Positive and negative frequency decompositions in curved spacetime

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In this note we derive a formula for the positive and negative frequency parts of a solution in terms of the Feynman propagator. Our arguments are valid in the presence of particle creation. We also derive a formula for an operator $\mathscr{J}$, that gives the particle creation rate. The formalism uses complex structures to capture the notion of positive and negative frequencies and thus avoids using analyticity arguments. The results obtained clarify the relation between approaches to quantum field theory based on the complex structure and approaches in which the propagator is the basic object. We will consider only scalar fields for simplicity.

I. INTRODUCTION

In this note we derive a formula for the complex structures associated with a quantum field theory in curved space–time. The action of the complex structures as well as the action of an operator describing particle creation are given in terms of the Feynman propagator. In an approach to field theory begun by Segal[1] and by Lichnerowicz,[2] the complex structure is used to capture the notion of positive and negative frequency decompositions of solutions of the wave equation. This approach has been extended by Ashtekar and Magnon-Ashtekar[3,4] and by Kay[5] to include particle creation effects. Unfortunately, their constructions are difficult to carry out in practice. The main results of this note is to give concrete expressions to their abstractly defined operators. These formulas allow one to see the relation between the approach to field theory of Segal[1] and Ashtekar and Magnon-Ashtekar[3,4] based on complex structures and the approach of Schwinger,[6] De Witt,[7] and Rumpf[8] based on Feynman propagators.

An application of these results would be to check that a propagator does define a positive and negative frequency decomposition. It has become popular to use Euclideanization[9,10] to define a propagator. In this approach the definitions of positive and negative frequencies are not explicit and it is of interest to obtain them. This is especially true in cosmological space–times where the definition of "early time" particle states is physically obscure.[10]

II. DEFINITIONS

Let M be a globally hyperbolic Lorentzian manifold of class C^∞ with a C^∞ metric g_{ab} defined on it. The free, neutral scalar field of mass m is described by the Klein–Gordon equation

$$(\Box - m^2)\phi = 0, \quad (1)$$

where $\Box$ is the Laplace–Beltrami operator and the field $\phi(x)$ is a real function on M. The Cauchy problem for Eq. (1) and for data on a space like hypersurface Σ is solved by[2]

$$\phi(x) = \int_\Sigma \{\phi(y)\,\nabla_y^a D(x,y) - D(x,y)\nabla^a\phi(y)\}\,d\sigma_a(y), \quad (2)$$

where $d\sigma^a$ is the volume element on Σ and $D(x,y)$ is the difference between the advanced and retarded Green functions of Eq. (1). $D(x,y)$ is skewed in its arguments. Associated with solutions of Eq. (1) is a canonical, skewed two-form Ω called the symplectic structure:

$$\Omega[\phi_1(x),\phi_2(x)] = \int_\Sigma \{\phi_1\,\nabla^a\phi_2 - \phi_2\,\nabla^a\phi_1\}\,d\sigma_a. \quad (3)$$

Because ϕ_1 and ϕ_2 are solutions of Eq. (1), Ω is independent of the hypersurface Σ.

The Feynman propagator $G_F(x,x')$ is defined as a (distributional) solution to

$$(\Box - m^2)G_F(x,x') = -\delta(x,x'), \quad (4)$$

which is symmetric in x and x'. To obtain a unique solution to Eq. (4) we must of course impose boundary conditions. If we have a positive and negative frequency decomposition (e.g., static space–times), then we can impose the "causal" boundary condition; positive frequencies are propagated into the future while negative frequencies are propagated into the past. If there is no such canonical decomposition, then one is forced to use other procedures which are known to be equivalent in flat space–time.[11] For a discussion of the construction of Feynman propagators in curved space–time see Ref. 12.

The Feynman propagator may be written as the sum of real and imaginary parts

$$G_F(x,x') = -\tfrac{1}{2}\widetilde{D}(x,x') + \tfrac{1}{2}i\gamma(x,x'), \quad (5)$$

where γ is a real symmetric solution to Eq. (1). We will consider the various distributions G_F, $\widetilde{D}$, γ, and D to act on smooth test functions of compact support and we will always denote such test functions by f, g, and h.

