Optimal Spectral-Norm Approximate Minimization of Weighted Finite Automata

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──── **Abstract** ────

We address the approximate minimization problem for weighted finite automata (WFAs) with weights in $\mathbb{R}$, over a one-letter alphabet: to compute the best possible approximation of a WFA given a bound on the number of states. This work is grounded in Adamyan-Arov-Krein approximation theory, a remarkable collection of results on the approximation of Hankel operators. In addition to its intrinsic mathematical relevance, this theory has proven to be very effective for model reduction. We adapt these results to the framework of weighted automata over a one-letter alphabet. We provide theoretical guarantees and bounds on the quality of the approximation in the spectral and ℓ^2 norm. We develop an algorithm that, based on the properties of Hankel operators, returns the optimal approximation in the spectral norm.

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1   Introduction

Weighted finite automata (WFAs) are an expressive class of models representing functions defined over sequences. The *approximate minimization problem* is concerned with finding an automaton that approximates the behaviour of a given minimal WFA, while being smaller in size. This second automaton recognizes a different language, and the objective is to minimize the approximation error [11, 12]. Approximate minimization becomes particularly useful in the context of spectral learning algorithms [5, 7, 9, 23]. When applied to a learning task, such algorithms can be viewed as working in two steps. First, they compute a minimal WFA that explains the training data exactly. Then, they obtain a model that generalizes to the unseen data by producing a smaller approximation to the minimal WFA. It is not just a question of saving space by having a smaller state space; the exact machine will *overfit* the data and generalize poorly. To obtain accurate results it is crucial to guess correctly the size of the minimal WFA, in particular when the data is generated by this type of machine.

The minimization task is greatly shaped by the way we decide to measure the approximation error. It is thus natural to wonder if there are norms that are preferable to others. We believe that the chosen norm should be computationally reasonable to minimize. For instance, the distance between WFAs can be computed using a metric based on bisimulation [8]. While this approach could still be interesting, the fact that this metric is hard to compute makes it unsuitable for our purposes. Moreover, this metric is specifically designed for WFAs, so it is not directly applicable to other models dealing with sequential data. We think that being transferable is a second important feature for a norm. In fact, being able to compare different classes of models is desirable for future applications of this method. For example, one can think of the burgeoning literature on approximating Recurrent Neural Networks (RNNs) using WFAs, where the objective is to extract from a trained RNN an automaton that accurately mimics its behaviour [37, 41, 32, 4, 19]. With this in mind, we think that it is preferable to consider a norm defined on the input-output function – or the Hankel matrix – rather than the parameters of the specific model considered. Finally, it is important to choose a norm that can be computed accurately. The spectral norm seems to be a good candidate for the task. In particular, it allows us to exploit the work of Adamyan, Arov and Krein which has come to be known as AAK theory [1]: a series of results connecting the theory of complex functions on the unit circle to Hankel matrices, a mathematical object representing functions defined over sequences. The core of this theory provides us with theoretical guarantees for the exact computation of the spectral norm of the error, and a method to construct the optimal approximation. We summarize our main contributions:

- We use AAK theory to study the approximate minimization problem of WFAs. To connect those areas, we establish a correspondence between the parameters of a WFA and the coefficients of a complex function on the unit circle. To the best of our knowledge, this paper represents the first attempt to apply AAK theory to WFAs.
- We present a theoretical analysis of the optimal spectral-norm approximate minimization problem for WFAs, based on their connection with finite-rank infinite Hankel matrices. We provide a closed form solution for real weighted automata $A = \langle \boldsymbol{\alpha}, \mathbf{A}, \boldsymbol{\beta} \rangle$ over a one-letter alphabet, under the assumption $\rho(\mathbf{A}) < 1$ on the spectral radius. We bound the approximation error, both in terms of spectral and ℓ^2 norm.
- We propose a self-contained algorithm that returns the unique optimal spectral-norm approximation of a given size.
- We tighten the connection, made in [12], between WFAs and discrete dynamical systems, by adapting some of the control theory concepts, *e.g.* the *allpass system*.

2   Background

2.1   Preliminaries

We denote with $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ the set of natural, integers and real numbers, respectively. We use bold letters for vectors and matrices; all vectors considered are column vectors. We denote with $\mathbf{1}$ the identity matrix, specifying its dimension only when not clear from the context. We refer to the i-th row and the j-th column of $\mathbf{M}$ by $\mathbf{M}(i,:)$ and $\mathbf{M}(:,j)$. Given a matrix $\mathbf{M} \in \mathbb{R}^{p \times q}$ of rank n, a *rank factorization* is a factorization $\mathbf{M} = \mathbf{P}\mathbf{Q}$, where $\mathbf{P} \in \mathbb{R}^{p \times n}$, $\mathbf{Q} \in \mathbb{R}^{n \times q}$ and $\mathrm{rank}(\mathbf{P}) = \mathrm{rank}(\mathbf{Q}) = n$. Let $\mathbf{M} \in \mathbb{R}^{p \times q}$ of rank n, the compact *singular value decomposition* SVD of $\mathbf{M}$ is the factorization $\mathbf{M} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$, where $\mathbf{U} \in \mathbb{R}^{p \times n}$, $\mathbf{D} \in \mathbb{R}^{n \times n}$, $\mathbf{V} \in \mathbb{R}^{q \times n}$ are such that $\mathbf{U}^\top \mathbf{U} = \mathbf{V}^\top \mathbf{V} = \mathbf{1}$, and $\mathbf{D}$ is a diagonal matrix. The columns of $\mathbf{U}$ and $\mathbf{V}$ are called left and right *singular vectors*, while the entries of $\mathbf{D}$ are the *singular values*. The *Moore-Penrose pseudo-inverse* $\mathbf{M}^+$ of $\mathbf{M}$ is the unique matrix such that $\mathbf{M}\mathbf{M}^+\mathbf{M} = \mathbf{M}$, $\mathbf{M}^+\mathbf{M}\mathbf{M}^+ = \mathbf{M}^+$, with $\mathbf{M}^+\mathbf{M}$ and $\mathbf{M}\mathbf{M}^+$ Hermitian [43]. The *spectral radius* $\rho(\mathbf{M})$ of a matrix $\mathbf{M}$ is the largest modulus among its eigenvalues.

A *Hilbert space* is a complete normed vector space where the norm arises from an inner product. A linear operator $T : X \to Y$ between Hilbert spaces is *bounded* if it has finite operator norm, *i.e.* $\|T\|_{op} = \sup_{\|g\|_X \leq 1} \|Tg\|_Y < \infty$. We denote by $\mathbf{T}$ the (infinite) matrix associated with T by some (canonical) orthonormal basis on H. An operator is *compact* if the image of the unit ball in X is relatively compact. Let $T : X \to Y$ be a compact operator between Hilbert spaces, the *adjoint operator* T^* is the linear operator $T^* : Y \to X$ such that $\langle Tx, y \rangle_Y = \langle x, T^*y \rangle_X$, with $x \in X$, $y \in Y$ and $\langle \cdot, \cdot \rangle$ denotes the inner product of the corresponding Hilbert space. The *singular numbers* $\{\sigma_n\}_{n \geq 0}$ of T are the square roots of the non-negative eigenvalues of the self-adjoint operator T^*T, arranged in decreasing order. A σ-*Schmidt pair* $\{\boldsymbol{\xi}, \boldsymbol{\eta}\}$ for T is a couple of norm 1 vectors such that: $\mathbf{T}\boldsymbol{\xi} = \sigma\boldsymbol{\eta}$ and $\mathbf{T}^*\boldsymbol{\eta} = \sigma\boldsymbol{\xi}$. The Hilbert-Schmidt decomposition provides a generalization of the compact SVD for the infinite matrix of a compact operator T using singular numbers and orthonormal Schmidt pairs: $\mathbf{T}\mathbf{x} = \sum_{n \geq 0} \sigma_n \langle \mathbf{x}, \boldsymbol{\xi}_n \rangle \boldsymbol{\eta}_k$ [43]. The *spectral norm* $\|\mathbf{T}\|$ of the matrix representing the operator T is the largest singular number of T. Note that the spectral norm of $\mathbf{T}$ corresponds to the operator norm of T.

Let ℓ^2 be the Hilbert space of square-summable sequences over Σ^*, with norm $\|f\|_2^2 = \sum_{x \in \Sigma^*} |f(x)|^2$ and inner product $\langle f, g \rangle = \sum_{x \in \Sigma^*} f(x)g(x)$ for $f, g \in \mathbb{R}^{\Sigma^*}$. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the complex unit circle, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the (open) complex unit disc. Let $1 < p < \infty$, $\mathcal{L}^p(\mathbb{T})$ be the space of measurable functions on $\mathbb{T}$ for which the p-th power of the absolute value is Lebesgue integrable. For $p = \infty$, we denote with $\mathcal{L}^\infty(\mathbb{T})$ the space of measurable functions that are bounded, with norm $\|f\|_\infty = \sup\{|f(x)| : x \in \mathbb{T}\}$.

2.2   Weighted Finite Automata

Let Σ be a fixed finite alphabet, Σ^* the set of all finite strings with symbols in Σ. We use ε to denote the empty string. Given $p, s \in \Sigma$, we denote with ps their concatenation.

A *weighted finite automaton* (WFA) of n states over Σ is a tuple $A = \langle \boldsymbol{\alpha}, \{\mathbf{A}_a\}, \boldsymbol{\beta} \rangle$, where $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^n$ are the vector of initial and final weights, respectively, and $\mathbf{A}_a \in \mathbb{R}^{n \times n}$ is the matrix containing the transition weights associated with each symbol a. Every WFA A with real weights realizes a function $f_A : \Sigma^* \to \mathbb{R}$, *i.e.* given a string $x = x_1 \cdots x_t \in \Sigma^*$, it returns $f_A(x) = \boldsymbol{\alpha}^\top \mathbf{A}_{x_1} \cdots \mathbf{A}_{x_t} \boldsymbol{\beta} = \boldsymbol{\alpha}^\top \mathbf{A}_x \boldsymbol{\beta}$. A function $f : \Sigma^* \to \mathbb{R}$ is called *rational* if there exists a WFA A that realizes it. The *rank* of the function is the size of the smallest WFA realizing f. Given $f : \Sigma^* \to \mathbb{R}$, we can consider a matrix $\mathbf{H}_f \in \mathbb{R}^{\Sigma^* \times \Sigma^*}$ having rows and columns indexed by strings and defined by $\mathbf{H}_f(p, s) = f(ps)$ for $p, s \in \Sigma^*$.

