On the Axiomatizability of Quantitative Algebras

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***Abstract*—Quantitative algebras (QAs) are algebras over metric spaces defined by quantitative equational theories as introduced by us in 2016. They provide the mathematical foundation for metric semantics of probabilistic, stochastic and other quantitative systems. This paper considers the issue of axiomatizability of QAs. We investigate the entire spectrum of types of quantitative equations that can be used to axiomatize theories: (i) simple quantitative equations; (ii) Horn clauses with no more than c equations between variables as hypotheses, where c is a cardinal and (iii) the most general case of Horn clauses. In each case we characterize the class of QAs and prove variety/quasivariety theorems that extend and generalize classical results from model theory for algebras and first-order structures.**

I. Introduction

In [1] we introduced the concept of a quantitative equational theory in order to support a quantitative algebraic theory of effects and address metric-semantics issues for probabilistic, stochastic and quantitative theories of systems. Probabilistic programming, in particular, has become very important recently [2], see, for example, the web site [3]. The need for semantics and reasoning principles for such languages is important as well and recently one can witness an increased interest of the research community in this topic. Equational reasoning is the most basic form of logical reasoning and it is with the aim of making this available in a metric context that we began this work.

A quantitative equational theory allows one to write equations of the form $s =_\epsilon t$, where ϵ is a rational number, in order to characterize metric structures in an algebraic context. We developed the analogue of universal algebras over metric spaces – called quantitative algebras (QAs), proved analogues of Birkhoff's completeness theorem and showed that quantitative equations defined monads on metric spaces. We also presented a number of examples of interesting quantitative algebras widely used in semantics. We presented variants of barycentric algebra [4] that model the space of probabilistic/subprobabilistic distributions with either the Kantorovich, Wasserstein or total variation metrics; the same algebras can also be used to characterize the space of Markov processes with the Kantorovich metric. We also gave a notion of quantitative semilattice that characterizes the space of closed subsets of an extended metric space with the Hausdorff metric.

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In all these examples we emphasized elegant axiomatizations characterizing these well-known metric spaces. In [5] the same tools are used to provide axiomatizations for a fixpoint semantics for Markov chains. Of course, some of these examples can be given by ordinary monads, as shown in [6], [7], but we are aiming to fully integrate metric reasoning into equational reasoning.

What was left open in our previous work was what kinds of metric-algebraic structures could be axiomatized. This is an important issue if we want a general theory for metric-based semantics, since we will need to understand whether the class of systems of interest with their natural metrics can, in fact, be axiomatized. In the present paper, we discuss the general question of what classes of quantitative algebras can be axiomatized by quantitative equations, or by more general axioms like Horn clauses.

The celebrated Birkhoff variety theorem [8] states that a class of algebras is equationally definable if and only if it is closed under homomorphic images, subalgebras, and products. Many extensions have been proved for more general kinds of axioms [9] and for coalgebras instead of algebras [10], [11], and see [12], [13], [14], [15] for a categorical perspective. It is natural to ask if there are corresponding results for quantitative equations and quantitative algebras. Since classical equations $s = t$ define a congruence over the algebraic structure, while quantitative equations $s =_\epsilon t$ define a pseudometric coherent with the algebraic structure, the classical results do not apply directly to our case. One therefore needs fully to understand how metric structures behave equationally to answer the question. This is the challenge we take up here.

The interesting examples that we present in [1] require not only axiomatizations involving quantitative equations of the form $s =_\epsilon t$, but also conditional equations, i.e., Horn clauses involving quantitative equations. Already the simple case of Horn clauses of the form $\{x_i =_{\epsilon_i} y_i \mid i \in I\}$ as hypotheses, where x_i, y_i are variables only, provides interesting examples. All this forces us to develop some new concepts and proof techniques that are innovative in a number of ways.

Firstly, we show that considering a metric structure on top of an algebraic structure, which implicitly requires one to replace the concept of *congruence* with a *pseudometric* coherent with the algebraic structure, is not a straightforward generalization.

Indeed, one can always think of a congruence $\cong$ on an algebra $\mathcal{A}$ as to the kernel of the pseudometric $p_{\cong}$ defined by $p_{\cong}(a,b) = 0$ iff $a \cong b$ and $p_{\cong}(a,b) = 1$ otherwise. Nevertheless, many standard model-theoretic results about axiomatizability of algebras are particular consequences of the discrete nature of this pseudometric. Many of these results fail when one takes a more complex pseudometric, even if its kernel remains a congruence.

Secondly, we show that in the case of quantitative algebras, quantitative equation-based axiomatizations behave very similarly to axiomatizations by *Horn clauses* involving only quantitative equations between variables as hypotheses. And this remains true even when one allows functions of countable arity in the signature. Horn clauses of this type are directly connected to *enriched Lawvere theories* [16]. We give a uniform treatment of all these cases by interpreting quantitative equations as Horn clauses with empty sets of hypotheses.

We discover, in this context, a special class of homomorphisms that we call *c-reflexive homomorphisms*, for a cardinal c, that play a crucial role. These homomorphisms preserve distances on selected subsets of cardinality less than c of the metric space, i.e., any c-space in the image pulls back (modulo non-expansiveness) to one in the domain. This concept generalizes the concept of homomorphism of quantitative algebras, since any homomorphism of quantitative algebras is 1-reflexive. The central role of c-reflexive homomorphisms is demonstrated by a *weak universality* property, proved below.

This result also shows that the classical canonical model construction for classes of universal algebras is mathematically inadequate and works in the traditional settings only because it is, coincidentally, a model isomorphic with the more general one that we present here. However, apart from the classic settings (of universal algebras and congruences) the standard construction fails to produce a model isomorphic with the "natural" one and consequently, it fails to reflect the weak universality properly up to c-reflexive homomorphisms.

Our main result in this first part of the paper is the c-variety theorem for a regular cardinal $c \leq \aleph_1$: a class of quantitative algebras can be axiomatized by Horn clauses, each axiom having fewer than c equations between variables as hypotheses, if, and only if, the class is closed under subobjects, products and c-reflexive homomorphisms. In particular, (i) the class is a 1-variety (closed under subobjects, products and homomorphisms) iff it can be axiomatized by quantitative equations; (ii) it is an $\aleph_0$-variety iff it can be axiomatized by Horn clauses with finite sets of quantitative equations between variables as hypotheses; and (iii) it is an $\aleph_1$-variety iff it can be axiomatized by Horn clauses with countable sets of quantitative equations between variables as hypotheses. Notice that in the light of the previously mentioned relation between congruences and pseudometrics, (i) generalizes the original Birkhoff result for universal algebras. Without the concept of c-reflexivity, one can only state a quasi-variety theorem under the very strong assumption that reduced products always exist, as happens, e.g., in [17].

Thirdly, we also study the axiomatizability of classes of quantitative algebras that admit Horn clauses as axioms, but which are not restricted to quantitative equations between variables as hypotheses. We prove that a class of quantitative algebras admits an axiomatization of this type, whenever it is closed under isomorphisms, subalgebras and what we call *subreduced products*. These are quantitative subalgebras of (a special type of) products of elements in the given class; however, while these products are always algebras, they are not always quantitative algebras, and this is where the new concept plays its role. This new type of closure condition allows us to generalize the usual quasivariety theorem of universal algebras.