When $D(x,x')$ acts on a test function f, it generates a solution of Eq. (1), since D is itself a solution of Eq. (1):

$$\phi_f(x) = \int D(x,x')f(x')d\tau(x'), \quad (6)$$

where $d\tau(x')$ is the covariant four volume element. Conversely, if we consider a solution $\phi(x)$ of Eq. (1) with compact spatial support, we can always find a test function $h(x)$ (not unique) such that

$$\int D(x,x')h(x')\,d\tau(x') = \phi(x). \quad (7)$$

To see this consider two spacelike hypersurfaces Σ_1 and Σ_2 with Σ_1 to the past of Σ_2. Define a new function $\hat{\phi}(x)$ by setting $\hat{\phi}(x) = \phi(x)$ to the future of Σ_2 and $\hat{\phi}(x) \equiv 0$ to the past of Σ_1. Between Σ_2 and Σ_1, $\hat{\phi}$ is constructed to be smooth and match its values on Σ_1 and Σ_2. This can be done in many ways and is the source of the nonuniqueness. Finally, set

$$h(x) = (\Box - m^2)\hat{\phi}(x). \tag{8}$$

The $h(x)$ defined by Eq. (8) will reproduce $\phi(x)$ when substituted back in Eq. (6). This can be shown as follows: Let $h(x)$ be obtained from $\phi(x)$ by the above construction. Then we have for Eq. (6)

$$\phi_h(x) = \int D(x,x')(\Box' - m^2)\hat{\phi}(x')\, d\tau(x'), \tag{9}$$

where the prime on $\Box$ means that it acts on the variable x'. We now pick hypersurfaces Σ_f in the future and Σ_p in the past and integrate Eq. (9) by parts twice over a region bounded by Σ_f and Σ_p. The volume term will vanish since $D(x,x')$ is a solution of Eq. (1) and for the boundary term we get

$$\int_{\Sigma_f} \{\hat{\phi}(x')\nabla'^a D(x,x') - D(x,x')\nabla'^a\hat{\phi}(x')\}\, d\sigma_a(x'). \tag{10}$$

The integral over Σ_p is zero since we set $\hat{\phi} = 0$ in the past. In the future, however, $\hat{\phi}(x') = \phi(x')$, so it is obvious from Eq. (2) that the solution ϕ_h is the same as the ϕ we started with. We will denote solutions by ϕ_h, ϕ_g, etc. to denote that they correspond [via Eq. (6)] to test functions.

The action of the symplectic structure on solutions ϕ_g and ϕ_h can be expressed as a volume integral over the test functions g and h. We substitute for ϕ_g and ϕ_h in Eq. (3) using Eq. (6) to get

$$\Omega(\phi_g,\phi_h) = \int_\Sigma \left\{\left[\int D(x,x')g(x')\, d\tau(x')\right] \overleftrightarrow{\nabla}'^a\left[\int D(x,x'')h(x'')d\tau(x'')\right]\right\} d\sigma_a(x), \tag{11}$$

where the double arrow on ∇ is an abbreviation

$$a\overleftrightarrow{\nabla}b = a\nabla b - b\nabla a. \tag{12}$$

In Eq. (11) we interchange the surface and volume integrals

$$\Omega(\phi_g,\phi_h) = \int\left[\int_\Sigma \{D(x,x')\overleftrightarrow{\nabla}^a D(x,x'')\}\, d\sigma_a(x)\right] \times g(x')h(x'')d\tau(x')d\tau(x''). \tag{13}$$

However, since D is itself a solution of Eq. (1), it satisfies Eq. (2) so the surface integral in Eq. (13) reduces to $D(x',x'')$ and we get

$$\Omega(\phi_g,\phi_h) = \int D(x',x'')g(x')h(x'')d\tau(x')d\tau(x''). \tag{14}$$

This process of changing surface integrals into volume integrals will be frequently carried out by using Eqs. (2) and (6) and the fact that D and γ are solutions of Eq. (1).

