators and Markov processes.

Depending on the type of monads, several specific conditions of existence and constructions appear in the literature [1] for the sum of monads. One of these, due to Hyland, Plotkin, and Power [9], recalled below for convenience, characterises the sum of a monad with a free one.

Theorem 7.1 ([9, Theorem 4])**.** *Given an endofunctor F and a monad T on a category* C*, if the free monads F^* and $(FT)^*$ exist and are algebraic, then the sum of monads $T + F^*$ exists in the category of monads over* C *and is given by a canonical monad structure on the composite $T(FT)^*$.*

In terms of the above result, given that $T_{\mathcal{B}+O(\Sigma)}$ is isomorphic to $\Pi + \tilde{\Sigma}^*$, if we prove that the free monad $(\tilde{\Sigma}\Pi)^*$ in **Met** exists and is algebraic, then $T_{\mathcal{B}+O(\Sigma)}$ would be also isomorphic to $\Pi(\tilde{\Sigma}\Pi)^*$. For this reason, in the following we characterise (and hence prove the existence of) the free algebra on $\tilde{\Sigma}\Pi$. For arbitrary $X \in \mathbf{Met}$, let P_X be the smallest set such that

- if $x \in X$, then $\underline{x} \in P_X$;
- if $\mu_1, \dots, \mu_n \in \Pi(P_X)$ and $f : n \in \Sigma$, then $f\langle \mu_1, \dots, \mu_n \rangle \in P_X$.

We define the metric $d_{P_X} : P_X \times P_X \to [0, 1]$ by induction on the complexity on the structure of the elements in P_X as follows[2], for arbitrary $x, x' \in X$, $\mu_1, \dots, \mu_n, \nu_1, \dots, \nu_k \in \Pi(P_X)$, and distinct operators $f, g \in \Sigma$ of arity $f : n$ and $g : k$ such that $n \leq k$, where

[2]The symmetric cases are omitted and defined as expected.

we assume $f : \langle n, c\rangle \in \Sigma$.

$$d_{P_X}(\underline{x}, \underline{x'}) = d_X(x, x'),$$
$$d_{P_X}(f\langle \mu_1, \dots, \mu_n\rangle, f\langle \nu_1, \dots, \nu_n\rangle) = c \cdot \max_{i=1}^{n} \mathcal{K}(d_{P_X})(\mu_i, \nu_i),$$
$$d_{P_X}(f\langle \mu_1, \dots, \mu_n\rangle, g\langle \nu_1, \dots, \nu_k\rangle) = 1,$$
$$d_{P_X}(\underline{x}, f\langle \mu_1, \dots, \mu_n\rangle) = 1.$$

Proposition 7.2. *For any $X \in$ **Met**, d_{P_X} is a well defined metric.*

For $X \in$ **Met**, define $\varrho_X : X \to P_X$ and $\vartheta_X : \tilde{\Sigma}\Pi P_X \to P_X$ as follows, for arbitrary $x \in X$, $f : n \in \Sigma$, and $\mu_1, \dots, \mu_n \in \Pi(P_X)$

$$\varrho_X(x) = \underline{x}, \qquad \vartheta_X(in_f(\mu_1, \dots, \mu_n)) = f\langle \mu_1, \dots, \mu_n\rangle.$$

By definition of d_{P_X}, it is straightforward to show that both ϱ_X and ϑ_X are non-expansive maps (actually, are isometric injections), thus they are morphisms in **Met**. In particular, ϑ_X is a universal $\tilde{\Sigma}\Pi$-algebra on P_X in **Met**.

Theorem 7.3 (Free Algebra). *For any $X \in$ **Met**, $\tilde{\Sigma}\Pi$-algebra (A, α), and non-expansive map $\beta : X \to A$, there exists a unique $\tilde{\Sigma}\Pi$-homomorphism $h : P_X \to A$ making the diagram below commute*

$$\begin{tikzcd} X \arrow[r, "\varrho_X"] \arrow[dr, "\beta"'] & P_X \arrow[d, dashed, "h"] & \tilde{\Sigma}\Pi P_X \arrow[l, "\vartheta_X"'] \arrow[d, "\tilde{\Sigma}\Pi h"] \\ & A & \tilde{\Sigma}\Pi A \arrow[l, "\alpha"'] \end{tikzcd}$$

Theorem 7.3 states that ϑ_X is the free $\tilde{\Sigma}\Pi$-algebra for $X \in$ **Met**, or equivalently, that the forgetful functor from the category of $\tilde{\Sigma}\Pi$-algebras to **Met** has a left adjoint. Thus, the free monad $(\tilde{\Sigma}\Pi)^*$ on $\tilde{\Sigma}\Pi$ in **Met** exists and is algebraic. Moreover, it acts on objects $X \in$ **Met** as $(\tilde{\Sigma}\Pi)^* X = P_X$.