Since all the isomorphisms of quantitative algebras are c-reflexive homomorphisms, and since a c-variety is closed under subalgebras and products, it is also closed under subreduced products, as they are quantitative subalgebras of the product. Hence, a c-variety is closed under these operators for any regular cardinal $c > 0$ and so our quasivariety theorem extends the c-variety theorem further. These all are novel generalizations of the classical results.

Last, but not least, to achieve the aforementioned results for general Horn clauses, we had to generalize concepts and results from model theory of first-order structures considering first-order model theory on metric structures. Thus, we extended to the general unrestricted case the pioneering work in [18] devoted to continuous logic over complete bounded metric spaces. We identified the first-order counterpart of a quantitative algebra, that we call a *quantitative first-order structure*, and prove that the category of quantitative algebras is isomorphic to the category of quantitative first-order structures. We have developed *first-order equational logic* for these structures and extended standard model theoretic results for quantitative first-order structures. Finally, the proof of the quasivariety theorem, which actively involves the new concept of subreduced product, is based on a more fundamental proof pattern that can be further used in model theory for other types of first-order structures. We essentially show how one can prove a quasivariety theorem for a restricted class of first-order structures that obey infinitary axiomatizations.

We have left behind an open question: the results regarding unrestricted Horn clauses have been proved under the restriction of having only finitary functions in the algebraic signature. This was required in order to use standard model theoretic techniques. We believe that a similar result might also hold for countable functions.

II. Preliminaries on Quantitative Algebras

In this section we recall some basic concepts from [1] and introduce a few more needed in our development.

A. Quantitative Equational Theories

Consider an algebraic similarity type Ω containing functions of finite or countable arity. Given a set X of variables, let $\mathbb{T}X$ be the Ω-*term algebra* over X. If c is the arity of the function f in Ω we write $f : c \in \Omega$; given an indexed family of terms $(t_i)_{i\in I}$ with $|I| = c$, we write $f((t_i)_{i\in I})$ for the term obtained by applying f to these terms. A *substitution* is any Ω-homomorphism $\sigma\colon \mathbb{T}X \longrightarrow \mathbb{T}X$; if $\Gamma \subseteq \mathbb{T}X$ and σ is a substitution, let $\sigma(\Gamma) = \{\sigma(t) \mid t \in \Gamma\}$.

Let $\mathcal{V}(X)$ denote the set of *indexed equations* of the form $x =_\epsilon y$ for $x, y \in X$ and $\epsilon \in \mathbb{Q}_+$; similarly, let $\mathcal{V}(\mathbb{T}X)$ denote the set of indexed equations of the form $t =_\epsilon s$ for $t, s \in \mathbb{T}X$, $\epsilon \in \mathbb{Q}_+$. We call them *quantitative equations*.

Let $\mathcal{E}(\mathbb{T}X)$ be the class of *conditional quantitative equations* on $\mathbb{T}X$, which are constructions of the form

$$\{s_i =_{\epsilon_i} t_i \mid i \in I\} \vdash s =_\epsilon t,$$

where I is countable[1], $(s_i)_{i\in I}, (t_i)_{i\in I} \subseteq \mathbb{T}X$ and $s, t \in \mathbb{T}X$. If $V \vdash \phi \in \mathcal{E}(\mathbb{T}X)$, we refer to the elements of V as the *hypotheses* of the conditional equation.

Given a cardinal $0 < c \leq \aleph_1$, a *c-basic conditional equation* on $\mathbb{T}X$ is a conditional quantitative equation of the form

$$\{x_i =_{\epsilon_i} y_i \mid i \in I\} \vdash s =_\epsilon t,$$

where $|I| < c$, $(x_i)_{i\in I}, (y_i)_{i\in I} \subseteq X$ and $s, t \in \mathbb{T}X$.

Note that the 1-*basic conditional equations* are the conditional equation with an empty set of hypotheses. We call them *unconditional equations*.

The $\aleph_0$-basic conditional equations are the conditional equations with a finite set of hypotheses, all equating variables only. We call them *finitary-basic quantitative equations*.

The $\aleph_1$-basic conditional equations are all the basic conditional equations, hence with countable (including finite, or empty) sets of equations between variables as hypotheses.

A *deducibility relation of type* Ω *over* X is a set of conditional equations on $\mathbb{T}X$ closed under the following rules stated for arbitrary $t, s, u \in \mathbb{T}X$, $(s_i)_{i\in I}, (t_i)_{i\in I} \subseteq \mathbb{T}X$, $\epsilon, \epsilon' \in \mathbb{Q}_+$, $\Gamma, \Gamma' \subseteq \mathcal{V}(\mathbb{T}X)$ and $\phi, \psi \in \mathcal{V}(\mathbb{T}X)$.

(Refl) $\emptyset \vdash t =_0 t$

(Symm) $\{t =_\epsilon s\} \vdash s =_\epsilon t$.

(Triang) $\{t =_\epsilon s, s =_{\epsilon'} u\} \vdash t =_{\epsilon+\epsilon'} u$.

(Max) For $\epsilon' > 0$, $\{t =_\epsilon s\} \vdash t =_{\epsilon+\epsilon'} s$.

(Arch) For $\epsilon \geq 0$, $\{t =_{\epsilon'} s \mid \epsilon' > \epsilon\} \vdash t =_\epsilon s$.

(NExp) For $f : |I| \in \Omega$,
$\{t_i =_\epsilon s_i \mid i \in I\} \vdash f((t_i)_{i\in I}) =_\epsilon f((s_i)_{i\in I})$

[1]Note that I countable subsumes the cases I finite and I void.

(Subst) If σ is a substitution,
$\Gamma \vdash t =_\epsilon s$ implies $\sigma(\Gamma) \vdash \sigma(t) =_\epsilon \sigma(s)$.

(Cut) If $\Gamma \vdash \phi$ for all $\phi \in \Gamma'$ and $\Gamma' \vdash \psi$, then $\Gamma \vdash \psi$.

(Assumpt) If $\phi \in \Gamma$, then $\Gamma \vdash \phi$.

Definition 2.1: Given a set $S \subseteq \mathcal{E}(\mathbb{T}X)$ of conditional equations on $\mathbb{T}X$, the *quantitative equational theory axiomatized by* S is the smallest deducibility relation that contains S.

B. Quantitative Algebras

An Ω-*quantitative algebra* (QA) is a tuple $\mathcal{A} = (A, \Omega^{\mathcal{A}}, d^{\mathcal{A}})$, where $(A, \Omega^{\mathcal{A}})$ is an Ω-algebra and $d^{\mathcal{A}} : A \times A \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ is a metric on A (possibly taking infinite values) such that all the operators $f : c \in \Omega^{\mathcal{A}}$ are *non-expansive*, i.e., for any $(a_i)_{i\in I}, (b_i)_{i\in I} \subseteq A$ with $|I| = c$, and any $\epsilon \geq 0$, if $d^{\mathcal{A}}(a_i, b_i) \leq \epsilon$ for all $i \in I$, then

$$d^{\mathcal{A}}(f((a_i)_{i\in I}), f((b_i)_{i\in I})) \leq \epsilon.$$

A quantitative algebra is *void* when its support is void and is *degenerate* if its support is a singleton.