A complex structure J acting on a real vector space V is a linear operator on V with the property that $J^2 = -1$. Such operators always exist on infinite dimensional spaces. Let V denote the space of real solutions to Eq. (1). Then if we have a decomposition of real solutions to Eq. (1) into positive and negative frequency parts $\phi^{(+)}$ and $\phi^{(-)}$, respectively, we can define a complex structure on V by

$$J\phi = i\phi^{(+)} - i\phi^{(-)}. \tag{15}$$

Note that $J\phi \in V$ even though $\phi^{(+)}$ and $\phi^{(-)}$ do not. $\phi^{(+)}$ and $\phi^{(-)}$ are complex solutions and belong to the complexified vector space $V_c = V' \oplus iV''$, where V' and V'' are copies of V. Conversely, if we have a complex structure J on V, we can define

$$P^+\phi = \phi^{(+)} = \frac{1}{2i}(i\phi + J\phi), \tag{16a}$$

$$P^-\phi = \phi^{(-)} = \frac{1}{2i}(i\phi - J\phi), \tag{16b}$$

where P^+ and P^- are maps from V into V_c. P^+ and P^- are the positive and negative frequency projection operators, respectively. It can be seen immediately that

$$P^{+2} = P^+, \quad P^{-2} = P^-, \quad \text{and}$$
$$P^+ + P^- = \text{identity}. \tag{17}$$

In stationary space–times there is a canonical complex structure.[3] In general, there is not. However, if the space–time has asymptotic static regimes in the past and future, we can define asymptotic complex structures in the past and future J_p and J_f, respectively. The fact that $J_p \neq J_f$ is what is responsible for particle creation.[3,4] Ashtekar and Magnon-Ashtekar have used the complex structure to define particle states and construct Fock spaces.[3,4] In their approach, they require that J and Ω be compatible in the sense that, for every real solutions ϕ of Eq. (1),

$$\Omega(\phi, J\phi) \geqslant 0. \tag{18}$$

This condition is imposed to ensure that the commutation relations between the field operators and the decomposition of a field operator into creation and annihilation operators are consistent.

We can define an operator

$$\mathscr{J} = (J_f - J_p)(J_f + J_p)^{-1}. \tag{19}$$

The existence of an S matrix relating the past and future Fock spaces depends on whether $\mathscr{J}$ is Hilbert–Schmidt.[13,14] The particle creation amplitudes can be given completely by $\mathscr{J}$ and vanish if $\mathscr{J}$ does. For details see Refs. 3–5 and 13.

III. RELATION BETWEEN PROPAGATOR AND COMPLEX STRUCTURE

We will first derive an expression for the Feynman propagator in terms of J_f and J_p. We will take the Feynman propagator to be given by

$$G_F(x,x') = \frac{i\langle \text{out} \mid T\phi(x)\phi(x') \mid \text{in}\rangle}{\langle \text{out} \mid \text{in}\rangle}. \tag{20}$$

This gives the usual Feynman propagator in flat space–time since $|\text{out}\rangle$ and $|\text{in}\rangle$ are the same there. In the formalism of Ref. 4 the J_f and J_p define $|\text{out}\rangle$ and $|\text{in}\rangle$, respectively. The quantum states are regarded as holomorphic functions on V, the space of classical solutions of Eq. (1). The usual Fock representation is recovered by taking the germ of the holomorphic function at the origin. The out vacuum state is the unit function while the image under S, of the in vacuum, is the function

$$S \mid \text{in}\rangle = K \exp(\tfrac{1}{2}\langle v, \mathscr{J} v\rangle_+), \tag{21}$$

where $v \in V$ and $\langle\ ,\ \rangle_+$ is the inner product in the future one particle Hilbert space given by

$$\langle a,b\rangle_+ = \frac{1}{2h}\Omega(a,J_f b) + \frac{1}{2h} i\Omega(a,b). \qquad (22)$$

The action of creation and annihilation operators $C(\phi)$ and $A(\phi)$, respectively, associated with a solution ϕ is

$$C(\phi)f(v) = \langle\phi,v\rangle_+ f(v), \qquad (23a)$$

$$A(\phi)f(v) = \mathscr{L}_\phi f(v), \qquad (23b)$$

where $v\in V$ and $f(v)$ is a holomorphic function on V.