Corollary 7.4. *The monads $T_{\mathcal{B}+O(\Sigma)}$ and $\Pi(\tilde{\Sigma}\Pi)^*$ in **Met** are isomorphic.*

Proof. Direct consequence of Theorems 4.4, 5.2, 6.3, 7.1, and 7.3. □

As observed in [9], the monad $T(FT)^*$ of Theorem 7.1 is simply another form of the generalised resumptions monad transformer of Cenciarelli and Moggi [5], sending T to $\mu y.T(Fy + -)$. Hence, by the characterisation above and guided by the same observations that lead to [9, Corollary 2], we obtain the following isomorphism.

Theorem 7.5. *The monad $T_{\mathcal{B}+O(\Sigma)}$ in **Met** is isomorphic to the canonical monad structure on $\mu y.\Pi(\tilde{\Sigma}y + -)$.*

Proof. By [22, Proposition 5.3], it is easy to show that $\mu y.\Pi(\tilde{\Sigma}y + -)$ exists if and only if $(\tilde{\Sigma}\Pi)^*$ does, and that $\Pi(\tilde{\Sigma}\Pi)^*$ and $\mu y.\Pi(\tilde{\Sigma}y + -)$ are then isomorphic. □

7.2 On Complete Separable Metric Spaces

We would like to apply Corollary 4.6 to provide a characterisation of the free monad on $\mathcal{B} + O(\Sigma)$ in the category **CSMet** as the sum of monads $\mathbb{C}T_{\mathcal{B}} + \mathbb{C}T_{O(\Sigma)}$.

To this end we have to verify that the conditions required by the corollary are satisfied. We already noted that the quantitative theories $\mathcal{B}$ and $O(\Sigma)$ are continuous and that the functors $T_{\mathcal{B}}$ and $T_{O(\Sigma)}$ preserve separability of the metric spaces. We are only left to prove that also $T_{\mathcal{B}+O(\Sigma)}$ preserves separability.

Lemma 7.6. *The functor $T_{\mathcal{B}+O(\Sigma)}$ in **Met** preserves separability.*

Thus, we can apply Corollary 4.6 to obtain the desired result.

Corollary 7.7. *The monad $\mathbb{C}T_{\mathcal{B}+O(\Sigma)}$ in **CSMet** is the sum of $\mathbb{C}T_{\mathcal{B}}$ and $\mathbb{C}T_{O(\Sigma)}$.*

An immediate consequence of the characterisation above and Theorems 5.3 and 6.6 is the following.

Corollary 7.8. *The monad $\mathbb{C}T_{\mathcal{B}+O(\Sigma)}$ in **CSMet** is isomorphic to $\Delta + \tilde{\Sigma}^*$.*

According to the above and similarly to what we have done in Section 7.1, we would like to prove for the free monad $\mathbb{C}T_{\mathcal{B}+O(\Sigma)}$ on $\mathcal{B} + O(\Sigma)$ in **CSMet** corresponding results to Corollary 7.4 and Theorem 7.5.

We will proceed again by using Theorem 7.1. Thus, as we did in Section 7.1, we need to show that the free monad on $\tilde{\Sigma}\Delta$ in **CSMet** exists and is algebraic. We already noted that the category **CSMet** is accessible and that $\tilde{\Sigma}$ is an accessible endofunctor on it. Moreover, by [25, Corollary 22], we also have that Δ on **CSMet** is accessible, so that their composition $\tilde{\Sigma}\Delta$ is accessible too. Therefore, by [8, Lemma 3.4] the free monad $(\tilde{\Sigma}\Delta)^*$ exists and is algebraic.

This observation, then leads to the desired characterisations.