Given two quantitative algebras of type Ω, $\mathcal{A}_i = (A_i, \Omega, d^{\mathcal{A}_i})$, $i = 1, 2$, a *homomorphism of quantitative algebras* is a homomorphism $h : A_1 \longrightarrow A_2$ of Ω-algebras, which is non-expansive, i.e., s.t., for arbitrary $a, b \in A_1$,

$$d^{\mathcal{A}_1}(a, b) \geq d^{\mathcal{A}_2}(h(a), h(b)).$$

Notice that identity maps are homomorphisms and that homomorphisms are closed under composition, hence quantitative algebras of type Ω and their homomorphisms form a category, written $\mathbf{QA}_{\mathbf{\Omega}}$.

Reflexive Homomorphisms. A key role is played by a special class of homomorphisms of QAs that we will call reflexive homomorphisms.

Hereafter we use $A \subseteq_c B$ for a cardinal $c > 0$ to mean that A is a subset of B and $|A| < c$.

Definition 2.2: A homomorphism $f : \mathcal{A} \longrightarrow \mathcal{B}$ of quantitative algebras is *c-reflexive*, where c is a cardinal, if for any subset $B' \subseteq_c \mathcal{B}$ there exists a set $A' \subseteq \mathcal{A}$ such that $f(A') = B'$ and for any $a, b \in A'$, $d^{\mathcal{A}}(a, b) = d^{\mathcal{B}}(f(a), f(b))$.

If $f : \mathcal{A} \longrightarrow \mathcal{B}$ is a c-reflexive homomorphism, $f(A)$ is a *c-reflexive homomorphic image* of A.

Note that any homomorphism of quantitative algebras is 1-reflexive. Moreover, for $c > c'$, a c-reflexive homomorphism is also c'-reflexive.

Subalgebras. A quantitative algebra $\mathcal{B} = (B, \Omega, d^{\mathcal{B}})$ is a quantitative subalgebra of the quantitative algebra $\mathcal{A} = (A, \Omega, d^{\mathcal{A}})$, denoted by $\mathcal{B} \leq \mathcal{A}$, if $\mathcal{B}$ is an Ω-subalgebra of $\mathcal{A}$ and for any $a, b \in B$, $d^{\mathcal{B}}(a, b) = d^{\mathcal{A}}(a, b)$.

Direct Products. Let $(\mathcal{A}_i)_{i\in I}$ be a family of quantitative algebras of type Ω, where $\mathcal{A}_i = (A_i, \Omega, d_i)$. The *(direct) product* $\mathcal{A} = (A, \Omega, d)$ is a quantitative algebra such that

- $A = \prod_{i \in I} A_i$ is the direct product of the sets A_i, for $i \in I$;
- for each $f : |J| \in \Omega$ and each $a_j = (b^i_j)_{i \in I}$, $j \in J$, $f^{\mathcal{A}}((a_j)_{j \in J}) = (f^{\mathcal{A}_i}((b^i_j)_{j \in J}))_{i \in I}$;
- for $a = (a_i)_{i \in I}$, $b = (b_i)_{i \in I}$, $d(a, b) = \sup_{i \in I} d_i(a_i, b_i)$.

The empty product $\prod \emptyset$ is the degenerate algebra with universe $\{\emptyset\}$.

The fact that this is a QA follows from the pointwise constructions of products in both the category of Ω-algebras and in the category of metric spaces with infinite values where the product metric is the pointwise supremum. The non-expansiveness of the functions in the product algebra follows from the non-expansiveness of the functions in the components. The product quantitative algebra is written $\prod_{i \in I} \mathcal{A}_i$.

Direct products have *projection maps* for each $k \in I$, $\pi_k : \prod_{i \in I} \mathcal{A}_i \to \mathcal{A}_k$ defined for arbitrary $a = (a_i)_{i \in I} \in \prod_{i \in I} A_i$ by $\pi_k(a) = a_k$. If none of the quantitative algebras in the family is void, the projection maps are always surjective homomorphisms of QAs.

Closure Operators. Consider the following operators mapping classes of QAs into classes of QAs. Given a class $\mathcal{K}$ of quantitative algebras and a cardinal c,

- $\mathcal{A} \in \mathbb{I}(\mathcal{K})$ iff $\mathcal{A}$ is isomorphic to some member of $\mathcal{K}$;
- $\mathcal{A} \in \mathbb{S}(\mathcal{K})$ iff $\mathcal{A}$ is a quantitative subalgebra of some member of $\mathcal{K}$;
- $\mathcal{A} \in \mathbb{H}_c(\mathcal{K})$ iff $\mathcal{A}$ is the c-reflexive homomorphic image of some algebra in $\mathcal{K}$; in particular, we denote $\mathbb{H}_1$ simply by $\mathbb{H}$ since it is the closure under homomorphic images;
- $\mathcal{A} \in \mathbb{P}(\mathcal{K})$ iff $\mathcal{A}$ is a direct product of a family of elements in $\mathcal{K}$;
- $\mathbb{V}_c(\mathcal{K})$ is the smallest class of quantitative algebras containing $\mathcal{K}$ and closed under subalgebras, products, and c-reflexive homomorphic images; such a class is called a *c-variety of quantitative algebras*. In particular, for $c = 1$ we also write $\mathbb{V}_1$ as $\mathbb{V}$ and call $\mathbb{V}(\mathcal{K})$ a variety.

For any operators $\mathbb{X}, \mathbb{Y} \in \{\mathbb{I}, \mathbb{S}, \mathbb{H}_c, \mathbb{P}, \mathbb{V}_c\}$, we write $\mathbb{X}\mathbb{Y}$ for their composition. Furthermore, for any compositions $\mathbb{X}, \mathbb{Y}$ of these we write $\mathbb{X} \subseteq \mathbb{Y}$ if $\mathbb{X}(\mathcal{K}) \subseteq \mathbb{Y}(\mathcal{K})$ for any class $\mathcal{K}$.

Lemma 2.3: The closure operators on classes of quantitative algebras enjoy the following properties:

1) whenever $c < c'$, $\mathbb{H}_c \subseteq \mathbb{H}_{c'}$;
2) whenever $c < c'$, if $\mathcal{K}$ is $\mathbb{H}_c$-closed, then it is $\mathbb{H}_{c'}$-closed; in particular, a $\mathbb{H}$-closed class is $\mathbb{H}_c$-closed for any c;
3) whenever $c < c'$, if $\mathcal{K}$ is c-variety, then it is a c'-variety; in particular, a variety is a c-variety for any c;
4) $\mathbb{S}\mathbb{H}_c \subseteq \mathbb{H}_c\mathbb{S}$; in particular, $\mathbb{S}\mathbb{H} \subseteq \mathbb{H}\mathbb{S}$;
5) $\mathbb{P}\mathbb{H}_c \subseteq \mathbb{H}_c\mathbb{P}$; in particular, $\mathbb{P}\mathbb{H} \subseteq \mathbb{H}\mathbb{P}$;
6) $\mathbb{V}_c = \mathbb{H}_c\mathbb{S}\mathbb{P}$; in particular, $\mathbb{V} = \mathbb{H}\mathbb{S}\mathbb{P}$;
7) $\mathbb{P}\mathbb{S} \subseteq \mathbb{S}\mathbb{P}$.

C. Algebraic Semantics for Quantitative Inferences

As expected, quantitative algebras are used to interpret quantitative equational theories.