We will calculate Eq. (20) with G_F "smeared out" with test functions h and g, where we have chosen h and g so that

$$\operatorname{supp} h \cap \text{past of } \operatorname{supp} g = \phi \qquad (24)$$

to incorporate the time ordering. We then obtain for the smeared out Feynman propagator

$$\begin{aligned} G_F(h,g) &= \int G_F(x,x')h(x)g(x')d\tau(x)d\tau(x') \\ &= i\langle\text{out}|S|\text{in}\rangle^{-1}\langle\text{out}|[C(\phi_h)+A(\phi_h)] \\ &\quad\times[C(\phi_g)+A(\phi_g)]S|\text{in}\rangle. \end{aligned} \qquad (25)$$

The inner products are taken in the future Fock space. Using the holomorphic function representation (21) for $S|\text{in}\rangle$ and Eq. (23) for the creation and annihilation operators,

$$\begin{aligned} G_F(h,g) &= iK\langle 1,[C(\phi_h)+A(\phi_h)][C(\phi_g)+A(\phi_g)] \\ &\quad\times\exp\tfrac{1}{2}\langle v,\mathscr{J}v\rangle\rangle, \end{aligned} \qquad (26)$$

where K is a constant. We note that only terms with annihilation opertors on the left survive [since $\langle\text{out}|C(\phi)=0$] so we are left with

$$G_F(h,g) = iK\langle 1,[A(\phi_h)C(\phi_g)+A(\phi_h)A(\phi_g)]\exp\tfrac{1}{2}\langle v,v\rangle\rangle. \qquad (27)$$

The first term is calculated easily using Eq. (23):

$$\begin{aligned} &A(\phi_h)C(\phi_g)\exp\tfrac{1}{2}\langle v,\mathscr{J}v\rangle \\ &= \mathscr{L}_{\phi_h}(\langle\phi_g,v\rangle\exp\tfrac{1}{2}\langle v,\mathscr{J}v\rangle) \\ &= \langle\phi_g,\phi_h\rangle\exp\tfrac{1}{2}\langle v,\mathscr{J}v\rangle + \langle\phi_g,v\rangle\mathscr{L}_{\phi_h}\exp\tfrac{1}{2}\langle v,\mathscr{J}v\rangle. \end{aligned} \qquad (28)$$

Since we are taking the inner product with the constant term, only the first term contributes, giving a term equal to $\langle\phi_g,\phi_h\rangle$. Similarly, the second term can be shown to give rise to a term $\langle\phi_g,\mathscr{J}\phi_h\rangle$. There is a factor of $\langle\text{out}|S|\text{in}\rangle$ in both terms which cancels the factor $K=\langle\text{out}|S|\text{in}\rangle^{-1}$. Using the definition of the inner product (22), we get

$$\begin{aligned} G_F(h,g) &= -\tfrac{1}{2}\Omega[(1+\mathscr{J})\phi_g,\phi_h] \\ &\quad+\tfrac{1}{2}i\Omega[(1+\mathscr{J})\phi_g,J_f\phi_h]. \end{aligned} \qquad (29)$$

Thus, we have expressed the Feynman propagator in terms of the complex structures and $\mathscr{J}$. Our subsequent analysis will consist of explicitly displaying the action of J_f, J_p, and $\mathscr{J}$ in terms of G_F. This will elucidate the relationship between the complex structure approach[1-4] and the propagator approach to field theory.