Corollary 7.9. *The monad $\mathbb{C}T_{\mathcal{B}+O(\Sigma)}$ in **CSMet** is isomorphic to the monads $\Delta(\tilde{\Sigma}\Delta)^*$ and $\mu y.\Delta(\tilde{\Sigma}y + -)$ with the canonical monad structures.*

8 The Algebras of Markov Processes

In this section we show how interpolative barycentric quantitative theories with operators can be used to axiomatise the probabilistic bisimilarity distance of Desharnais et al. [7] over Markov processes.

For any $0 < c < 1$, we define the finite signature of contractive operators

$$\mathcal{M}_c = \{\mathbf{0} : \langle 0, c\rangle\} \cup \{\diamond : \langle 1, c\rangle\},$$

consisting of one constant symbol $\mathbf{0}$, representing termination, and a unary operator $\diamond(t)$ expressing the capability of doing a transition to t. Both operators are associated with the same contractive factor.

The interpolative barycentric quantitative theory with operators in $\mathcal{M}_c$, given as the disjoint union $\mathcal{B} + O(\mathcal{M}_c)$ of theories is generated by the following set of axioms

(B1) $\vdash x +_1 y \equiv_0 x$,

(B2) $\vdash x +_e x \equiv_0 x$,

(SC) $\vdash x +_e y \equiv_0 y +_{1-e} x$,

(SA) $\vdash (x +_e y) +_{e'} z \equiv_0 x +_{ee'} (y +_{\frac{e'-ee'}{1-ee'}} z)$, for $e, e' \in [0, 1)$,

(IB) $\{x \equiv_\varepsilon y, x' \equiv_{\varepsilon'} y'\} \vdash x +_e x' \equiv_\delta y +_e y'$, for $\delta \geq e\varepsilon + (1-e)\varepsilon'$,

($\diamond$-Lip) $\{x =_\varepsilon y\} \vdash \diamond(x) \equiv_\delta \diamond(y)$, for $\delta \geq c\varepsilon$.

Note that, the constant $\mathbf{0}$ has no explicit associated ($\mathbf{0}$-Lip) axiom since it is equivalent to (Refl).

8.1 Markov Processes over Metric Spaces

We briefly recall the definitions of Markov processes over metric spaces and discounted probabilistic bisimilarity distance on them, presented following the pattern proposed in [25, Section 6].

Definition 8.1. A (sub-probabilistic) *Markov process* over a metric space is a tuple (X, τ) consisting of a metric space X of *states* and non-expansive *Markov kernel* $\tau : X \to \Delta(1 + X)$.

It is clear that these structures correspond to the coalgebras for the (sub-probabilistic) Giry functor $\Delta(1 + Id)$ in **Met**.

In [25], van Breugel et al. proved that the final coalgebra for $\Delta(1 + Id)$ in **Met** exists and they characterised the probabilistic bisimilarity distance on Markov processes as the pseudometric induced by the unique homomorphism to the final coalgebra.

We will do the same here by slightly extending their approach to encompass the case when the probabilistic bisimilarity distance is discounted by a factor $0 < c < 1$. Explicitly, the only difference consists in considering coalgebras for the functor $\Delta(1 + c \cdot Id)$ in **Met**. For simplicity we call these structures c-Markov processes. Note that Markov processes is a proper subclass: one can turn any Markov process into a c-Markov process as $(X, \Delta(1 + id^c_X) \circ \tau)$, where $id^c : Id \Rightarrow c \cdot Id$ is the obvious natural transformation acting as the identity on the elements of the metric space and allowing for the change of "type".

The final coalgebra (Z_c, ω_c) for $\Delta(1 + c \cdot Id)$ exists by similar arguments to [25, Section 6]. Then, for an arbitrary c-Markov process (X, τ), the *c-discounted probabilistic bisimilarity pseudometric* on (X, τ) is defined as the function $\mathbf{d}^c_\tau : X \times X \to [0, 1]$ given as

$$\mathbf{d}^c_\tau(x, x') = d_{Z_c}(h(x), h(x')),$$

where $h : X \to Z_c$ is the unique homomorphism of coagebras from (X, τ) to (Z_c, ω_c).

Since terminal objects are unique up to isomorphism, the definition of the distance function $\mathbf{d}^c_\tau$ does not depend on which terminal $\Delta(1 + c \cdot Id)$-coalgebra is chosen. Clearly, since d_{Z_c} is a 1-bounded metric, then $\mathbf{d}^c_\tau$ is a well defined 1-bounded pseudometric.