Given a quantitative algebra $\mathcal{A} = (A, \Omega^{\mathcal{A}}, d^{\mathcal{A}})$ of type Ω and a set X of variables, an *assignment* on $\mathcal{A}$ is an Ω-homomorphism $\alpha : \mathbb{T}X \to A$; it is used to interpret abstract terms in $\mathbb{T}X$ as concrete elements in $\mathcal{A}$. We denote by $\mathbb{T}(X|\mathcal{A})$ the set of assignments on $\mathcal{A}$.

Definition 2.4: A quantitative algebra $\mathcal{A}$ under the assignment $\alpha \in \mathbb{T}(X|\mathcal{A})$ *satisfies* a conditional quantitative equation $\Gamma \vdash s =_\epsilon t \in \mathcal{E}(\mathbb{T}X)$, written $\Gamma \models_{\mathcal{A},\alpha} s =_\epsilon t$, whenever

$$[d^{\mathcal{A}}(\alpha(t'), \alpha(s')) \leq \epsilon' \text{ for all } s' =_{\epsilon'} t' \in \Gamma]$$

implies $d^{\mathcal{A}}(\alpha(s), \alpha(t)) \leq \epsilon$.

$\mathcal{A}$ *satisfies* $\Gamma \vdash s =_\epsilon t \in \mathcal{E}(\mathbb{T}X)$, written $\Gamma \models_{\mathcal{A}} s =_\epsilon t$, if $\Gamma \models_{\mathcal{A},\alpha} s =_\epsilon t$, for all assignments $\alpha \in \mathbb{T}(X|\mathcal{A})$; in this case $\mathcal{A}$ is a *model* of the conditional quantitative equation.

Similarly, for a set Γ of conditional quantitative equations (e.g., a quantitative equational theory), we say that $\mathcal{A}$ is a model of Γ if $\mathcal{A}$ satisfies each element of Γ.

If $\mathcal{K}$ is a class of quantitative algebras we write $\Gamma \models_{\mathcal{K}} s =_\epsilon t$, if for any $\mathcal{A} \in \mathcal{K}$, $\Gamma \models_{\mathcal{A}} s =_\epsilon t$. Furthermore, if $\mathcal{U}$ is a quantitative equational theory we write $\mathcal{K} \models \mathcal{U}$ if all algebras in $\mathcal{K}$ are models for $\mathcal{U}$.

For the case of unconditional equations, note that the left-hand side of the implication in the previous definition is vacuously satisfied. For these, instead of $\emptyset \models_{\mathcal{A},\alpha} s =_\epsilon t$ and $\emptyset \models_{\mathcal{A}} s =_\epsilon t$ we will often write $\mathcal{A}, \alpha \models s =_\epsilon t$ and $\mathcal{A} \models s =_\epsilon t$ respectively. Furthermore, for a class $\mathcal{K}$ of QAs, $\mathcal{K} \models s =_\epsilon t$ denotes that $\mathcal{A} \models s =_\epsilon t$ for all $\mathcal{A} \in \mathcal{K}$.

Definition 2.5: For a signature Ω and a set $\mathcal{U} \subseteq \mathcal{E}(\mathbb{T}X)$ of conditional quantitative equations over the Ω-terms $\mathbb{T}X$, the *conditional equational class induced by $\mathcal{U}$* is the class of quantitative algebras of signature Ω satisfying $\mathcal{U}$.

We denote this class as well as the full subcategory of Ω-quantitative algebras satisfying $\mathcal{U}$ by $\mathbb{K}(\Omega, \mathcal{U})$. We say that a class of algebras that is a conditional equational class is *conditional-equationally definable*.

Lemma 2.6: Given a set $\mathcal{U}$ of conditional quantitative equations of type Ω over $\mathbb{T}X$, $\mathbb{K}(\Omega, \mathcal{U})$ is closed under taking isomorphic images and subalgebras. Consequently, if $\mathcal{K}$ is a class of quantitative algebras over Ω, then $\mathcal{K}$, $\mathbb{I}(\mathcal{K})$ and $\mathbb{S}(\mathcal{K})$ satisfy the same conditional quantitative equations.

III. The Variety Theorem for Basic Conditional Equations

In this section we focus on the quantitative equational theories that admit an axiomatization containing only basic conditional equations, i.e., conditional equations of type

$$\{x_i =_{\epsilon_i} y_i \mid i \in I\} \vdash s =_\epsilon t,$$

for $x_i, y_i \in X$, $s, t \in \mathbb{T}X$ and $\epsilon_i, \epsilon \in \mathbb{Q}_+$. We shall call such a theory *basic equational theory*.

For a cardinal $c \leq \aleph_1$, a basic equational theory is a *c-basic equational theory* if it admits an axiomatization containing only c-basic conditional equations, i.e., of type

$$\{x_i =_{\epsilon_i} y_i \mid i \in I\} \vdash s =_\epsilon t,$$

for $|I| < c$, $x_i, y_i \in X$, $s, t \in \mathbb{T}X$ and $\epsilon_i, \epsilon \in \mathbb{Q}_+$.

An $\aleph_0$-basic equational theory is called a *finitary-basic equational theory*; it admits an axiomatization containing only finitary-basic conditional equations, i.e., of type

$$\{x_i =_{\epsilon_i} y_i \mid i \in 1, .., n\} \vdash s =_\epsilon t,$$

for $n \in \mathbb{N}$, $x_i, y_i \in X$, $s, t \in \mathbb{T}X$ and $\epsilon_i, \epsilon \in \mathbb{Q}_+$.

A 1-basic equational theory is called an *unconditional equational theory*; it admits an axiomatization containing only unconditional equations of type $\emptyset \vdash s =_\epsilon t$, for $s, t \in \mathbb{T}X$ and $\epsilon \in \mathbb{Q}_+$.

A. Products and Homomorphisms

Lemma 3.1: If $\mathcal{U}$ is a basic equational theory (in particular, finitary-basic or unconditional), then $\mathbb{K}(\Omega, \mathcal{U})$ is closed under direct products.

The c-reflexive homomorphisms play a central role in characterizing the basic equational theories in the case of the regular cardinals[2]. In fact, because our signature admits only functions of countable (including finite) arities, we will only focus on three regular cardinals: 1, $\aleph_0$ and $\aleph_1$.

Lemma 3.2: If $\mathcal{U}$ is a c-basic equational theory, where c is a non-null regular cardinal, then $\mathbb{K}(\Omega, \mathcal{U})$ is closed under c-reflexive homomorphic images. In particular,

- if $\mathcal{U}$ is an unconditional equational theory, then $\mathbb{K}(\Omega, \mathcal{U})$ is closed under homomorphic images;
- if $\mathcal{U}$ is a finitary-basic equational theory, then $\mathbb{K}(\Omega, \mathcal{U})$ is closed under $\aleph_0$-reflexive homomorphic images;
- if $\mathcal{U}$ is a basic equational theory, then $\mathbb{K}(\Omega, \mathcal{U})$ is closed under $\aleph_1$-reflexive homomorphic images.

Corollary 3.3: Let $\mathcal{K}$ be a class of quantitative algebras over the same signature and $c \leq \aleph_1$ a regular non-null cardinal. Then $\mathcal{K}$, $\mathbb{P}(\mathcal{K})$, $\mathbb{H}_c(\mathcal{K})$ and $\mathbb{V}_c(\mathcal{K})$ satisfy all the same c-basic conditional equations.

[2]The regular cardinals are the cardinals that cannot be obtained by using arithmetic involving smaller cardinals. Thus, for example, 23 is not a regular cardinal but 1, $\aleph_0$ or $\aleph_1$ are, because none of them can be written as a smaller sum of smaller cardinals.