Consider the case where there is a unique complex structure J, i.e., $\mathscr{J}=0$ and there is no particle creation. Then we obtain the following expression by setting $\mathscr{J}=0$ in Eq. (29):

$$\begin{aligned} &\int G_F(x,x')h(x)g(x')d\tau(x)d\tau(x') \\ &\quad= -\tfrac{1}{2}\Omega(\phi_g,\phi_h)+\tfrac{1}{2}i\Omega(\phi_g,J\phi_h), \end{aligned} \qquad (30)$$

where we have chosen g and h as in Eq. (24). Comparing Eqs. (30) and (5), we see that in this case

$$\int\gamma(x,x')h(x)g(x')d\tau(x)d\tau(x') = \Omega(\phi_g,J\phi_h). \qquad (31)$$

We invert Eq. (31) to obtain J as follows: Pick a spacelike hypersurface Σ. Then define the action of J on ϕ by

$$J\phi(x) = \int_\Sigma\{\phi(x')\nabla'^a\gamma(x,x')-\gamma(x,x')\nabla'^a\phi(x')\}d\sigma_a(x'). \qquad (32)$$

For J to be a complex structure we must have $J^2=-1$. Imposing this condition and using Eq. (2), we obtain

$$\begin{aligned} &\int_\Sigma\{\gamma(x,y)\nabla^a_y\gamma(x',y)-\gamma(x',y)\nabla^a_y\gamma(x,y)\}d\sigma_a(y) \\ &\quad= D(x,x'). \end{aligned} \qquad (33)$$

Thus, if γ is to define the action of J via Eq. (32), it must satisfy Eq. (33). We thus impose Eq. (33) as a condition on γ and are then assured that Eq. (32) defines a complex structure.

To obtain compatibility in the sense of Eq. (18) we demand that γ be positive definite in the sense that

$$\int\gamma(x,x')g(x)g(x')\,d\tau(x)\,d\tau(x')\geqslant 0 \qquad (34)$$

for any real test function $g(x)$. It also follows that

$$\Omega(\phi_g,J\phi_h) = \Omega(\phi_h,J\phi_g) \qquad (35)$$

from the symmetry of γ. By comparison of Eqs. (31) and (14) we see that if D and γ annihilate the same test functions (which they must since J is a linear operator), we must have the following condition:

$$\ker D\subset\ker\gamma. \qquad (36)$$

The complex structure we have defined is clearly the same one as was implicit in Eq. (30), as can be seen by computing $\Omega(\phi_g,J\phi_h)$ using Eqs. (32) and (14) and the fact that γ satisfies Eq. (2).

We can express Eq. (32) as a four volume integral. If we use Eq. (6) in (32), we find that

$$\begin{aligned} J\phi_f(x) &= \int_\Sigma\Big[\int D(x',y)f(y)d\tau(y)\Big]\nabla'^a\gamma(x,x') \\ &\quad-\gamma(x,x')\nabla'^a\Big[\int D(x',y)f(y)d\tau(y)\Big]d\sigma_a(x'), \end{aligned} \qquad (37)$$

interchanging the orders of integration

$$\begin{aligned} J\phi_f(x) &= \int\Big[\int_\Sigma\{D(x',y)\nabla'^a\gamma(x,x') \\ &\quad-\gamma(x,x')\nabla'^a D(x',y)\}d\sigma_a(x')\Big]f(y)d\tau(y). \end{aligned} \qquad (38)$$

Since γ is itself a solution of Eq. (1), it satisfies Eq. (2) so we get

$$J\phi_f(x) = \int\gamma(x,y)f(y)d\tau(y). \qquad (39)$$

The fact that this J is unique is guaranteed[13,15] if γ is Lie derived by the timelike Killing field. To see that the γ that we have produced is indeed the one which defines the vacuum in Eq. (20) we note that if we have some complex structure which defines a vacuum and we use Eq. (20) to define G_F, then 2 $\mathrm{Im}G_F$ is $\Omega(\phi_h, J\phi_g)$ by the argument following Eq. (20). If we use our definition of J and directly compute $\Omega(\phi_h, J\phi_g)$, a straightforward calculation reveals that we recover γ. Thus, the J we have defined is indeed the one implicit in γ.