Proposition 8.2 ([25]). *Let (X, τ) be a c-Markov process. Then, for all $x, x' \in X$, $\mathbf{d}^c_\tau(x, x') = 0$ if and only if x and x' are probabilistically bisimilar.*

Moreover, this distance has a characterisation as the least fixed point of a monotone function on a complete lattice of 1-bounded pseudometrics.

Theorem 8.3 ([25]). *The c-discounted probabilistic bisimilarity pseudometric $\mathbf{d}^c_\tau$ on (X, τ) is the least fixed point of the following operator on the complete lattice of 1-bounded pseudometrics d on X with pointwise order $\sqsubseteq$, such that $d \sqsubseteq d_X$,*

$$\Psi^c_\tau(d)(x, x') = \sup_{f \in \Phi_{1+c \cdot X}} \left| \int f \,\mathrm{d}\tau(x) - \int f \,\mathrm{d}\tau(x') \right|,$$

with supremum ranging over the set $\Phi_{1+c\cdot X}$ of non-expansive positive 1-bounded real valued functions $f : 1 + c \cdot X \to [0, 1]$.

8.2 On Metric Spaces

In this section we want to relate c-Markov processes and their c-discounted probabilistic bisimilarity pseudometric with the free monad arising from the quantitative theory $\mathcal{B} + O(\mathcal{M}_c)$ in **Met**.

First note that the functor associated to the signature $\mathcal{M}_c$ is

$$\tilde{\mathcal{M}_c} = 1 + c \cdot Id.$$

where 1 is the terminal object in **Met** (*i.e.*, the singleton metric space)[3]. Thus, by Theorem 7.5, the free monad on $\mathcal{B} + O(\mathcal{M}_c)$ corresponds to the canonical monad structure on $\mu y.\Pi(1 + c \cdot y + -)$.

[3]Here we are implicitly applying the isomorphism $1 \cong c \cdot 1$.

Explicitly, this means that, the free monad on $\mathcal{B} + O(\mathcal{M}_c)$ assigns to an arbitrary metric space $M \in$ **Met** the *initial solution* of the following functorial equation in **Met**

$$FP_M \cong \Pi(1 + c \cdot FP_M + M).$$

In particular, when $M = 0$ is the empty metric space (*i.e.*, the initial object) the above corresponds to the isomorphism on the initial $\Pi(1 + c \cdot Id)$-algebra. This gives rise to a $\Pi(1 + c \cdot Id)$-coalgebra structure on FP_0, which in turn can be converted into a c-Markov process via a post-composition with the subspace inclusion $\Pi(-) \hookrightarrow \Delta(-)$. Let us write this c-Markov process as

$$(FP, \alpha : FP \to \Delta(1 + c \cdot FP)).$$

The following states that the metric on FP corresponds to the c-discounted probabilistic bisimilarity (pseudo)metric on (FP, α).

Lemma 8.4. $d_{FP} = \mathbf{d}^c_\alpha$

Proof. By Theorem 8.3 we need to prove that d_{FP} is the least fixed point of Ψ^c_α. This follows trivially by definition of the functor $\Pi(1 + c \cdot Id)$ and because (FP, α^{-1}) is the initial $\Pi(1 + c \cdot Id)$-algebra. □

Next we would like to give a more explicit characterisation of the elements in FP. By recalling the characterisation of the metric term monad in [13], the elements in FP can be represented by equivalence classes of terms generated by the following grammar

$$f ::= \mathbf{0} \mid \diamond(f) \mid f +_e f. \qquad \text{for } e \in [0, 1]$$

with respect to the kernel of the distance. In this specific case the distance corresponds to $\mathbf{d}^c_\alpha$, with transition probability function α defined as follows:

$$\alpha(\mathbf{0}) = \delta_\perp, \quad \alpha(\diamond(f)) = \delta_f, \quad \alpha(f +_e g) = \alpha(f) +_e \alpha(g).$$

Thus, by Proposition 8.2, we can interpret the element in FP as pointed (or rooted) Markov processes constructed over the above grammar and quotiented by bisimilarity. It is not difficult to see that these structures correspond to the class of *rooted acyclic finite Markov processes* from [6].

8.3 On Complete Separable Metric Spaces

We would like to relate c-Markov processes and the c-discounted probabilistic bisimilarity pseudometric with the free monad in **CSMet**.