B. Canonical Model and Weak Universality

In this subsection we give the quantitative analogue of the canonical model construction and prove weak universality. Before we begin the detailed arguments, we note a few points. In the original variety theorem one proceeds by looking at all congruences on the term algebra and quotienting by the coarsest. This strategy does not work in the present case. We need to consider the pseudometrics induced by all assignments of variables; next, instead of quotienting by the kernel of the coarsest pseudometric, as the analogy with the usual case would suggest, we need to take the product of the quotient algebras indexed by these pseudometrics. We note that this is indeed a generalization of the non-quantitative case where, coincidentally, this product algebra is isomorphic to the quotient algebra by the coarsest congruence. However, our proof here shows that the natural construction that guarantees the weak universality, even when one considers reflexive homomorphisms, is the product of the quotient algebras.

Consider, as before, an algebraic similarity type Ω and a set X of variables. Let $\mathcal{P}_{\mathbb{T}X}$ be the set of all pseudometrics $p : \mathbb{T}X^2 \to \mathbb{R}_+ \cup \{\infty\}$ such that all the functions in Ω are non-expansive with respect to p. For arbitrary $p \in \mathcal{P}_{\mathbb{T}X}$, let

$$\mathbb{T}X|_p = (\mathbb{T}X|_{ker(p)}, \Omega, p)$$

be the quantitative algebra obtained by taking the quotient of $\mathbb{T}X$ with respect to the congruence relation[3]

$$ker(p) = \{(s, t) \in \mathbb{T}X^2 \mid p(s, t) = 0\}.$$

Let $\mathcal{K}$ be a family of quantitative algebras of type Ω and

$$\mathcal{P}_\mathcal{K} = \{p \in \mathcal{P}_{\mathbb{T}X} \mid \mathbb{T}X|_p \in \mathbb{IS}(\mathcal{K})\}.$$

We begin by showing that $\mathcal{P}_\mathcal{K} \neq \emptyset$ whenever $\mathcal{K} \neq \emptyset$.

Consider an algebra $\mathcal{A} \in \mathcal{K}$, let $\alpha \in \mathbb{T}(X|\mathcal{A})$ be an arbitrary assignment and $[\alpha] : \mathbb{T}X^2 \to \mathbb{R}_+ \cup \{\infty\}$ a pseudometric defined for arbitrary $s, t \in \mathbb{T}X$ by

$$[\alpha](s, t) = \inf\{\epsilon \mid \mathcal{A}, \alpha \models s =_\epsilon t\}.$$

Lemma 3.4: If $\mathcal{A} \in \mathcal{K}$ and $\alpha \in \mathbb{T}(X|\mathcal{A})$, then $[\alpha] \in \mathcal{P}_\mathcal{K}$. Moreover, $\mathbb{T}X|_{[\alpha]}$ is a quantitative algebra isomorphic to $\alpha(\mathbb{T}X)$.

Proof. The fact that $[\alpha]$ is a pseudometric follows directly from the algebraic semantics.

Let $f : |I| \in \Omega$ and $(s_i)_{i \in I}, (t_i)_{i \in I} \subseteq \mathbb{T}X$. Assume that $[\alpha](s_i, t_i) \leq \epsilon$ for all $i \in I$. This means that for each $i \in I$, $\mathcal{A}, \alpha \models s_i =_\delta t_i$ for any $\delta \in \mathbb{Q}_+$ with $\delta \geq \epsilon$. The soundness of (NExp) provides $\mathcal{A}, \alpha \models f((s_i)_{i \in I}) =_\delta f((t_i)_{i \in I})$, i.e.,

[3]The non-expansiveness of p w.r.t. all the functions in Ω guarantees that $ker(p)$ is a congruence with respect to Ω.

$[\alpha](f((s_i)_{i\in I}), f((t_i)_{i\in I})) \leq \delta$ for any $\delta \geq \epsilon$. And this proves that $[\alpha] \in \mathcal{P}_{\mathbb{T}X}$.

We know that $\alpha : \mathbb{T}X \to \mathcal{A}$ is a homomorphism of quantitative algebras, hence $\alpha(\mathbb{T}X) \leq \mathcal{A}$ and $\hat{\alpha} : \mathbb{T}X \to \alpha(\mathbb{T}X)$ defined by $\hat{\alpha}(s) = \alpha(s)$ for any $s \in \mathbb{T}X$ is a surjection. Since from the way we have defined $[\alpha]$ we have that

$$\hat{\alpha}(s) = \hat{\alpha}(t) \quad \text{iff} \quad [\alpha](s,t) = 0,$$

we obtain that the map $\overline{\alpha} : \mathbb{T}X|_{[\alpha]} \to \alpha(\mathbb{T}X)$ defined by $\overline{\alpha}(s|_{[\alpha]}) = \alpha(s)$ for any $s \in \mathbb{T}X$, where $s|_{[\alpha]}$ denotes the $ker([\alpha])$-congruence class of s, is a QAs isomorphism. ∎

The previous lemma states that for any algebra $\mathcal{A} \in \mathcal{K}$ and any assignment $\alpha \in \mathbb{T}(X|\mathcal{A})$, $\mathbb{T}X|_{[\alpha]} \simeq \alpha(\mathbb{T}X) \leq \mathcal{A}$. Since a consequence of it is $\mathcal{P}_{\mathcal{K}} \neq \emptyset$ whenever $\mathcal{K} \neq \emptyset$, we can define a pointwise supremum over the elements in $\mathcal{P}_{\mathcal{K}}$:

$$d^{\mathcal{K}}(s,t) = \sup_{p\in\mathcal{P}_{\mathcal{K}}} p(s,t), \text{ for arbitrary } s,t \in \mathbb{T}X.$$

It is not difficult to notice that, $d^{\mathcal{K}} \in \mathcal{P}_{\mathbb{T}X}$.

Let $\mathbb{T}_{\mathcal{K}}X = (\prod_{p\in\mathcal{P}_{\mathcal{K}}} \mathbb{T}X|_p, \Omega, d^{\mathcal{K}})$ be the product quantitative algebra with the index set $\mathcal{P}_{\mathcal{K}}$.

For arbitrary $s \in \mathbb{T}X$, let $\langle s\rangle \in \mathbb{T}_{\mathcal{K}}X$ be the element such that for any $p \in \mathcal{P}_{\mathcal{K}}$, $\pi_p(\langle s\rangle) = s|_p$, where $s|_p \in \mathbb{T}X|_p$ denotes the $ker(p)$-equivalence class of s.

Now note that, if $\mathcal{K}$ is a class of quantitative algebras of the same type containing non-degenerate elements, then the map $\gamma : \mathbb{T}X \to \mathbb{T}_{\mathcal{K}}X$ defined by $\gamma(t) = \langle t\rangle$ for any $t \in \mathbb{T}X$ is an injective homomorphism of Ω-algebras.

In order to state now the weak universality property for a class $\mathcal{K}$ of QAs, we need firstly to identify a cardinal that plays a key role in our statement as an upper bound for the reflexive homomorphisms. We shall denote it by $r(\mathcal{K})$:

$$r(\mathcal{K}) = \begin{cases} \aleph_1 & \text{if } \exists \mathcal{A} \in \mathcal{K},\ |\mathcal{A}|^+ \geq \aleph_1 \\ \sup\{|\mathcal{A}|^+ \mid \mathcal{A} \in \mathcal{K}\} & \text{otherwise} \end{cases}$$

where $|\mathcal{A}|$ denotes of the cardinal of the support set of $\mathcal{A}$ and c^+ denotes the successor of the cardinal c.