▶ **Definition 1.** *A (bi-infinite) matrix* $\mathbf{H} \in \mathbb{R}^{\Sigma^* \times \Sigma^*}$ *is* **Hankel** *if for all* $p, p', s, s' \in \Sigma^*$ *such that* $ps = p's'$, *we have* $\mathbf{H}(p, s) = \mathbf{H}(p', s')$. *Given a Hankel matrix* $\mathbf{H} \in \mathbb{R}^{\Sigma^* \times \Sigma^*}$, *there exists a unique function* $f : \Sigma^* \to \mathbb{R}$ *such that* $\mathbf{H}_f = \mathbf{H}$.

▶ **Theorem 2** ([16, 20]). *A function* $f : \Sigma^* \to \mathbb{R}$ *is realized by a WFA A if and only if* $\mathbf{H}_f$ *has finite rank n. In that case, n is the minimal number of states of any A such that* $f = f_A$.

Given a WFA $A = \langle \boldsymbol{\alpha}, \{\mathbf{A}_a\}, \boldsymbol{\beta} \rangle$, the *forward matrix* of A is the infinite matrix $\mathbf{F}_A \in \mathbb{R}^{\Sigma^* \times n}$ given by $\mathbf{F}_A(p, :) = \boldsymbol{\alpha}^\top \mathbf{A}_p$ for any $p \in \Sigma^*$, while the *backward matrix* of A is $\mathbf{B}_A \in \mathbb{R}^{\Sigma^* \times n}$, given by $\mathbf{B}_A(s, :) = (\mathbf{A}_s \boldsymbol{\beta})^\top$ for any $s \in \Sigma^*$. Let $\mathbf{H}_f$ be the Hankel matrix of f, its forward-backward (FB) factorization is: $\mathbf{H}_f = \mathbf{F}\mathbf{B}^\top$. A WFA with n states is *reachable* if rank$(\mathbf{F}_A) = n$, while it is *observable* if rank$(\mathbf{B}_A) = n$. A WFA is *minimal* if it is reachable and observable. If A is minimal, the FB factorization is a rank factorization [7].

We recall the definition of the singular value automaton, a canonical form for WFAs [11].

▶ **Definition 3.** *Let* $f : \Sigma^* \to \mathbb{R}$ *be a rational function and suppose* $\mathbf{H}_f$ *admits an SVD,* $\mathbf{H}_f = \mathbf{U}\mathbf{D}\mathbf{V}^\top$. *A* **singular value automaton** *(SVA) for f is the minimal WFA A realizing f such that* $\mathbf{F}_A = \mathbf{U}\mathbf{D}^{1/2}$ *and* $\mathbf{B}_A = \mathbf{V}\mathbf{D}^{1/2}$.

The SVA can be computed with an efficient algorithm relying on the following matrices [12].

▶ **Definition 4.** *Let* $f : \Sigma^* \to \mathbb{R}$ *be a rational function,* $\mathbf{H}_f = \mathbf{F}\mathbf{B}^\top$ *a FB factorization. If the matrices* $\mathbf{P} = \mathbf{F}^\top \mathbf{F}$ *and* $\mathbf{Q} = \mathbf{B}^\top \mathbf{B}$ *are well defined, we call* $\mathbf{P}$ *the* **reachability Gramian** *and* $\mathbf{Q}$ *the* **observability Gramian**.

Note that if A is an SVA, then the Gramians associated with its FB factorization satisfy $\mathbf{P}_A = \mathbf{Q}_A = \mathbf{D}$, where $\mathbf{D}$ is the matrix of singular values of its Hankel matrix. The Gramians can alternatively be characterized (and computed [12]) using fixed point equations, corresponding to Lyapunov equations when $|\Sigma| = 1$ [28].

▶ **Theorem 5.** *Let* $|\Sigma| = 1$, $A = \langle \boldsymbol{\alpha}, \mathbf{A}, \boldsymbol{\beta} \rangle$ *a WFA with n states and well-defined Gramians* $\mathbf{P}$, $\mathbf{Q}$. *Then* $X = \mathbf{P}$ *and* $Y = \mathbf{Q}$ *solve the equations* $X - \mathbf{A}X\mathbf{A}^\top = \boldsymbol{\beta}\boldsymbol{\beta}^\top$ *and* $Y - \mathbf{A}^\top Y \mathbf{A} = \boldsymbol{\alpha}\boldsymbol{\alpha}^\top$.

Finally, we recall the definition of *generative probabilistic automata* (GPA). A WFA $A = \langle \boldsymbol{\alpha}, \{\mathbf{A}_a\}, \boldsymbol{\beta} \rangle$ is a GPA if $f_A(x) \geq 0$ for every x and $\sum_{x \in \Sigma^*} f_A(x) = 1$, *i.e.* if f_A computes a probability distribution over Σ^*.

▶ **Example 6.** Let $|\Sigma| = 1$, $\Sigma = \{x\}$. The WFA A: $\mathbf{A} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$, $\boldsymbol{\alpha} = \boldsymbol{\beta} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ 0 \end{pmatrix}$, is a GPA. Ideed, $f_A(x) \geq 0$ and $\sum_{x \in \Sigma^*} f_A(x) = 1$, since the rational function is:

$$f_A(x \cdots x) = f_A(k) = \boldsymbol{\alpha}^\top \mathbf{A}^k \boldsymbol{\beta} = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \frac{3}{4}2^{-k} & \text{if } k \text{ is even} \end{cases}$$

where k corresponds to the string where x is repeated k-times. We remark that A is minimal and in its SVA form, with Gramians $\mathbf{P} = \mathbf{Q} = \begin{pmatrix} \frac{4}{5} & 0 \\ 0 & \frac{1}{5} \end{pmatrix}$, and f_A has rank 2. Finally, the corresponding Hankel matrix, also of rank 2, is:

$$\mathbf{H} = \begin{pmatrix} f_A(0) & f_A(1) & f_A(2) & \cdots \\ f_A(1) & f_A(2) & f_A(3) & \cdots \\ f_A(2) & f_A(3) & f_A(4) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & 0 & \frac{3}{16} & \cdots \\ 0 & \frac{3}{16} & 0 & \cdots \\ \frac{3}{16} & 0 & \frac{3}{64} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1}$$

2.3 AAK Theory

Theorem 2 provides us with a way to associate a minimal WFA A with n states to a Hankel matrix $\mathbf{H}$ of rank n. The approach we propose to approximate A is to find the WFA corresponding to the matrix that minimizes $\mathbf{H}$ in the spectral norm. We recall the fundamental result of Schmidt, Eckart, Young and Mirsky [18].

▶ **Theorem 7** ([18]). *Let $\mathbf{H}$ be a Hankel matrix corresponding to a compact Hankel operator of rank n, and σ_m, with $0 \leq m < n$ and $\sigma_0 \geq \cdots \geq \sigma_{n-1} > 0$, its singular numbers. Then, if $\mathbf{R}$ is a matrix of rank k, we have: $\|\mathbf{H} - \mathbf{R}\| \geq \sigma_k$. The equality is attained when $\mathbf{R}$ corresponds to the truncated SVD of $\mathbf{H}$.*

Note that a low-rank approximation obtained by truncating the SVD is not in general a Hankel matrix. This is problematic, since $\mathbf{G}$ needs to be Hankel in order to be the matrix of a WFA. Surprisingly, we can obtain a result comparable to the one of Theorem 7 while preserving the Hankel property. This is the possible thanks to AAK theory [1], a theory of optimal approximation. To apply this theory, we will need to rewrite the approximation problem in functional analysis terms. First, we will associate a linear operator to the Hankel matrix. Then, we will use Fourier analysis to reformulate the problem in a function space. A comprehensive presentation of the concepts recalled in this section can be found in [31, 34, 29].

Let $f : \Sigma^* \to \mathbb{R}$ be a rational function, we interpret the corresponding Hankel matrix $\mathbf{H}_f$ as the expression of a linear (Hankel) operator $H_f : \ell^2 \to \ell^2$ in terms of the canonical basis. We recall that a Hankel operator H_f is bounded if and only if $f \in \ell^2$ [12]. This property, together with the fact that we only consider finite rank operators (corresponding to the Hankel matrices of WFAs), is sufficient to guarantee compactness.

To introduce AAK theory, we need to consider a second realization of Hankel operators on complex spaces. Since in this paper we work with two classes of functions – functions over sequences and complex functions – to avoid any confusion we will make explicit the dependence on the complex variable $z = e^{it}$. We start by recalling a few fundamental definitions from the theory of complex functions. Note that a function $\phi(z) \in \mathcal{L}^2(\mathbb{T})$ can be represented, using the orthonormal basis $\{z^n\}_{n \in \mathbb{Z}}$, by means of its Fourier series: $\phi(z) = \sum_{n \in \mathbb{Z}} \widehat{\phi}(n) z^n$, with Fourier coefficients $\widehat{\phi}(n) = \int_{\mathbb{T}} \phi(z) \bar{z}^n dz$, $n \in \mathbb{Z}$. This establishes an isomorphism between the function $\phi(z)$ and the sequence of the corresponding Fourier coefficients $\widehat{\phi}$. Thus, we can partition the function space $\mathcal{L}^2(\mathbb{T})$ into two subspaces.

▶ **Definition 8.** *For $0 < p \leq \infty$, the **Hardy space** $\mathcal{H}^p$ and the **negative Hardy space** $\mathcal{H}^p_-$ on $\mathbb{T}$ are the subspaces of $\mathcal{L}^p(\mathbb{T})$ defined as:*

$$\mathcal{H}^p = \{\phi(z) \in \mathcal{L}^p(\mathbb{T}) : \widehat{\phi}(n) = 0, n < 0\}, \qquad \mathcal{H}^p_- = \{\phi(z) \in \mathcal{L}^p(\mathbb{T}) : \widehat{\phi}(n) = 0, n \geq 0\}.$$

Interestingly, the elements of the Hardy space can be canonically identified with the set of functions analytic in $\mathbb{D}$, with the property that the p-th power of their absolute value is integrable on $\mathbb{T}$ (a proof can be found in [31]). Thus, we will make no difference between these functions in the unit disc and their boundary value on the circle.