We can summarize the situation as follows: A propagator G_F arises from a single complex structure J if and only if it satisfies in addition

(a) $\mathrm{Re}G_F(x,x') = \frac{1}{2}D(x,x'), \quad \text{if } x \gg x',$
$\quad = -\frac{1}{2}D(x,x') \quad \text{if } x \ll x',$

and zero if x and x' are spacelike related;

(b) $\gamma(x,x') \equiv 2\,\mathrm{Im}G_F(x,x')$ is symmetric in x and x' and satisfies Eq. (1);

(c) $\mathrm{Ker}\,\gamma(x,x') = \mathrm{Ker}D(x,x')$;

(d) Eq. (33) is satisfied by $\gamma(x,x')$.

Condition (d) is stated by Lichnerowicz[2] and the existence of a γ satisfying these conditions is discussed by Moreno.[16]

We can obtain explicitly the positive and negative frequency parts of a solution $\phi_h(x)$ by using Eq. (16). One obtains the result (for x to the future of supp h but not in supp h).

$$\phi_h^{(+)}(x) = \int G_F(x,x')h(x')d\tau(x'), \tag{40a}$$

$$\phi_h^{(-)}(x) = \int G_F^*(x,x')h(x')d\tau(x'), \tag{40b}$$

where the * denotes complex conjugation. To obtain the solutions everywhere we use Eq. (2).

We now consider the case where there are two asymptotic complex structures[4] J_p and J_f. In this case there will be particle creation by the space–time geometry since $\mathscr{J} \neq 0$. Also, the expression for the propagator given by Eq. (29) is appropriate. This is the most general situation that can be described by an S matrix connecting particle states in the distant past and the distant future. We see immediately from Eq. (29) that if there is particle creation, the real part of G_F is not simply the symplectic structure. If we define

$$\widetilde{D}(x,x') = -2\,\mathrm{Re}G_F(x,x'), \tag{41}$$

we can tell whether there is particle creation by comparing $\widetilde{D}$ and D.

If there is particle creation, we can extract the action as follows: Define $\widehat{D}$ as follows:

$$\widehat{D}(x,x') = \widetilde{D}(x,x') - D(x,x'). \tag{42}$$

where $\widehat{D}$ acts on test functions g and h we obtain

$$\widehat{D}(\phi_g,\phi_h) = \int \widehat{D}(x,x')g(x)h(x')d\tau(x)d\tau(x') = \Omega(\mathscr{J}\phi_g,\phi_h). \tag{43}$$

Let $\tilde{g}$ be a test function with the property

$$\int D(x,x')\tilde{g}(x')d\tau(x') = \mathscr{J}\phi_g. \tag{44}$$

Using Eq. (44) to rewrite the RHS of Eq. (43) in the form of Eq. (14),

$$\Omega(\mathscr{J}\phi_g,\phi_h) = \int D(x,x')\tilde{g}(x)h(x')d\tau(x)d\tau(x') = \int \widehat{D}(x,x')g(x)h(x')d\tau(x)d\tau(x'). \tag{45}$$

Since this is valid for arbitrary test functions h, we get

$$\mathscr{J}\phi_g(x) = \int \widehat{D}(x,x')g(x')d\tau(x'). \tag{46}$$

For this to be well defined we must required

$$\mathrm{Ker}D \subset \mathrm{Ker}\widehat{D}. \tag{47}$$

We now turn to the question of recovering J_f and J_p from G_F. The Feynman propagator defines the positive and negative frequencies in Eq. (40). In analogy with this we define

$$\phi_h^{(+)f}(x) = \int G_F(x,x')h(x')d\tau(x'), \tag{48a}$$

$$\phi_h^{(-)f}(x) = \int G_F^*(x,x')h(x')d\tau(x') \tag{48b}$$

to the future of but not including supp h. We obtain the solutions everywhere by choosing a spacelike hypersurface to the future of supp h and inducing the appropriate Cauchy data on it. We then use Eq. (2) to solve the Cauchy problem. Similarly, we obtain the past decomposition by defining