The following theorem is a central result of this paper. One might be tempted to just use a quotient by $ker(d^{\mathcal{K}})$ but in that case the homomorphism that one gets by weak universality does not satisfy the c-reflexive condition.

Theorem 3.5 (Weak Universality): Consider a class $\mathcal{K}$ of quantitative algebras containing non-degenerate elements. For any $\mathcal{A} \in \mathcal{K}$ and any map $\alpha : X \to \mathcal{A}$ there exists a $r(\mathcal{K})$-reflexive homomorphism $\beta : \mathbb{T}_{\mathcal{K}}X \to \mathcal{A}$ such that

$$\text{for any } x \in X,\ \beta(\langle x\rangle) = \alpha(x).$$

Proof. Let $\mathbf{QA}_{\mathbf{\Omega}}$ be the category of Ω-quantitative algebras.

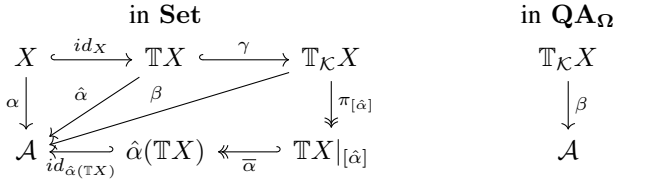
The map $\alpha : X \to \mathcal{A}$ can be canonically extended to an Ω-homomorphism $\hat{\alpha} : \mathbb{T}X \to \mathcal{A}$.

Let $\gamma : \mathbb{T}X \hookrightarrow \mathbb{T}_{\mathcal{K}}X$ be the aforementioned injective homomorphism of Ω-algebras.

From Lemma 3.4 we know that $\mathbb{T}X|_{[\hat{\alpha}]} \simeq \hat{\alpha}(\mathbb{T}X) \leq \mathcal{A}$. So, we consider the projection $\pi_{[\hat{\alpha}]} : \mathbb{T}_{\mathcal{K}}X \twoheadrightarrow \mathbb{T}X|_{[\hat{\alpha}]}$ which is a surjective morphism of quantitaive algebras.

Let $\overline{\alpha} : \mathbb{T}X|_{[\hat{\alpha}]} \to \hat{\alpha}(\mathbb{T}X)$ be the isomorphism of quantitaive algebras defined in (the proof of) Lemma 3.4.

These maps give us the following commutative diagram.



The diagonal of this diagram is a map β defined for arbitrary $u \in \mathbb{T}_{\mathcal{K}}X$ as follows:

$$\beta(u) = \overline{\alpha} \circ \pi_{[\hat{\alpha}]}(u).$$

Note that if $u = \langle s\rangle$ for some $s \in \mathbb{T}X$, then

$$\beta(\langle s\rangle) = \overline{\alpha}(\pi_{[\hat{\alpha}]}(\langle s\rangle)) = \overline{\alpha}(s|_{[\hat{\alpha}]}) = \hat{\alpha}(s)$$

and further more, if $x \in X$,

$$\beta(\langle x\rangle) = \overline{\alpha}(\pi_{[\hat{\alpha}]}(\langle x\rangle)) = \overline{\alpha}(x|_{[\hat{\alpha}]}) = \hat{\alpha}(x) = \alpha(x).$$

Since β is the composition of two homomorphisms of quantitative algebras, it is a homomorphism of quantitative algebras.

Finally we show that β is a $r(\mathcal{K})$-reflexive. To start with, note that $\hat{\alpha}(\mathbb{T}X) \leq \mathcal{A}$ is the image of $\mathbb{T}_{\mathcal{K}}X$ through β. Since $|\hat{\alpha}(\mathbb{T}X)| < r(\mathcal{K})$, it only remains to prove that there exists a subset in $\mathbb{T}_{\mathcal{K}}X$ such that for any $a, b \in \hat{\alpha}(\mathbb{T}X)$ we find two elements u, v in this subset such that $\beta(u) = a$, $\beta(v) = b$ and

$$d^{\mathcal{A}}(a,b) = d^{\mathcal{K}}(u,v).$$

Let $s, t \in \mathbb{T}X$ be such that $\hat{\alpha}(s) = a$ and $\hat{\alpha}(t) = b$. Let $u, v \in \mathbb{T}_{\mathcal{K}}X$ such that $\pi_{[\hat{\alpha}]}(u) = s|_{[\hat{\alpha}]}$, $\pi_{[\hat{\alpha}]}(v) = t|_{[\hat{\alpha}]}$ and for any $p \neq [\hat{\alpha}]$, $\pi_p(u) = \pi_p(v)$.

Since $d^{\mathcal{K}}(u,v) = \sup_{p\in\mathcal{P}_{\mathcal{K}}} p(\pi_p(u), \pi_p(v))$ and $\pi_p(u) = \pi_p(v)$ for $p \neq [\hat{\alpha}]$, we obtain that indeed

$$d^{\mathcal{K}}(u,v) = [\hat{\alpha}](s,t) = d^{\mathcal{A}}(a,b).$$ ∎

Observe that the homomorphism β is not unique, since any pseudometric $p \in \mathcal{P}_{\mathcal{K}}$ can be associated to a projection π_p that will eventually define a homomorphism of type β making the diagram commutative - hence, we have weak-universality. However, only for β associated to $[\alpha]$, can we guarantee that β is $r(\mathcal{K})$-reflexive.

The weak universality reflects a fundamental relation between $\mathbb{T}_{\mathcal{K}}X$ and the $r(\mathcal{K})$-reflexive closure operator $\mathbb{H}_{r(\mathcal{K})}$, as stated below.

Corollary 3.6: If $\mathcal{A} \in \mathcal{K}$, then for X sufficiently large,

$$\mathcal{A} \in \mathbb{H}_{r(\mathcal{K})}(\{\mathbb{T}_{\mathcal{K}} X\}).$$

Corollary 3.7: Suppose that $\mathbb{T}X \neq \emptyset \neq \mathcal{K}$. Then,

$$\mathbb{T}_{\mathcal{K}} X \in \mathbb{H}_{r(\mathcal{K})}\mathbb{S}\mathbb{P}(\mathcal{K}).$$

Hence, if $\mathcal{K}$ is closed under $\mathbb{H}_{r(\mathcal{K})}$, $\mathbb{S}$ and $\mathbb{P}$, then $\mathbb{T}_{\mathcal{K}} X \in \mathcal{K}$.

The following theorem explains why we refer to $\mathbb{T}_{\mathcal{K}} X$ as to the canonical model.