We can now embed the sequence space ℓ^2 into $\ell^2(\mathbb{Z})$ by "duplicating" each vector, *i.e.* by associating $\boldsymbol{\mu} = (\mu_0, \mu_1, \dots) \in \ell^2$ to $\boldsymbol{\mu}^{(2)} = (\dots, \mu_1, \mu_0, \mu_1, \dots) \in \ell^2(\mathbb{Z})$. Then, we can use the Fourier isomorphism to map the vector $\boldsymbol{\mu}^{(2)} \in \ell^2(\mathbb{Z})$ to the function space $\mathcal{L}^2(\mathbb{T})$. In this way each vector $\boldsymbol{\mu} \in \ell^2$ corresponds to two functions in the Hardy spaces:

$$\mu^-(z) = \sum_{j=0}^{\infty} \boldsymbol{\mu}_j z^{-j-1} \in \mathcal{H}^2_-, \qquad \mu^+(z) = \sum_{j=0}^{\infty} \boldsymbol{\mu}_j z^j \in \mathcal{H}^2. \tag{2}$$

This leads to an alternative characterization of Hankel operators in Hardy spaces.

▶ **Definition 9.** *Let $\phi(z)$ be a function in the space $\mathcal{L}^2(\mathbb{T})$. A **Hankel operator** is an operator $H_\phi : \mathcal{H}^2 \to \mathcal{H}^2_-$ defined by $H_\phi f(z) = \mathbb{P}_- \phi f(z)$, where $\mathbb{P}_-$ is the orthogonal projection from $\mathcal{L}^2(\mathbb{T})$ onto $\mathcal{H}^2_-$. The function $\phi(z)$ is called a **symbol** of the Hankel operator H_ϕ.*

We briefly recall the following theorem, due to Nehari [30], characterizing bounded Hankel operators.

▶ **Theorem 10** ([30]). *Let $\phi \in \mathcal{L}^2(\mathbb{T})$ be a symbol of the Hankel operator on Hardy spaces $H_\phi : \mathcal{H}^2 \to \mathcal{H}^2_-$. Then, H_ϕ is bounded on $\mathcal{H}^2$ if and only if there exists $\psi \in \mathcal{L}^\infty(\mathbb{T})$ such that $\widehat{\psi}(m) = \widehat{\phi}(m)$ for all $m < 0$. If the conditions above are satisfied, then:*

$$\|H_\phi\| = \inf\{\|\psi\|_\infty : \widehat{\psi}(m) = \widehat{\phi}(m),\ m < 0\}. \tag{3}$$

As a consequence of Theorem 10, if H_ϕ is a bounded operator, we can consider without loss of generality $\phi(z) \in \mathcal{L}^\infty(\mathbb{T})$. We remark that a Hankel operator has infinitely many different symbols, since $H_\phi = H_{\phi+\psi}$ for $\psi(z) \in \mathcal{H}^\infty$.

▶ Remark 11. Note that, in the standard orthonormal bases $\{z^k\}_{k\geq 0}$ of $\mathcal{H}^2$ and $\{z^{-(j+1)}\}_{j\geq 0}$ of $\mathcal{H}^2_-$, H_ϕ has Hankel matrix $\mathbf{H}(j,k) = \widehat{\phi}(-j-k-1)$ for $j,k \geq 0$.

▶ **Example 12.** In the case of the Hankel matrix in Example 6, we have that $\mathbf{H}(j,k) = 0$ if $j+k$ is odd, and $\mathbf{H}(j,k) = \frac{3}{4}2^{-(j+k)}$ if $j+k$ is even. Since $\mathbf{H}(j,k) = \widehat{\phi}(-j-k-1)$, we can recover the corresponding symbol: $\mathbb{P}_- \phi = \sum_{n\geq 0} \widehat{\phi}(-n-1)z^{-n-1} = \sum_{n\geq 0} \frac{3}{4}2^{-2n}z^{-2n-1} = \frac{3z}{4z^2-1}$.

▶ **Definition 13.** *The complex function $\phi(z)$ is **rational** if $\phi(z) = p(z)/q(z)$, with $p(z)$ and $q(z)$ polynomials. The rank of $\phi(z)$ is the maximum between the degrees of $p(z)$ and $q(z)$. A rational function is **strictly proper** if the degree of $p(z)$ is strictly smaller than that of $q(z)$.*

We remark that the poles of a complex function $\phi(z)$ correspond to the zeros of $1/\phi(z)$. The following result of Kronecker relates finite-rank infinite Hankel matrices to rational functions.

▶ **Theorem 14** ([25]). *Let H_ϕ be a bounded Hankel operator with matrix $\mathbf{H}$. Then $\mathbf{H}$ has finite rank if and only if $\mathbb{P}_- \phi$ is a strictly proper rational function. Moreover the rank of $\mathbf{H}$ is equal to the number of poles (with multiplicities) of $\mathbb{P}_- \phi$ inside the unit disc.*

▶ **Example 15.** The function in Example 12 is rational with degree 2 and has two poles inside the unit disc at $z = \pm\frac{1}{2}$. Thus, the Hankel matrix associated has rank 2.

We state as remark an important takeaway from this section.

▶ Remark 16. Given a rank n Hankel matrix $\mathbf{H}$, we can look at it in two alternative ways. On the one hand we can consider the Hankel operator over sequences $H_f : \ell^2 \to \ell^2$, associated to a function $f : \Sigma^* \to \mathbb{R}$. In this case $\mathbf{H}(i,j) = f(i+j)$ for $i,j \geq 0$, and f is rational in the sense that it is realized by a WFA of size n. On the other hand, we can consider the Hankel operator over complex (Hardy) spaces $H_\phi : \mathcal{H}^2 \to \mathcal{H}^2_-$, associated to a function $\phi(z) \in \mathcal{L}^2(\mathbb{T})$, the symbol. In this case $\mathbf{H}(j,k) = \widehat{\phi}(-j-k-1)$ for $j,k \geq 0$, and $\mathbb{P}_- \phi = \widehat{\phi}(-j-k-1)$ is rational of rank n in the sense of Definition 13.

We can introduce now the main result of Adamyan, Arov and Krein [1]. The theorem, stated for Hankel operators over Hardy spaces, shows that for infinite dimensional Hankel matrices the constraint of preserving the Hankel property does not affect the achievable approximation error.

▶ **Theorem 17** (AAK-1[1])**.** *Let H_ϕ be a compact Hankel operator of rank n and singular numbers σ_m, with $0 \le m < n$ and $\sigma_0 \ge \cdots \ge \sigma_{n-1} > 0$. Then there exists a unique Hankel operator G of rank $k < n$ such that:*

$$\|\mathbf{H} - \mathbf{G}\| = \sigma_k. \tag{4}$$

We can reformulate the theorem in terms of symbols. Let $\mathcal{R}_k \subset \mathcal{H}_-^\infty$ be the set of strictly proper rational functions of rank k, and $\mathcal{H}_k^\infty = \{\psi \in \mathcal{L}^\infty(\mathbb{T}) : \ \exists g \in \mathcal{R}_k, \exists l \in \mathcal{H}^\infty, \ \psi = g + l\}$.

▶ **Theorem 18** (AAK-2 [1])**.** *Let $\phi(z) \in \mathcal{L}^\infty(\mathbb{T})$. Then, there exists $\psi(z) \in \mathcal{H}_k^\infty$ such that:*

$$\|\phi(z) - \psi(z)\|_\infty = \sigma_k(H_\phi). \tag{5}$$

The solutions of Theorem 17 and 18 are strictly related.

▶ **Corollary 19.** *Let $\psi(z) = g(z) + l(z) \in \mathcal{H}_k^\infty$, with $g(z) \in \mathcal{R}_k$, $l(z) \in \mathcal{H}^\infty$. If $\psi(z)$ solves Equation 5, then $G = H_g$ is the unique Hankel operator from Theorem 17.*

Proof of Corollary 19. Let H_ϕ be a Hankel operator with symbol $\phi(z) \in \mathcal{L}^\infty(\mathbb{T})$ and matrix $\mathbf{H}$. Let $\psi(z) = g(z) + l(z) \in \mathcal{H}_k^\infty$ be the solution of Equation 5. We have:

$$\|H_\phi - H_\psi\| = \|H_{\phi - \psi}\| \tag{6}$$
$$= \|H_{\sigma_k \eta_k^-(z)/\xi_k^+(z)}\| \tag{7}$$
$$\le \sigma_k \|\eta_k^-(z)/\xi_k^+(z)\|_\infty = \sigma_k \tag{8}$$

where first we used Corollary 20 and then we applied Theorem 10. Now, using the definition of Hankel operator, we have:

$$\|H_\phi - H_\psi\| = \|H_\phi - H_g\| = \|\mathbf{H} - \mathbf{G}\| \le \sigma_k. \tag{9}$$

Since $\|\mathbf{H} - \mathbf{G}\| \ge \sigma_k$ (from Eckart-Young theorem [18]), it follows that $\|\mathbf{H} - \mathbf{G}\| = \sigma_k$. Note that $\mathbf{G}$ has rank k, as required, because $g \in \mathcal{R}_k$ (Theorem 14). ◀

We state as corollary the key point of the proof of AAK Theorem, that provides us with a practical way to find the best approximating symbol.

▶ **Corollary 20.** *Let $\phi(z)$ and $\{\boldsymbol{\xi}_k, \boldsymbol{\eta}_k\}$ be a symbol and a σ_k-Schmidt pair for H_ϕ. A function $\psi(z) \in \mathcal{L}^\infty$ is the best AAK approximation according to Theorem 18, if and only if:*

$$(\phi(z) - \psi(z))\xi_k^+(z) = \sigma_k \eta_k^-(z). \tag{10}$$

Moreover, the function $\psi(z)$ does not depend on the particular choice of the pair $\{\boldsymbol{\xi}_k, \boldsymbol{\eta}_k\}$.

3 Approximate Minimization

3.1 Assumptions

To apply AAK theory to the approximate minimization problem, we consider only automata with real weights, defined over a one-letter alphabet. In this case, the free monoid generated by the single letter can be identified with $\mathbb{N}$, and canonically embedded into $\mathbb{Z}$. This passage is fundamental to use Fourier analysis and the isomorphism that leads to Theorem 17 and 18. Unfortunately, this idea cannot be directly generalized to bigger alphabets, since in this case we would obtain a free non-abelian monoid (not identifiable with $\mathbb{Z}$).