$$\phi_h^{(+)p}(x) = \int G_F(x,x')h(x')d\tau(x'), \tag{49a}$$

$$\phi_h^{(-)p}(x) = \int G_F^*(x,x')h(x')d\tau(x') \tag{49b}$$

in the past of but not including supp h. The Cauchy problem can again be used to obtain the solutions everywhere. We recover J_f and J_p by using Eq. (15). Thus,

$$J_f\phi_h(x) = \int \gamma(x,x')h(x')d\tau(x') \tag{50}$$

to the future of supp h, while to the past of supp h

$$J_p\phi_h(x) = \int \gamma(x,x')h(x')d\tau(x'). \tag{51}$$

These complex structures are not the same since the real part of G_F no longer governs the Cauchy evolution. Since we want $J_f\phi$ and $J_p\phi$ to be solutions of Eq. (1), we must demand that γ is a solution of Eq. (1). Then we can use Eq. (2) in Eqs. (50) and (51), and using Eq. (6) we can write the volume integrals as surface integrals to obtain

$$J_f\phi_h(x) = \int_\Sigma \{\phi_h(x')\nabla'^a\gamma(x,x') - \gamma(x,x')\nabla'^a\phi_h(x')\}d\sigma_a(x') \tag{52a}$$

for Σ being a hypersurface to the future of supp h and x to the future of Σ. Similarly, we have

$$J_p\phi_h(x) = \int_\Sigma \{\phi_h(x')\nabla'^a\gamma(x,x') - \gamma(x,x')\nabla'^a\phi_h(x')\}d\sigma_a(x') \tag{52b}$$

for Σ to the past of supp h and x to the past of Σ. We use these forms to impose $J_p^2 = J_f^2 = -1$ and obtain convolution conditions exactly like Eq. (33). This is the case even though γ now involves as well as Ω and J. The formulas (48) and (49) can also be written in terms of surface integrals. We regard $G_F(x,x')$ as a function of x' for fixed x. This is a well behaved (nonsingular) solution of Eq. (1) for all $x' \neq x$. Thus, if we restrict x' to the future of x, we may use Eq. (2) in (48) or (49). We then perform the four volume integrals so that $\int D(x,x')h(x')d\tau(x')$ becomes $\phi_h(x)$ and obtain

$$\phi_h^{(+)\mathrm{f}}(x) = \int_\Sigma \{G_F(x,x')\nabla'^a\phi_h(x') - \phi_h(x')\nabla'^a G_F(x,x')\}d\sigma_a(x'), \quad (53a)$$

$$\phi_h^{(+)\mathrm{f}}(x) = \int_\Sigma \{G_F(x,x')\nabla'^a\phi_h(x') - \phi_h(x')\nabla'^a G_F(x,x')\}d\sigma_a(x'), \quad (53b)$$

for Σ being a hypersurface to the future of x. There is an analogous formula for the past decomposition. In this form the positive and negative frequency parts can be defined without reference to the test functions. Rumpf had earlier used these formulas to define positive and negative frequency parts.[8] His arguments used the analyticity properties of the propagator regarded as a function of m^2. As in the one complex structure case we require for consistency (a) $\mathrm{Ker}D \subset \mathrm{Ker}\hat{D}$, (b) γ is a symmetric solution of Eq. (1), and (c) γ obeys Eq. (33) on hypersurfaces in the distant past or distant future.

There are three principal approaches to quantum field theory in curved space–time: That of Lichnerowicz[2,16] based on Eq. (33), that of Segal,[1] Ashtekar,[3,4] and Kay[5] using complex structures, and that of DeWitt[7] and Rumpf[8] in which the propagator is the fundamental object. In this paper we have indicated the relations between these three approaches by explicitly displaying the complex structures in terms of the propagator and obtaining Eq. (33) as a necessary condition. We have also obtained conditions which a propagator must satisfy in order to qualify as being a legitimate propagator. Finally, since the propagator can often be explicitly calculated we can explicitly determine $\mathscr{J}$ and hence the S matrix, a calculation which is very difficult in the original formulation of Ashtekar and Magnon-Ashtekar[3,4] and Kay.[5]

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