Theorem 3.8: Let $\mathcal{K}$ be a class of quantitative algebras containing non-degenerate elements and $c \leq r(\mathcal{K})$ a non-null regular cardinal. Let $\{x_i =_{\epsilon_i} y_i \mid i \in I\} \vdash s =_\epsilon t$ be an arbitrary c-basic conditional equation on $\mathbb{T}X \neq \emptyset$. Then,

$$\{x_i =_{\epsilon_i} y_i \mid i \in I\} \models_{\mathcal{K}} s =_\epsilon t \text{ iff}$$

$$\{x_i =_{\epsilon_i} y_i \mid i \in I\} \models_{\mathbb{T}_{\mathcal{K}} X} s =_\epsilon t.$$

Corollary 3.9: Let $\mathcal{K}$ be a class of quantitative algebras containing non-degenerate elements and $\mathbb{T}X \neq \emptyset$. Then for arbitrary $s, t \in \mathbb{T}X$ and arbitrary $\epsilon \in \mathbb{Q}_+$,

$$\mathcal{K} \models s =_\epsilon t \text{ iff } \mathbb{T}_{\mathcal{K}} X \models s =_\epsilon t \text{ iff } d^{\mathcal{K}}(\langle s \rangle, \langle t \rangle) \leq \epsilon.$$

C. Variety Theorem

With these results in hand, we are ready to prove a general variety theorem for quantitative algebras.

Hereafter the signature Ω remains fixed; so, if S is an axiomatization for $\mathcal{U}$, we use $\mathbb{K}(S)$ to denote the class $\mathbb{K}(\Omega, \mathcal{U})$.

If S is a set of c-basic conditional equations, we say that $\mathbb{K}(S)$ is a *c-basic conditional equational class*. We call an $\aleph_1$-basic conditional equational class simply *basic equational class*. A *finitary-basic equational class* is an $\aleph_0$-basic conditional equational class. An *unconditional equational class* is a 1-basic conditional equational class.

We propose now a symmetric concept: if $\mathcal{K}$ is a set of quantitative algebras and $0 < c \leq \aleph_1$ is a cardinal, let $\mathcal{E}^c_X(\mathcal{K})$ be the set of all c-basic conditional equations over the set X of variables that are satisfied by all the elements of $\mathcal{K}$.

Lemma 3.10: If $\mathcal{K}$ is a non-void c-variety for a regular cardinal $0 < c \leq r(\mathcal{K})$ and X is an infinite set of variables, then

$$\mathcal{K} = \mathbb{K}(\mathcal{E}^c_X(\mathcal{K})).$$

Proof. Let $\mathcal{K}' = \mathbb{K}(\mathcal{E}^c_X(\mathcal{K}))$. Obviously $\mathcal{K} \subseteq \mathcal{K}'$.

We prove for the beginning that $\mathcal{E}^c_X(\mathcal{K}) = \mathcal{E}^c_X(\mathcal{K}')$.

Since $\mathcal{K} \subseteq \mathcal{K}'$, $\mathcal{E}^c_X(\mathcal{K}) \supseteq \mathcal{E}^c_X(\mathcal{K}')$.

Let $\Gamma \vdash \phi \in \mathcal{E}^c_X(\mathcal{K})$ be a c-basic quantitative inference. Then, for any $\mathcal{A} \in \mathcal{K}$, $\Gamma \models_{\mathcal{A}} \phi$. Consider an arbitrary $\mathcal{B} \in \mathcal{K}'$. Since $\mathcal{K}' = \mathbb{K}(\mathcal{E}^c_X(\mathcal{K}))$, B must satisfy all the c-basic conditional equations in $\mathcal{E}^c_X(\mathcal{K})$; in particular, $\Gamma \models_{\mathcal{B}} \phi$. Hence, $\mathcal{E}^c_X(\mathcal{K}) \subseteq \mathcal{E}^c_X(\mathcal{K}')$.

Consider now an arbitrary $\mathcal{A}' \in \mathcal{K}'$.

From Corollary 3.6, for a suitable set Y of variables such that $|Y| \geq r(\mathcal{K}')$, we can define a surjection $\alpha : \mathbb{T}Y \twoheadrightarrow \mathcal{A}'$.

For arbitrary $s \in \mathbb{T}Y$, let $s|_{\mathcal{K}} \in \prod_{p \in \mathcal{P}_{\mathcal{K}}} \mathbb{T}Y|_p$ be the element[4] such that for any $p \in \mathcal{P}_{\mathcal{K}}$, $\pi_p(s|_{\mathcal{K}}) = s|_p$ and similarly $s|_{\mathcal{K}'} \in \prod_{p \in \mathcal{P}_{\mathcal{K}'}} \mathbb{T}Y|_p$ be the element such that for any $p \in \mathcal{P}_{\mathcal{K}'}$, $\pi_p(s|_{\mathcal{K}'}) = s|_p$.

Theorem 3.5 provides an injection $\gamma' : \mathbb{T}Y \hookrightarrow \mathbb{T}_{\mathcal{K}'} Y$ defined by $\gamma'(s) = s|_{\mathcal{K}'}$ for any $s \in \mathbb{T}Y$; and a $r(\mathcal{K}')$-reflexive homomorphism $\beta' : \mathbb{T}_{\mathcal{K}'} Y \twoheadrightarrow \mathcal{A}'$ which has the property that $\beta'(s|_{\mathcal{K}'}) = \alpha(s)$. Moreover, β' is a surjection since α is.

Because $c \leq r(\mathcal{K}) \leq r(\mathcal{K}')$, β' is also $r(\mathcal{K})$-reflexive and c-reflexive. Note now that also $\hat{\beta}' : \gamma'(\mathbb{T}Y) \to \mathcal{A}'$, which is defined by $\hat{\beta}'(u) = \beta'(u)$ for any $u \in \gamma'(\mathbb{T}Y)$, is a surjective c-reflexive homomorphism of quantitative algebras such that $\hat{\beta}'(s|_{\mathcal{K}'}) = \alpha(s)$.

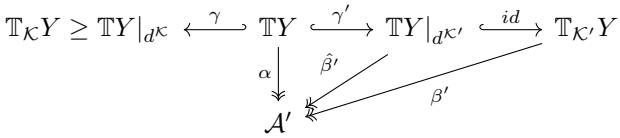
Similarly, there exists an injection $\gamma : \mathbb{T}Y \hookrightarrow \mathbb{T}_{\mathcal{K}} Y$ defined by $\gamma(s) = s|_{\mathcal{K}}$ for any $s \in \mathbb{T}Y$.

Consider now the following two quantitative algebras

$$\mathbb{T}Y|_{d^{\mathcal{K}}} = (\mathbb{T}Y|_{ker(d^{\mathcal{K}})}, \Omega, d^{\mathcal{K}}) \text{ and}$$

$$\mathbb{T}Y|_{d^{\mathcal{K}'}} = (\mathbb{T}Y|_{ker(d^{\mathcal{K}'})}, \Omega, d^{\mathcal{K}'}).$$

Note that the functions $\theta : \mathbb{T}Y|_{d^{\mathcal{K}}} \to \gamma(\mathbb{T}Y)$ defined by $\theta(s|_{d^{\mathcal{K}}}) = \gamma(s)$ and $\theta' : \mathbb{T}Y|_{d^{\mathcal{K}'}} \to \gamma'(\mathbb{T}Y)$ defined by $\theta'(s|_{d^{\mathcal{K}'}}) = \gamma'(s)$ are isomorphisms of quantitative algebras.