Theorem 17 requires the Hankel operator H to be bounded. To ensure that a minimal WFA $A = \langle \boldsymbol{\alpha}, \mathbf{A}, \boldsymbol{\beta} \rangle$ satisfies this condition, we assume $\rho(\mathbf{A}) < 1$, where ρ is the spectral radius of $\mathbf{A}$ [12]. As a matter of fact, to guarantee the boundness of the Hankel operator it is enough that the considered WFA computes a function $f \in \ell^2$ [12]. However, the stricter assumption on the spectral radius is needed when computing the symbol associated to a WFA. This condition directly implies the existence of the SVA, and of the Gramian matrices $\mathbf{P}$ and $\mathbf{Q}$, with $\mathbf{P} = \mathbf{Q}$ and diagonal [12]. We assume $A = \langle \boldsymbol{\alpha}, \mathbf{A}, \boldsymbol{\beta} \rangle$ is in SVA form. In this case, given the size of the alphabet, the Hankel matrix $\mathbf{H}$ is symmetric, so $\mathbf{A} = \mathbf{A}^\top$. Moreover, if we denote with λ_i the i-th non-zero eigenvalue of $\mathbf{H}$, and we consider the coordinates of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, we have that $\boldsymbol{\alpha}_i = \operatorname{sgn}(\lambda_i) \boldsymbol{\beta}_i$, where $\operatorname{sgn}(\lambda_i) = \lambda_i / |\lambda_i|$.

For example, we note that a minimal GPA computes a function $f \in \ell^1$, so the condition on $\rho(\mathbf{A})$ is automatically satisfied by this class of WFAs [12]. Possible relaxations of the spectral radius assumption are discussed in Section 5, together with an alternative method to find the optimal spectral-norm approximation of a Hankel matrix without extracting a WFA.

Finally, in this paper we consider only automata with weights in $\mathbb{R}$, though results remain true for complex numbers. The method we present can be easily extended to vector-valued automata [36], but the solution to the optimal approximation problem will not be unique [34].

3.2   Problem Formulation

Let $A = \langle \boldsymbol{\alpha}, \mathbf{A}, \boldsymbol{\beta} \rangle$ be a minimal WFA with n states in SVA form, defined over a one-letter alphabet. Let $\mathbf{H}$ be the Hankel matrix of A, we denote with σ_i, for $0 \leq i < n$, the singular numbers. Given a target number of states $k < n$, we say that a WFA $\widehat{A}_k$ with k states solves the *optimal spectral-norm approximate minimization* problem if the Hankel matrix $\mathbf{G}$ of $\widehat{A}_k$ satisfies $\|\mathbf{H} - \mathbf{G}\| = \sigma_k(\mathbf{H})$. Note that the content of the "optimal spectral-norm approximate minimization" is equivalent to the problem solved by Theorem 17, with the exception that here we insist on representing the inputs and outputs of the problem effectively by means of WFAs. Based on the AAK theory sketched in Section 2.3, we draw the following steps:

1. *Compute a symbol $\phi(z)$ for H using Remark 16.* We obtain the negative Fourier coefficients of $\phi(z)$ and derive its Fourier series.
2. *Compute the optimal symbol $\psi(z)$ using Corollary 20.* The main challenge here is to find a suitable representation for the functions $\psi(z)$ and $e(z) = \phi(z) - \psi(z)$. We define them in terms of two auxiliary WFAs. The key point is to select constraints on their parameters to leverage the properties of weighted automata, while still keeping the formulation general.
3. *Extracting the rational component by solving for $g(z)$ in Corollary 19.* This step is arguably the most conceptually challenging, as it requires to identify the position of the function's poles. In fact, we know from Theorem 14 that $g(z)$ has k poles, all inside the unit disc.
4. *Find a WFA representation for $g(z)$.* Since in Step 2 we parametrized the functions using WFAs, the expression of $g(z)$ directly reveals the WFA $\widehat{A}_k$.

3.3   Spectral-Norm Approximate Minimization

In the following sections we will consider a minimal WFA $A = \langle \boldsymbol{\alpha}, \mathbf{A}, \boldsymbol{\beta} \rangle$ with n states in SVA form, defined over a one-letter alphabet $\Sigma = \{a\}$, its Hankel matrix $\mathbf{H}$, corresponding to the bounded operator H, and the singular numbers σ_i, for $0 \leq i < n$. Let $f : \Sigma^* \to \mathbb{R}$ be the function realized by A. We denote by x the string where a is repeated x times, so we have $f(x) = \boldsymbol{\alpha}^\top \mathbf{A}^x \boldsymbol{\beta}$.

3.3.1 Computation of a Symbol for A

To determine the symbol $\phi(z)$ of H, we recall that each entry of the Hankel matrix corresponds simultaneously to the values of f and to the negative Fourier coefficients of $\phi(z)$. In fact, as seen in Remark 16, we have:

$$\mathbf{H} = \begin{pmatrix} f_A(0) & f_A(1) & \cdots \\ f_A(1) & f_A(2) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} \widehat{\phi}(-1) & \widehat{\phi}(-2) & \cdots \\ \widehat{\phi}(-2) & \widehat{\phi}(-3) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \tag{11}$$

We obtain:

$$\mathbb{P}_-\phi(z) = \sum_{k \geq 0} f(k)z^{-k-1} = \sum_{k \geq 0} \boldsymbol{\alpha}^\top \mathbf{A}^k \boldsymbol{\beta} z^{-k-1} = \boldsymbol{\alpha}^\top (z\mathbf{1} - \mathbf{A})^{-1}\boldsymbol{\beta} \tag{12}$$

where we use the fact that $\rho(A) < 1$ for the last equality. Since the function obtained is already bounded, we can directly consider $\phi(z) = \boldsymbol{\alpha}^\top (z\mathbf{1} - \mathbf{A})^{-1}\boldsymbol{\beta}$ as a symbol for H.

▶ **Example 21.** If we apply the formula in Equation 12 to the GPA in Example 6, we recover the rational function $\phi(z) = \frac{3z}{4z^2-1}$ found in Example 12.

3.3.2 Computation of the Optimal Symbol

We consider two auxiliary WFAs. Let $\widehat{A} = \langle \widehat{\boldsymbol{\alpha}}, \widehat{\mathbf{A}}, \widehat{\boldsymbol{\beta}} \rangle$ be a WFA with $j \geq k$ states, satisfying the following properties:

1. 1 is not an eigenvalue of $\widehat{\mathbf{A}}$
2. the automaton $E = \langle \boldsymbol{\alpha}_e, \mathbf{A}_e, \boldsymbol{\beta}_e \rangle$ is minimal, with

$$\mathbf{A}_e = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \widehat{\mathbf{A}} \end{pmatrix}, \quad \boldsymbol{\alpha}_e = \begin{pmatrix} \boldsymbol{\alpha} \\ -\widehat{\boldsymbol{\alpha}} \end{pmatrix}, \quad \boldsymbol{\beta}_e = \begin{pmatrix} \boldsymbol{\beta} \\ \widehat{\boldsymbol{\beta}} \end{pmatrix}. \tag{13}$$

Using the parameters of the automaton $\widehat{A}$ and a constant C, we define a function $\psi(z) = \widehat{\boldsymbol{\alpha}}^\top (z\mathbf{1} - \widehat{\mathbf{A}})^{-1}\widehat{\boldsymbol{\beta}} + C$. We remark that the poles of $\psi(z)$ correspond to the eigenvalues of $\widehat{\mathbf{A}}$, counted with their multiplicities. By assumption, 1 is not an eigenvalue of $\widehat{A}$, so $\psi(z)$ does not have any poles on the unit circle, and therefore $\psi(z) \in \mathcal{L}^\infty(\mathbb{T})$. Analogously, the function $e(z) = \phi(z) - \psi(z) = \boldsymbol{\alpha}_e^\top (z\mathbf{1} - \mathbf{A}_e)^{-1}\boldsymbol{\beta}_e - C$ is also bounded on the circle.

Our objective is to compute the parameters of $\widehat{A} = \langle \widehat{\boldsymbol{\alpha}}, \widehat{\mathbf{A}}, \widehat{\boldsymbol{\beta}} \rangle$ that make $\psi(z)$ the best approximation of $\phi(z)$ according to Theorem 18. In particular, we will use Corollary 20 to find the triple $\widehat{\boldsymbol{\alpha}}, \widehat{\mathbf{A}}, \widehat{\boldsymbol{\beta}}$ such that $\psi(z)$ satisfies Equation 10. Note that, with this purpose, the constant term $C \in H^\infty$ becomes necessary to characterize $\psi(z)$. In fact, while the H^∞-component of the symbol does not affect the Hankel norm, it plays a role in the computation of the $\mathcal{L}^\infty$-norm (in Equation 5) according to Theorem 10, so it cannot be dismissed.

The following theorem provides us with an explicit expression for the functions in the Hardy space corresponding to a σ_k-Schmidt pair.

▶ **Theorem 22.** *Let σ_k be a singular number of the Hankel operator H. The singular functions associated with the σ_k-Schmidt pair $\{\boldsymbol{\xi}_k, \boldsymbol{\eta}_k\}$ of H are:*

$$\xi_k^+(z) = \sigma_k^{-1/2}\boldsymbol{\alpha}^\top (\mathbf{1} - z\mathbf{A})^{-1}\mathbf{e}_k \tag{14}$$

$$\eta_k^-(z) = \sigma_k^{-1/2}\boldsymbol{\beta}^\top (z\mathbf{1} - \mathbf{A})^{-1}\mathbf{e}_k. \tag{15}$$

If $\psi(z)$ is the best approximation to the symbol, then $\sigma_k^{-1}e(z)$ has modulus 1 almost everywhere on the unit circle (i.e. it is unimodular).