By Corollary 7.9, we know that the free monad on $\mathcal{B} + O(\mathcal{M}_c)$ in **CMet** corresponds to the canonical monad structure on $\mu y.\Delta(1 + c \cdot y + -)$.

Explicitly, this means that, for the case of complete metric spaces the free monad on $\mathcal{B} + O(\mathcal{M}_c)$ assigns to any arbitrary metric space $M \in$ **CSMet** the *initial solution* of the following functorial equation in **CSMet**

$$MP_M \cong \Delta(1 + c \cdot MP_M + M).$$

Observe that the map $\omega_M : MP_M \to \Delta(1 + c \cdot MP_M + M)$ arising from the above isomorphism is a coalgebra structure for the functor $\Delta(1 + c \cdot Id + M)$. Next we show that (MP_M, ω_M) is actually the final coalgebra in **CSMet**.

We will do this by using the following result from [24, Section 7].

Theorem 8.5 ([24]). *Every locally contractive endofunctor H on* **CMet** *has a unique fixed point which is both an initial algebra and a final coalgebra for H.*

Note that if the fixed point lies in a subcategory of **CMet**, it is unique also in that subcategory. Hence, our goal is to prove that, for any $M \in \mathbf{CMet}$, the functor $\Delta(1 + c \cdot Id + M)$ is locally contractive.

In **CMet** the homsets $\mathbf{CMet}(X, Y)$ are themselves complete separable metric spaces, with distance

$$d_{X \to Y}(f, g) = \sup_{x \in X} d_Y(f(x), g(x)) .$$

An endofunctor H on **CMet** is *locally c-Lipschitz continuous* if for all $X, Y \in \mathbf{CMet}$ and non-expansive maps $f, g \colon X \to Y$,

$$d_{HX \to HY}(H(f), H(g)) \leq c \cdot d_{X \to Y}(f, g) .$$

We say that H is *locally non-expansive* if it is locally 1-Lipschitz continuous, and *locally contractive* if is locally c-Lipschitz continuous, for some $0 \leq c < 1$.

Examples of locally contractive functors are the constant functors and the rescaling functor $c \cdot Id$, for $0 \leq c < 1$. Locally contractiveness is preserved by products and coproducts and composition. Moreover, if H is locally non-expansive and G is locally contractive, then HG is locally contractive.

Lemma 8.6. *The endofunctor Δ on* **CMet** *is locally non-expansive.*

Thus, for the free monad on $\mathcal{B} + O(\mathcal{M}_c)$ in **CSMet**, the following holds.

Theorem 8.7. *For every $M \in$* **CSMet***, (MP_M, ω_M) is the final coalgebra of the functor $\Delta(1 + c \cdot Id + M)$ in* **CSMet**.

Proof. This is a direct consequence of Theorem 8.5 and Lemma 8.6, since, $1 + c \cdot Id + M$ is locally contractive and the composition of a locally contractive functor with a locally non-expansive one is locally contractive. □

Note that, when $M = 0$ is the empty metric space, the coalgebras of this functor correspond to the final c-Markov process we have used in Section 8.1 to characterise the c-discounted probabilistic bisimilarity distance. When M is not the empty space, we obtain coalgebraic structures that can be interpreted as Markov process with M-labelled terminal states; one can view the labels in M as describing different kind of termination of the process.

Hence, in the light of Theorem 8.7, we have shown that for the case of complete metric spaces $\mathcal{B} + O(\mathcal{M}_c)$ axiomatises the c-probabilistic bisimilarity distance on the final Markov process.

9 Conclusions

The main contribution of this paper was extending the notion of "sum of theories" from [9] to the quantitative setting. This, we feel opens the way to developing combinations of quantitative effects just as [9] did for combining effects in the ordinary sense. The Markov process example developed in this paper is of interest in its own right as it is the underlying operational semantics for probabilistic programming languages.

A significant novelty of this paper is a treatment of Markov processes that presents them both as algebras and as coalgebras. The algebra structure arises by combining the quantitative equational theory of probability distributions equipped with the Kantorovich metric whereas the coalgebra structure corresponds to the final coalgebra equipped with the discounted probabilistic bisimilarity distance. Such algebra-coalgebra duality has been used before [4] but much more could be done and in future work we hope to use this connection to reason about properties of probabilistic programs.

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