Proof. Let $\mathbf{F}$ and $\mathbf{B}$ be the forward and backward matrices, respectively, with $\mathbf{H} = \mathbf{FB}^\top$, $\mathbf{P} = \mathbf{F}^\top \mathbf{F}, \mathbf{Q} = \mathbf{B}^\top \mathbf{B}$. We consider the σ_k-Schmidt pair $\{\boldsymbol{\xi}_k, \boldsymbol{\eta}_k\}$. By definition, we have $\mathbf{H}^\top \mathbf{H} \boldsymbol{\xi}_k = \sigma_k^2 \boldsymbol{\xi}_k$. Rewriting in terms of the FB factorization, we obtain:

$$\mathbf{H}^\top \mathbf{H} \boldsymbol{\xi}_k = \sigma_k^2 \boldsymbol{\xi}_k \tag{16}$$

$$\mathbf{BF}^\top \mathbf{FB}^\top \boldsymbol{\xi}_k = \sigma_k^2 \boldsymbol{\xi}_k \tag{17}$$

$$\mathbf{BPB}^\top \boldsymbol{\xi}_k = \sigma_k^2 \boldsymbol{\xi}_k \tag{18}$$

$$\mathbf{BPe}_k = \sigma_k^2 \boldsymbol{\xi}_k \tag{19}$$

where in the last step we set $\mathbf{e}_k = \mathbf{B}^\top \boldsymbol{\xi}_k$, to reduce the SVD problem of $\mathbf{H}$ to the one of $\mathbf{QP}$. Note that, since $\mathbf{P}$ and $\mathbf{Q}$ are diagonal, $\mathbf{e}_k$ is the k-th coordinate vector $(0, \ldots, 0, 1, 0, \ldots, 0)^\top$. Since $\mathbf{e}_k$ is an eigenvector of $\mathbf{QP}$ for σ_k^2, we get:

$$\mathbf{BQ}^{-1} \mathbf{QPe}_k = \sigma_k^2 \boldsymbol{\xi}_k \tag{20}$$

$$\mathbf{BQ}^{-1} \mathbf{e}_k = \boldsymbol{\xi}_k. \tag{21}$$

Moreover, $\mathbf{H}$ is symmetric, so we have that the singular vectors $\boldsymbol{\eta}_k$ and $\boldsymbol{\xi}_k$ have the same coordinates up to the sign of the corresponding eigenvalues. We obtain:

$$\xi_k^+(z) = \sum_{j=0}^\infty \sigma_k^{-1/2} \boldsymbol{\alpha}^\top \mathbf{A}^j \mathbf{e}_k z^j = \sigma_k^{-1/2} \boldsymbol{\alpha}^\top (\mathbf{1} - z\mathbf{A})^{-1} \mathbf{e}_k \tag{22}$$

$$\eta_k^-(z) = \sum_{j=0}^\infty \sigma_k^{-1/2} \boldsymbol{\beta}^\top \mathbf{A}^j \mathbf{e}_k z^{-j-1} = \sigma_k^{-1/2} \boldsymbol{\beta}^\top (z\mathbf{1} - \mathbf{A})^{-1} \mathbf{e}_k \tag{23}$$

where the singular functions have been computed following Equation 2. If r is the multiplicity of σ_k, from Corollary 20 we get the following fundamental equation:

$$(\phi(z) - \psi(z)) \boldsymbol{\alpha}^\top (\mathbf{1} - z\mathbf{A})^{-1} \mathbf{V} = \sigma_k \boldsymbol{\beta}^\top (z\mathbf{1} - \mathbf{A})^{-1} \mathbf{V}$$

where $\mathbf{V} = \begin{pmatrix} \mathbf{0} & \mathbf{1}_r \end{pmatrix}^\top$ is a $n \times r$ matrix. Consequently, we obtain the unimodular function:

$$\sigma_k^{-1} e(z) = \frac{\boldsymbol{\beta}^\top (z\mathbf{1} - \mathbf{A})^{-1} \mathbf{V}}{\boldsymbol{\alpha}^\top (\mathbf{1} - z\mathbf{A})^{-1} \mathbf{V}}$$

which is unimodular, since $\boldsymbol{\alpha}_i = \text{sgn}(\lambda_i) \boldsymbol{\beta}_i$. ◀

It is reasonable to wonder how the fact that $\sigma_k^{-1} e(z)$ is unimodular reflects on the structure of the WFA $E = \langle \boldsymbol{\alpha}_e, \mathbf{A}_e, \boldsymbol{\beta}_e \rangle$ associated with it. We remark that, *a priori*, the controllability and observability Gramians of E might not be well defined. The following theorem provides us with two matrices $\mathbf{P}_e$ and $\mathbf{Q}_e$ satisfying properties similar to those of the Gramians. This theorem is the analogous of a control theory result [17], rephrased in terms of WFAs. The proof, that relies on the minimality of the WFA E [39], can be found in [17].

▶ **Theorem 23** ([17])**.** *Consider the function $e(z) = \boldsymbol{\alpha}_e^\top (z\mathbf{1} - \mathbf{A}_e)^{-1} \boldsymbol{\beta}_e - C$ and the corresponding minimal WFA $E = \langle \boldsymbol{\alpha}_e, \mathbf{A}_e, \boldsymbol{\beta}_e \rangle$ associated with it. If $\sigma_k^{-1} e(z)$ is unimodular, then there exists a unique pair of symmetric invertible matrices $\mathbf{P}_e$ and $\mathbf{Q}_e$ satisfying:*
(a) $\mathbf{P}_e - \mathbf{A}_e \mathbf{P}_e \mathbf{A}_e^\top = \boldsymbol{\beta}_e \boldsymbol{\beta}_e^\top$
(b) $\mathbf{Q}_e - \mathbf{A}_e^\top \mathbf{Q}_e \mathbf{A}_e = \boldsymbol{\alpha}_e \boldsymbol{\alpha}_e^\top$
(c) $\mathbf{P}_e \mathbf{Q}_e = \sigma_k^2 \mathbf{1}$

We can now derive the parameters of the WFA $\widehat{A} = \langle \widehat{\boldsymbol{\alpha}}, \widehat{\mathbf{A}}, \widehat{\boldsymbol{\beta}} \rangle$.

▶ **Theorem 24.** *Let $A = \langle \boldsymbol{\alpha}, \mathbf{A}, \boldsymbol{\beta} \rangle$ be a minimal WFA with n states in its SVA form, and let $\phi(z) = \boldsymbol{\alpha}^\top (z\mathbf{1} - \mathbf{A})^{-1} \boldsymbol{\beta}$ be a symbol for its Hankel operator H. Let σ_k be a singular number of multiplicity r for H, with $\sigma_0 \geq \cdots > \sigma_k = \cdots = \sigma_{k+r-1} > \sigma_{k+r} \geq \cdots \geq \sigma_{n-1} > 0$. We can partition the Gramian matrices $\mathbf{P}$, $\mathbf{Q}$ and, conformally to them, $\mathbf{A}$, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, as follows:*

$$\mathbf{P} = \mathbf{Q} = \begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{0} \\ \mathbf{0} & \sigma_k \mathbf{1}_r \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^\top & \mathbf{A}_{22} \end{pmatrix}, \quad \boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{pmatrix}. \tag{24}$$

where $\boldsymbol{\Sigma} \in \mathbb{R}^{(n-r) \times (n-r)}$ is the diagonal matrix containing the remaining singular numbers. Let $\mathbf{R} = \sigma_k^2 \mathbf{1}_{n-r} - \boldsymbol{\Sigma}^2$, we denote by $(\cdot)^+$ the Moore-Penrose pseudo-inverse. If the function $\psi(z) = \widehat{\boldsymbol{\alpha}}^\top (z\mathbf{1} - \widehat{\mathbf{A}})^{-1} \widehat{\boldsymbol{\beta}} + C$ is the best approximation of $\phi(z)$, then:

▬ *If $\boldsymbol{\alpha}_2 \neq \mathbf{0}$:*

$$\begin{cases} \widehat{\boldsymbol{\beta}} = -\widehat{\mathbf{A}} \mathbf{A}_{12}(\boldsymbol{\beta}_2^\top)^+ \\ \widehat{\boldsymbol{\alpha}} = \widehat{\mathbf{A}}^\top \mathbf{R} \mathbf{A}_{12}(\boldsymbol{\alpha}_2^\top)^+ \\ \widehat{\mathbf{A}}(\mathbf{A}_{11} - \mathbf{A}_{12}(\boldsymbol{\beta}_2^\top)^+ \boldsymbol{\beta}_1^\top) = \mathbf{1} \end{cases} \tag{25}$$

▬ *If $\boldsymbol{\alpha}_2 = \mathbf{0}$:*

$$\begin{cases} \widehat{\boldsymbol{\beta}} = (\mathbf{1} - \widehat{\mathbf{A}} \mathbf{A}_{11})(\boldsymbol{\beta}_1^\top)^+ \\ \widehat{\boldsymbol{\alpha}} = -(\mathbf{R} - \widehat{\mathbf{A}}^\top \mathbf{R} \mathbf{A}_{11})(\boldsymbol{\alpha}_1^\top)^+ \\ \widehat{\mathbf{A}} \mathbf{A}_{12} = \mathbf{0} \end{cases} \tag{26}$$

Proof. Since $\sigma^{-1} e(z) = \phi(z) - \psi(z)$ is unimodular, from Theorem 23 there exist two symmetric nonsingular matrices $\mathbf{P}_e$, $\mathbf{Q}_e$ satisfying the fixed point equations:

$$\mathbf{P}_e - \mathbf{A}_e \mathbf{P}_e \mathbf{A}_e^\top = \boldsymbol{\beta}_e \boldsymbol{\beta}_e^\top \tag{27}$$

$$\mathbf{Q}_e - \mathbf{A}_e^\top \mathbf{Q}_e \mathbf{A}_e = \boldsymbol{\alpha}_e \boldsymbol{\alpha}_e^\top \tag{28}$$

and such that $\mathbf{P}_e \mathbf{Q}_e = \sigma_k^2 \mathbf{1}$. We can partition $\mathbf{P}_e$ and $\mathbf{Q}_e$ according to the definition of $\mathbf{A}_e$ (see Eq. 13):

$$\mathbf{P}_e = \begin{pmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{12}^\top & \mathbf{P}_{22} \end{pmatrix}, \quad \mathbf{Q}_e = \begin{pmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{12}^\top & \mathbf{Q}_{22} \end{pmatrix}.$$

From Equation 27 and 28, we note that $\mathbf{P}_{11}$ and $\mathbf{Q}_{11}$ corresponds to the controllability and observability Gramians of A:

$$\mathbf{P}_{11} = \mathbf{Q}_{11} = \mathbf{P} = \begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{0} \\ \mathbf{0} & \sigma_k \mathbf{1} \end{pmatrix}.$$

Moreover, since $\mathbf{P}_e \mathbf{Q}_e = \sigma_k^2 \mathbf{1}$, we get $\mathbf{P}_{12} \mathbf{Q}_{12}^\top = \sigma_k^2 \mathbf{1} - \mathbf{P}^2$. It follows that $\mathbf{P}_{12} \mathbf{Q}_{12}^\top$ has rank $n - r$. Without loss of generality we can set $\dim \widehat{\mathbf{A}} = j = n - r$, and choose an appropriate basis for the state space such that $\mathbf{P}_{12} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \end{pmatrix}^\top$ and $\mathbf{Q}_{12} = \begin{pmatrix} \mathbf{R} & \mathbf{0} \end{pmatrix}^\top$, with $\mathbf{R} = \sigma_k^2 \mathbf{1} - \boldsymbol{\Sigma}^2$. Once $\mathbf{P}_{12}$ and $\mathbf{Q}_{12}$ are fixed, the values of $\mathbf{P}_{22}$ and $\mathbf{Q}_{22}$ are automatically determined. We obtain:

$$\mathbf{P}_e = \begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \sigma_k \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & -\boldsymbol{\Sigma} \mathbf{R}^{-1} \end{pmatrix}, \quad \mathbf{Q}_e = \begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{0} & \mathbf{R} \\ \mathbf{0} & \sigma_k \mathbf{1} & \mathbf{0} \\ \mathbf{R} & \mathbf{0} & -\boldsymbol{\Sigma} \mathbf{R} \end{pmatrix}. \tag{29}$$

Now that we have an expression for the matrices $\mathbf{P}_e$ and $\mathbf{Q}_e$ of Theorem 23, we can rewrite the fixed point equations to derive the parameters $\widehat{\boldsymbol{\alpha}}$, $\widehat{\mathbf{A}}$ and $\widehat{\boldsymbol{\beta}}$. We obtain the following systems:

$$
\begin{cases}
\mathbf{P} - \mathbf{A}\mathbf{P}\mathbf{A} = \boldsymbol{\beta}\boldsymbol{\beta}^\top \\
\mathbf{N} - \mathbf{A}\mathbf{N}\widehat{\mathbf{A}}^\top = \boldsymbol{\beta}\widehat{\boldsymbol{\beta}}^\top \\
-\boldsymbol{\Sigma}\mathbf{R}^{-1} + \widehat{\mathbf{A}}\boldsymbol{\Sigma}\mathbf{R}^{-1}\widehat{\mathbf{A}}^\top = \widehat{\boldsymbol{\beta}}\widehat{\boldsymbol{\beta}}^\top
\end{cases}
\qquad
\begin{cases}
\mathbf{P} - \mathbf{A}\mathbf{P}\mathbf{A} = \boldsymbol{\alpha}\boldsymbol{\alpha}^\top \\
\mathbf{M} - \mathbf{A}^\top\mathbf{M}\widehat{\mathbf{A}} = -\boldsymbol{\alpha}\widehat{\boldsymbol{\alpha}}^\top \\
-\boldsymbol{\Sigma}\mathbf{R} + \widehat{\mathbf{A}}^\top\boldsymbol{\Sigma}\mathbf{R}\widehat{\mathbf{A}} = \widehat{\boldsymbol{\alpha}}\widehat{\boldsymbol{\alpha}}^\top
\end{cases}
\tag{30}
$$

where $\mathbf{N} = \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix}$ and $\mathbf{M} = \begin{pmatrix} \mathbf{R} \\ \mathbf{0} \end{pmatrix}$. We can rewrite the second equation of each system as follows:

$$
\begin{cases}
\mathbf{1} - \mathbf{A}_{11}\widehat{\mathbf{A}}^\top = \boldsymbol{\beta}_1\widehat{\boldsymbol{\beta}}^\top \\
-\mathbf{A}_{12}^\top\widehat{\mathbf{A}}^\top = \boldsymbol{\beta}_2\widehat{\boldsymbol{\beta}}^\top
\end{cases}
\qquad
\begin{cases}
\mathbf{R} - \mathbf{A}_{11}\mathbf{R}\widehat{\mathbf{A}} = -\boldsymbol{\alpha}_1\widehat{\boldsymbol{\alpha}}^\top \\
\widehat{\mathbf{A}}^\top\mathbf{R}\mathbf{A}_{12} = \widehat{\boldsymbol{\alpha}}\boldsymbol{\alpha}_2^\top
\end{cases}
\tag{31}
$$

If $\boldsymbol{\alpha}_2 \neq \mathbf{0}$, then also $\boldsymbol{\beta}_2 \neq \mathbf{0}$ (recall that $\boldsymbol{\alpha}_i = \operatorname{sgn}(\lambda_i)\boldsymbol{\beta}_i$), and we have:

$$
\begin{cases}
\widehat{\boldsymbol{\beta}} = -\widehat{\mathbf{A}}\mathbf{A}_{12}(\boldsymbol{\beta}_2^\top)^+ \\
\widehat{\boldsymbol{\alpha}} = \widehat{\mathbf{A}}^\top\mathbf{R}\mathbf{A}_{12}(\boldsymbol{\alpha}_2^\top)^+ \\
\widehat{\mathbf{A}}(\mathbf{A}_{11} - \mathbf{A}_{12}(\boldsymbol{\beta}_2^\top)^+\boldsymbol{\beta}_1^\top) = \mathbf{1}
\end{cases}
\tag{32}
$$

with $(\boldsymbol{\alpha}_2^\top)^+ = \frac{\boldsymbol{\alpha}_2}{\boldsymbol{\alpha}_2^\top\boldsymbol{\alpha}_2}$ and $(\boldsymbol{\beta}_2^\top)^+ = \frac{\boldsymbol{\beta}_2}{\boldsymbol{\beta}_2^\top\boldsymbol{\beta}_2}$.

If $\boldsymbol{\alpha}_2 = \mathbf{0}$, we have $\widehat{\mathbf{A}}\mathbf{A}_{12} = \mathbf{0}$. We remark that $\widehat{\mathbf{A}}$ has size $(n-r) \times (n-r)$, while $\mathbf{A}_{12}$ is $(n-r) \times r$, so the system of equations corresponding to $\widehat{\mathbf{A}}\mathbf{A}_{12} = \mathbf{0}$ is underdetermined if $r < \frac{n}{2}$, in which case we can find an alternative set of solutions:

$$
\begin{cases}
\widehat{\boldsymbol{\beta}} = (\mathbf{1} - \widehat{\mathbf{A}}\mathbf{A}_{11})(\boldsymbol{\beta}_1^\top)^+ \\
\widehat{\boldsymbol{\alpha}} = -(\mathbf{R} - \widehat{\mathbf{A}}^\top\mathbf{R}\mathbf{A}_{11})(\boldsymbol{\alpha}_1^\top)^+ \\
\widehat{\mathbf{A}}\mathbf{A}_{12} = \mathbf{0}
\end{cases}
\tag{33}
$$

with $\widehat{\mathbf{A}} \neq \mathbf{0}$. On the other hand, if $r \geq \frac{n}{2}$, i.e. if the multiplicity of the singular number σ_k is more than half the size of the original WFA, the system might not have any solution unless $\widehat{\mathbf{A}} = \mathbf{0}$ (or unless $\mathbf{A}_{12}$ was zero to begin with). In this setting, the method proposed returns $\widehat{\mathbf{A}} = \mathbf{0}$. An alternative in this case is to search for an approximation of size $k-1$ or $k+1$, so that $r < \frac{n}{2}$, and the system in Equation 33 is underdetermined. ◀

3.3.3   Extraction of the Rational Component

The function $\psi(z) = \widehat{\boldsymbol{\alpha}}^\top(z\mathbf{1} - \widehat{\mathbf{A}})^{-1}\widehat{\boldsymbol{\beta}} + C$ found in Theorem 24 corresponds to the solution of Theorem 18. To find the solution to the approximation problem we only need to "isolate" the function $g(z) \in \mathcal{R}_k$, i.e. the *rational component*. To do this, we study the position of the poles of $\psi(z)$, since the poles of a strictly proper rational function lie in the unit disc (Theorem 14). As noted before, we parametrized $\psi(z)$ so that its poles correspond to the eigenvalues of $\widehat{A}$. After a change of basis (detailed in Paragraph 3.3.3), we can rewrite $\widehat{\mathbf{A}}$ in block-diagonal form:

$$
\widehat{\mathbf{A}} = \begin{pmatrix} \widehat{\mathbf{A}}_+ & \mathbf{0} \\ \mathbf{0} & \widehat{\mathbf{A}}_- \end{pmatrix}
\tag{34}
$$

where the modulus of the eigenvalues of $\widehat{\mathbf{A}}_+$ (resp. $\widehat{\mathbf{A}}_-$) is smaller (resp. greater) than one. We apply the same change of coordinates on $\widehat{\boldsymbol{\alpha}}$ and $\widehat{\boldsymbol{\beta}}$.

To conclude the study of the eigenvalues of $\widehat{\mathbf{A}}$, we need the following auxiliary result from Ostrowski [33]. A proof of this theorem can be found in [42].

▶ **Theorem 25** ([33]). *Let $|\Sigma| = 1$, and let $\mathbf{P}$ be a solution to the fixed point equation $X - \mathbf{A}X\mathbf{A}^\top = \boldsymbol{\beta}\boldsymbol{\beta}^\top$ for the WFA $A = \langle \boldsymbol{\alpha}, \mathbf{A}, \boldsymbol{\beta} \rangle$. If A is reachable, then:*

- *The number of eigenvalues λ of $\mathbf{A}$ such that $|\lambda| < 1$ is equal to the number of positive eigenvalues of $\mathbf{P}$.*
- *The number of eigenvalues λ of $\mathbf{A}$ such that $|\lambda| > 1$ is equal to the number of negative eigenvalues of $\mathbf{P}$.*

We can finally find the rational component of the function $\psi(z)$, *i.e.* the function $g(z)$ from Corollary 19 necessary to solve that approximate minimization problem.

▶ **Theorem 26.** *Let $\widehat{\mathbf{A}}_+, \widehat{\boldsymbol{\alpha}}_+, \widehat{\boldsymbol{\beta}}_+$ be as in Theorem 24. The rational component of $\psi(z)$ is the function $g(z) = \widehat{\boldsymbol{\alpha}}_+^\top (z\mathbf{1} - \widehat{\mathbf{A}}_+)^{-1}\widehat{\boldsymbol{\beta}}_+$.*

Proof. Clearly $\psi(z) = g(z) + l(z)$, with $l(z) = \widehat{\boldsymbol{\alpha}}_-^\top (z\mathbf{1} - \widehat{\mathbf{A}}_-)^{-1}\widehat{\boldsymbol{\beta}}_-$, $l \in \mathcal{H}^\infty$. To conclude the proof we need to show that $g(z)$ has k poles inside the unit disc, and therefore has rank k. We do this by studying the position of the eigenvalues of $\widehat{\mathbf{A}}_+$.

Since $\widehat{A}$ is reachable (follows from the minimality of E), we can use Theorem 25 and solve the problem by directly examining the eigenvalues of $-\boldsymbol{\Sigma}\mathbf{R}$. From the proof of Theorem 24 we have $-\boldsymbol{\Sigma}\mathbf{R} = \boldsymbol{\Sigma}(\boldsymbol{\Sigma}^2 - \sigma_k^2\mathbf{1})$, where $\boldsymbol{\Sigma}$ is the diagonal matrix having as elements the singular numbers of H different from σ_k. It follows that $-\boldsymbol{\Sigma}\mathbf{R}$ has only k strictly positive eigenvalues, and $\widehat{\mathbf{A}}$ has k eigenvalues with modulus smaller than 1. Thus, $\widehat{\mathbf{A}}_+$ has k eigenvalues, corresponding to the poles of $g(z)$. ◀

Block Diagonalization

In this paragraph we detail the technical steps necessary to rewrite $\widehat{\mathbf{A}}$ in block-diagonal form. The problem of computing the Jordan form of a matrix is ill-conditioned, hence it is not suitable for our algorithmic purposes [40]. Instead, we compute the Schur decomposition, *i.e.* we find an orthogonal matrix $\mathbf{U}$ such that $\mathbf{U}^\top \widehat{\mathbf{A}} \mathbf{U}$ is upper triangular, with the eigenvalues of $\widehat{\mathbf{A}}$ on the diagonal. We obtain:

$$\mathbf{T} = \mathbf{U}^\top \widehat{\mathbf{A}} \mathbf{U} = \begin{pmatrix} \widehat{\mathbf{A}}_+ & \widehat{\mathbf{A}}_{12} \\ \mathbf{0} & \widehat{\mathbf{A}}_- \end{pmatrix} \tag{35}$$

where the eigenvalues are arranged in increasing order of modulus, and the modulus of those in $\widehat{\mathbf{A}}_+$ (resp. $\widehat{\mathbf{A}}_-$) is smaller (resp. greater) than one. To transform this upper triangular matrix into a block-diagonal one, we use the following result.

▶ **Theorem 27** ([38]). *The matrix $\mathbf{X}$ is a solution of the equation $\widehat{\mathbf{A}}_+\mathbf{X} - \mathbf{X}\widehat{\mathbf{A}}_- + \widehat{\mathbf{A}}_{12} = \mathbf{0}$ if and only if $\mathbf{M} = \begin{pmatrix} \mathbf{1} & \mathbf{X} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$ and $\mathbf{M}^{-1} = \begin{pmatrix} \mathbf{1} & -\mathbf{X} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$ satisfy $\mathbf{M}^{-1}\mathbf{T}\mathbf{M} = \begin{pmatrix} \widehat{\mathbf{A}}_+ & \mathbf{0} \\ \mathbf{0} & \widehat{\mathbf{A}}_- \end{pmatrix}$, where $\mathbf{T}$ is the matrix defined in Equation 35.*

Setting $\boldsymbol{\Gamma} = \begin{pmatrix} \mathbf{1}_k & \mathbf{0} \end{pmatrix}$ we can now derive the rational component of the WFA:

$$\widehat{\mathbf{A}}_+ = \boldsymbol{\Gamma}\mathbf{M}^{-1}\mathbf{U}^\top \widehat{\mathbf{A}} \mathbf{U}\mathbf{M}\boldsymbol{\Gamma}^\top, \quad \widehat{\boldsymbol{\alpha}}_+ = \boldsymbol{\Gamma}\mathbf{M}^\top \mathbf{U}^\top \widehat{\boldsymbol{\alpha}}, \quad \widehat{\boldsymbol{\beta}}_+ = \boldsymbol{\Gamma}\mathbf{M}^{-1}\mathbf{U}^\top \widehat{\boldsymbol{\beta}}. \tag{36}$$

3.3.4   Solution to the Approximation Problem

In the previous sections, we have derived the rational function $g(z)$ corresponding to the symbol of G, the operator that solves Theorem 17. To find the solution to the approximation problem we only need to find the parameters of $\widehat{A}_k$, the optimal approximating WFA. These are directly revealed by the expression of $g(z)$, thanks to the way we parametrized the functions.

▶ **Theorem 28.** *Let $A = \langle \boldsymbol{\alpha}, \mathbf{A}, \boldsymbol{\beta} \rangle$ be a minimal WFA with n states over a one-letter alphabet. Let A be in its SVA form. The optimal spectral-norm approximation of rank k is given by the WFA $\widehat{A}_k = \langle \widehat{\boldsymbol{\alpha}}_+, \widehat{\mathbf{A}}_+, \widehat{\boldsymbol{\beta}}_+ \rangle$.*

Proof. From Corollary 19 we know that $g(z)$ is the rational function associated with the Hankel matrix of the best approximation. Given the correspondence between the Fourier coefficients of $g(z)$ and the entries of the matrix (Remark 16), we have:

$$g(z) = \widehat{\boldsymbol{\alpha}}_+^\top (z\mathbf{1} - \widehat{\mathbf{A}}_+)^{-1} \widehat{\boldsymbol{\beta}}_+ = \sum_{k \geq 0} \widehat{\boldsymbol{\alpha}}_+^\top \widehat{\mathbf{A}}_+^k \widehat{\boldsymbol{\beta}}_+ z^{-k-1} = \sum_{k \geq 0} \bar{f}(k) z^{-k-1} \tag{37}$$

where $\bar{f} : \Sigma^* \to \mathbb{R}$ is the function computed by $\widehat{A}_k$, and $\widehat{\boldsymbol{\alpha}}_+, \widehat{\mathbf{A}}_+, \widehat{\boldsymbol{\beta}}_+$ are the parameters.   ◀

3.4   Error Analysis

The theoretical foundations of AAK theory guarantee that the construction detailed in Section 3.3 produces the rank k optimal spectral-norm approximation of a WFA satisfying our assumptions, and the singular number σ_k provides the exact error.

Similarly to the case of SVA truncation [12], due to the ordering of the singular numbers, the error decreases when k increases, meaning that allowing $\widehat{A}_k$ to have more states guarantees a better approximation of A. We remark that, while the solution we propose is optimal in the spectral norm, the same is not necessarily true in other norms. Nonetheless, we have the following bound between ℓ^2-norm and spectral-norm.

▶ **Theorem 29.** *Let A be a minimal WFA computing $f : \Sigma^* \to \mathbb{R}$, with matrix $\mathbf{H}$. Let $\widehat{A}_k$ be its optimal spectral-norm approximation, computing $g : \Sigma^* \to \mathbb{R}$, with matrix $\mathbf{G}$. Then:*

$$\|f - g\|_{\ell^2} \leq \|\mathbf{H} - \mathbf{G}\| = \sigma_k. \tag{38}$$

Proof. Let $\mathbf{e}_0 = \begin{pmatrix} 1 & 0 & \cdots \end{pmatrix}^\top$, $f : \Sigma^* \to \mathbb{R}$, $g : \Sigma^* \to \mathbb{R}$ with Hankel matrices $\mathbf{H}$ and $\mathbf{G}$, respectively. We have:

$$\|f - g\|_{\ell^2} = \left(\sum_{n=0}^{\infty} |f_n - g_n|^2 \right)^{1/2} = \|(\mathbf{H} - \mathbf{G})\mathbf{e}_0\|_{\ell^2} \leq \sup_{\|\mathbf{x}\|_{\ell^2}=1} \|(\mathbf{H} - \mathbf{G})\mathbf{x}\|_{\ell^2} = \|\mathbf{H} - \mathbf{G}\| = \sigma_k$$

where the second equation follows by definition and by observing that matrix difference is computed entry-wise.   ◀

4   Algorithm

In this section we describe the algorithm for spectral-norm approximate minimization. The algorithm takes as input a target number of states $k < n$, a minimal WFA A with $\rho(\mathbf{A}) < 1$, $\boldsymbol{\alpha}_2 \neq 0$, n states and in SVA form, and its Gramian $\mathbf{P}$. Note that, in the case of $\boldsymbol{\alpha}_2 = 0$, it is

◼ **Algorithm 1** `AAKapproximation`.

---

input : A minimal WFA A, with $\boldsymbol{\alpha}_2 \neq 0$, n states and in SVA form,
          its Gramian $\mathbf{P}$, a target number of states $k < n$
output : A WFA $\widehat{A}_k$ with k states

1 Let $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \mathbf{A}_{11}, \mathbf{A}_{12}, \mathbf{A}_{22}, \boldsymbol{\Sigma}$ be the blocks defined in Eq. 24

2 Let $(\boldsymbol{\alpha}_2^\top)^+ = \frac{\boldsymbol{\alpha}_2}{\boldsymbol{\alpha}_2^\top \boldsymbol{\alpha}_2}$, $(\boldsymbol{\beta}_2^\top)^+ = \frac{\boldsymbol{\beta}_2}{\boldsymbol{\beta}_2^\top \boldsymbol{\beta}_2}$

3 Let $\mathbf{R} = \sigma_k^2 \mathbf{1} - \boldsymbol{\Sigma}^2$

4 Let $\widehat{\mathbf{A}} = (\mathbf{A}_{11} - \mathbf{A}_{12}(\boldsymbol{\beta}_2^\top)^+ \boldsymbol{\beta}_1^\top)^{-1}$

5 Let $\widehat{\boldsymbol{\alpha}} = \widehat{\mathbf{A}}^\top \mathbf{R} \mathbf{A}_{12}(\boldsymbol{\alpha}_2^\top)^+$

6 Let $\widehat{\boldsymbol{\beta}} = -\widehat{\mathbf{A}} \mathbf{A}_{12}(\boldsymbol{\beta}_2^\top)^+$

7 Let $\widehat{A} = \langle \widehat{\boldsymbol{\alpha}}, \widehat{\mathbf{A}}, \widehat{\boldsymbol{\beta}} \rangle$

8 Let $\widehat{A}_k \leftarrow \texttt{BlockDiagonalize}(\widehat{A})$

9 **return** $\widehat{A}_k$

◼ **Algorithm 2** `BlockDiagonalize`.

---

input : A WFA $\widehat{A}$
output : A WFA $\widehat{A}_k$ wit $\rho < 1$

1 Compute the Schur decomposition of $\widehat{\mathbf{A}} = \mathbf{U}\mathbf{T}\mathbf{U}^\top$, where $|T_{11}| \leq |T_{22}| \leq \dots$

2 Solve $\widehat{\mathbf{A}}_{11}\mathbf{X} - \mathbf{X}\widehat{\mathbf{A}}_{22} + \widehat{\mathbf{A}}_{12} = \mathbf{0}$ for $\mathbf{X}$

3 Let $\mathbf{M} = \begin{pmatrix} \mathbf{1} & \mathbf{X} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$ and $\mathbf{M}^{-1} = \begin{pmatrix} \mathbf{1} & -\mathbf{X} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$

4 Let $\boldsymbol{\Gamma} = \begin{pmatrix} \mathbf{1}_k & \mathbf{0} \end{pmatrix}$

5 Let $\widehat{\mathbf{A}}_+ = \boldsymbol{\Gamma} \mathbf{M}^{-1} \mathbf{U}^\top \widehat{\mathbf{A}} \mathbf{U} \mathbf{M} \boldsymbol{\Gamma}^\top$

6 Let $\widehat{\boldsymbol{\alpha}}_+ = \boldsymbol{\Gamma} \mathbf{M}^\top \mathbf{U}^\top \widehat{\boldsymbol{\alpha}}$

7 Let $\widehat{\boldsymbol{\beta}}_+ = \boldsymbol{\Gamma} \mathbf{M}^{-1} \mathbf{U}^\top \widehat{\boldsymbol{\beta}}$

8 Let $\widehat{A}_k = \langle \widehat{\boldsymbol{\alpha}}_+, \widehat{\mathbf{A}}_+, \widehat{\boldsymbol{\beta}}_+ \rangle$

9 **return** $\widehat{A}_k$

---

enough to substitute the Steps $4, 5, 6$ with the analogous from Equation 26. The constraints on the WFA A to be minimal and in SVA form are non essential. In fact a WFA with n states can be minimized in time $O(n^3)$ [15], and the SVA computed in $O(n^3)$ [12].

Using the results of Theorem 24, we outline in Algorithm 1, `AAKapproximation`, the steps necessary to extract the best spectral-norm approximation of a WFA.

The algorithm involves a call to Algorithm 2, `BlockDiagonalize`. This corresponds to the steps, outlined in Paragraph 3.3.3, necessary to derive the WFA $\widehat{A}_k$ corresponding to the rational function $g(z)$. We remark that Step 2 in `BlockDiagonalize` can be performed using the Bartels-Stewart algorithm [14].

To compute the computational cost we recall the following facts [40]:

- The product of two $n \times n$ matrices can be computed in time $O(n^3)$ using a standard iterative algorithm, but can be reduced to $O(n^\omega)$ with $\omega < 2.4$.
- The inversion of a $n \times n$ matrix can be computed in time $O(n^3)$ using Gauss-Jordan elimination, but can be reduced to $O(n^\omega)$ with $\omega < 2.4$.
- The computation of the Schur decomposition of a $n \times n$ matrix can be done with a two-step algorithm, where each step takes $O(n^3)$, using the Hessenberg form of the matrix.
- The Bartels-Stewart algorithm applied to upper triangular matrices to find a matrix $m \times n$ takes $O(mn^2 + nm^2)$.

The running time of `BlockDiagonalize` with input a WFA $\widehat{A}$ with $(n - r)$ states is thus in $O((n - r)^3)$, where r is the multiplicity of the singular value considered. The running time of `AAKapproximation` for an input WFA $\widehat{A}$ with n states is in $O((n - r)^3)$.

5 Possible Extensions

5.1 Relaxing the Spectral Radius Assumption

It is possible to extend part of our method to WFAs over a one-letter alphabet with $\rho(\mathbf{A}) \neq 1$, but the approximation recovered is not optimal in the spectral norm.

Let $A = \langle \boldsymbol{\alpha}, \mathbf{A}, \boldsymbol{\beta} \rangle$, with $\rho(\mathbf{A}) \neq 1$, be a WFA with n states that we want to minimize. The idea is to block-diagonalize $\mathbf{A}$ like we did in Section 3.3.3, and tackle each component separately. The case of $A_+ = \langle \boldsymbol{\alpha}_+, \mathbf{A}_+, \boldsymbol{\beta}_+ \rangle$, the component having $\rho(\mathbf{A}) < 1$, can be dealt with in the way presented in the previous sections. This means that we can find an optimal spectral-norm approximation of the desired size for A_+. Now we can consider the second component, $A_- = \langle \boldsymbol{\alpha}_-, \mathbf{A}_-, \boldsymbol{\beta}_- \rangle$. The key idea is to apply the transformation $z^{j-1} \mapsto z^{-j}$ for $j \geq 1$ to the function $\phi''(z)$ associated to A_-. Then, the function

$$\phi''(z^{-1}) = \sum_{k \geq 0} \boldsymbol{\alpha}_-^\top \mathbf{A}_-^k z^k \boldsymbol{\beta}_- = \boldsymbol{\alpha}_-^\top (\mathbf{1} - z\mathbf{A}_-)^{-1} \boldsymbol{\beta}_- \tag{39}$$

is well defined, as the series converges for z with small enough modulus. Using this transformation we obtain a function with poles inside the unit disc and we can apply the method presented in the paper. An important choice to make is the size of the approximation of A_-, as it can influence the quality of the approximation. Analyzing the effects of this parameter on the approximation error constitutes an interesting direction for future work, both in the theoretical and experimental side. We refer the reader to the control theory literature [21], where some theoretical work has been done to study an analogous approach for continuous time systems and their approximation error.

5.2 Polynomial method

We remark that Equation 10 from Corollary 20 can be rewritten as

$$\psi(z) = \phi(z) - \frac{H\xi_k^+(z)}{\xi_k^+(z)}, \tag{40}$$

where $\xi_k^+(z)$ is the function in $\mathcal{H}^2$ associated to the vector $\boldsymbol{\xi}_k \in \mathrm{Ker}(\mathbf{H}^*\mathbf{H} - \sigma_k^2 \mathbf{1})$ (and $\psi(z)$ does not depend on the choice of the specific $\boldsymbol{\xi}_k$). There is an alternative way to find the best approximation, particularly useful when the objective is to approximate a finite-rank infinite Hankel matrix with another Hankel matrix, without necessarily extract a WFA. We can consider the adjoint operator H^* and its matrix $\mathbf{H}^*$. The singular numbers and singular vectors of H correspond to the eigenvalues and eigenvectors of $\mathbf{R} = (\mathbf{H}^*\mathbf{H})^{1/2}$. Hence, it is possible to compute σ_k and a corresponding singular vector $\boldsymbol{\xi}_k$. The function $\xi_k^+(z)$ is then obtained following Equation 2. Then, the Hankel matrix $\mathbf{G}$ that best approximates $\mathbf{H}$ is given by $\mathbf{G} = \mathbf{H} - \mathbf{M}$, where $\mathbf{M}$ is the Hankel matrix having $\frac{H\xi_k^+(z)}{\xi_k^+(z)}$ as symbol.

6 Related Work

The study of approximate minimization for WFAs is very recent, and only a few works have been published on the subject. In [11, 12] the authors present an approximate minimization technique using a canonical expression for WFAs, and provide bounds on the error in the

ℓ^2 norm. The result is supported by strong theoretical guarantees, but it is not optimal in any norm. An extension of this method to the case of Weighted Tree Automata can be found in [13]. A similar problem is addressed in [26], with less general results. In [27], the authors connect spectral learning to the approximate minimization problem of a small class of Hidden Markov models, bounding the error in terms of the total variation distance.

The control theory community has largely studied approximate minimization in the context of linear time-invariant systems [3]. A parallel with those results can be drown by noting that the impulse response of a discrete Single-Input-Single-Output SISO system can be parametrized as a WFA over a one-letter alphabet. In [21] Glover presents a state-space solution for the case of continuous Multi-Input-Multi-Output MIMO systems. His method led to a widespread application of these results, thanks to its computational and theoretical simplicity. This stems from the structure of the continuous Lyapunov equations. For discrete systems, though, the quadratic nature of the Lyapunov equations does not allow for a simple closed form formula for the state space solution [17]. Thus, most of the results for the discrete case work with a suboptimal version of the problem [6, 2, 24]. A solution for the SISO case can be found using a polynomial approach, but it does not provide an explicit representation of the state space nor it generalizes to the MIMO setting. The first to actually extend Glover results is Gu, who provides an elegant solution for the MIMO discrete problem [22]. Glover and Gu's solutions rely on building an *all-pass system*, equivalent to the WFA E in our case. Part of our contribution is the adaptation of some of the control theory tools to WFAs.

7 Conclusion

In this paper we applied the AAK theory for Hankel operators and complex functions to the framework of WFAs in order to construct the best possible approximation to an automaton given a bound on the size. We provide an algorithm to find the parameters of the best WFA approximation in the spectral norm, and bounds on the error. Our method applies to real WFAs $A = \langle \boldsymbol{\alpha}, \mathbf{A}, \boldsymbol{\beta} \rangle$, defined over a one-letter alphabet, with $\rho(\mathbf{A}) < 1$. While this setting is certainly restricted, we believe that this work constitutes a first fundamental step towards optimal approximation. Furthermore, the use of AAK techniques has proven to be very fruitful in related areas like control theory; we think that automata theory can also benefit from it. The use of such methods can help deepen the understanding of the behaviour of rational functions. This paper highlights and strengthens the interesting connections between functional analysis, automata theory and control theory, unifying tools from different domains in one formalism.

A compelling direction for future work is to extend our results to the multi-letter case. The work of Adamyan, Arov and Krein provides us with a powerful theory connecting sequences to the study of complex functions. We note that, unfortunately, this approach cannot be directly generalized to the multi-letter case because of the non-commutative nature of the monoid considered. Extending this work would require adapting standard harmonic analysis results to the non-abelian case. A recent line of work in functional analysis is centered around extending this theory to the case of non-commutative operators, and in [35] a non-commutative version of the AAK theorem is presented. However, those results are non-constructive, making this direction, already challenging, even harder to pursue.

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