Repeatedly applying Corollary 3.9 we get that for arbitrary $s, t \in \mathbb{T}Y$, $d^{\mathcal{K}}(s|_{\mathcal{K}}, t|_{\mathcal{K}}) = 0$ iff $\mathbb{T}_{\mathcal{K}} Y \models s =_0 t$, iff $\mathcal{K} \models s =_0 t$, iff $\emptyset \vdash s =_0 t \in \mathcal{E}^c_Y(\mathcal{K})$ (since $\mathcal{E}^c_Y(\mathcal{K}) = \mathcal{E}^c_Y(\mathcal{K}')$), iff $\emptyset \vdash s =_0 t \in \mathcal{E}^c_Y(\mathcal{K}')$, iff $\mathcal{K}' \models s =_0 t$, iff $\mathbb{T}_{\mathcal{K}'} Y \models s =_0 t$, iff $d^{\mathcal{K}'}(s|_{\mathcal{K}'}, t|_{\mathcal{K}'}) = 0$.

Hence, $ker(d^{\mathcal{K}}) = ker(d^{\mathcal{K}'})$ implying that $\mathbb{T}Y|_{d^{\mathcal{K}}}$ and $\mathbb{T}Y|_{d^{\mathcal{K}'}}$ are isomorphic Ω-algebras.

Similarly, we can apply Corollary 3.9 for arbitrary $s, t \in \mathbb{T}Y$ and $\epsilon \in \mathbb{Q}_+$, as we did it before for $\epsilon = 0$, and obtain: $d^{\mathcal{K}}(s|_{\mathcal{K}}, t|_{\mathcal{K}}) \leq \epsilon$ iff $\mathbb{T}_{\mathcal{K}} Y \models s =_\epsilon t$, iff $\mathcal{K} \models s =_\epsilon t$, iff $\emptyset \vdash s =_\epsilon t \in \mathcal{E}^c_Y(\mathcal{K})$, iff $\emptyset \vdash s =_\epsilon t \in \mathcal{E}^c_Y(\mathcal{K}')$, iff $\mathcal{K}' \models s =_\epsilon t$, iff $\mathbb{T}_{\mathcal{K}'} Y \models s =_\epsilon t$, iff $d^{\mathcal{K}'}(s|_{\mathcal{K}'}, t|_{\mathcal{K}'}) \leq \epsilon$; and since this is true for any $\epsilon \in \mathbb{Q}_+$, we obtain

$$d^{\mathcal{K}}(s|_{\mathcal{K}}, t|_{\mathcal{K}}) = d^{\mathcal{K}'}(s|_{\mathcal{K}'}, t|_{\mathcal{K}'}).$$

[4]Observe that $s|_{\mathcal{K}}$ has been denoted by $\langle s \rangle$ previously, when $\mathcal{K}$ was fixed. We change the notation here because we need to speak of such elements for various classes $\mathcal{K}, \mathcal{K}'$.

Hence, $\mathbb{T}Y|_{d^{\mathcal{K}}}$ and $\mathbb{T}Y|_{d^{\mathcal{K}'}}$ are isomorphic quantitative algebras implying further that $\gamma(\mathbb{T}Y)$ is isomorphic to $\gamma'(\mathbb{T}Y)$.

Now, since $\mathcal{A}'$ is the c-homomorphic image of $\gamma'(\mathbb{T}Y)$, it is also a c-homomorphic image of $\gamma(\mathbb{T}Y)$. But $\gamma(TY) \leq \mathbb{T}_{\mathcal{K}}Y$ and since $\mathcal{K}$ is a c-variety, from Lemma 3.7 we know that $\mathbb{T}_{\mathcal{K}}Y \in \mathcal{K}$, hence $\gamma(\mathbb{T}Y) \in \mathcal{K}$.

Consequently, $\mathcal{A}' \in \mathbb{H}_c(\mathcal{K})$ and since $\mathcal{K}$ is a c-variety, $\mathcal{A}' \in \mathcal{K}$, from which we conclude $\mathcal{K}' \subseteq \mathcal{K}$. ∎

Now we prove the variety theorem for QAs.

Theorem 3.11 (c-Variety Theorem): Let $\mathcal{K}$ be a class of quantitative algebras and $0 < c \leq r(\mathcal{K})$ a regular cardinal. Then, $\mathcal{K}$ is a c-equational class iff $\mathcal{K}$ is a c-variety. In particular,

1) $\mathcal{K}$ is an unconditional equational class iff it is a variety;
2) $\mathcal{K}$ is a finitary-basic equational class iff it is an $\aleph_0$-variety;
3) $\mathcal{K}$ is a basic equational class iff it is an $\aleph_1$-variety.

Proof. ($\Longrightarrow$): $\mathcal{K} = \mathbb{K}(\mathcal{U})$ for some set $\mathcal{U}$ of c-basic conditional equations. Then, $\mathbb{V}_c(\mathcal{K}) \models \mathcal{U}$ implying further that $\mathbb{V}_c(\mathcal{K}) \subseteq \mathbb{K}(\mathcal{U}) = \mathcal{K}$. Hence, $\mathbb{V}_c(\mathcal{K}) = \mathcal{K}$.

($\Longleftarrow$): this is guaranteed by Lemma 3.10. ∎

Birkhoff Theorem in perspective. Before concluding this section, we notice that our variety theorem also generalizes the original Birkhoff theorem. This is because any congruence $\cong$ on an Ω-algebra $\mathcal{A}$ can be seen as the kernel of the pseudometric $p_{\cong}$ defined by $p_{\cong}(a,b) = 0$ whenever $a \cong b$ and $p_{\cong}(a,b) = 1$ otherwise. The quotient algebra $\mathcal{A}|_{\cong}$ is a quantitative algebra. Any quantitative equational theory satisfied by $\mathcal{A}|_{\cong}$ can be axiomatized by equations involving only $=_0$ and $=_1$, since 0 and 1 are the only possible distances between its elements. However, this algebra also satisfies the equation $x =_1 y$ for any two variables x and y, because 1 is the diameter of its support. Consequently, the only non-redundant equations satisfied by such an algebra are of type $s =_0 t$, and these correspond to the equations of the form $s = t$.

IV. The Quasivariety Theorem for General Conditional Equations

In this section we study the axiomatizability of classes of quantitative algebras that can be axiomatized by conditional quantitative equations, but not necessarily by basic conditional quantitative equations. Thus, we are now looking for more relaxed types of axioms and consequently we will identify more relaxed closure conditions.

We prove that a class $\mathcal{K}$ of Ω-quantitative algebras admits an axiomatization consisting of conditional quantitative equations, whenever it is closed under isomorphisms, subalgebras and what we call subreduced products. A subreduced product is a quantitative subalgebra of (a special type of) product of elements in $\mathcal{K}$; however, while these products are always Ω-algebras, they are not always quantitative algebras. This closure condition allow us to generalize the classical quasivariety theorem that characterizes the classes of universal algebras with an axiomatization consisting of Horn clauses.

It is not trivial to see that a c-variety is closed under these operators for any regular cardinal $c > 0$ and so our quasivariety theorem extends the c-variety theorem presented in the previous section. Indeed, all isomorphisms are c-reflexive homomorphisms and since a c-variety is closed under subalgebras and products, it must be closed under subreduced products, as they are quantitative subalgebras of the product.

However, to achieve these results we had to involve and generalize concepts and results from model theory of first-order structures. This required us to restrict ourselves to the signatures Ω containing only functions of finite arity.

A. Preliminaries in Model Theory

In this subsection we recall some basic concepts and results about the model theory of first order structures.

A *first-order language* is a tuple $\mathcal{L} = (\Omega, \mathcal{R})$ where Ω is an algebraic similarity type containing functions of finite arity and $\mathcal{R}$ is a set of relation symbols of finite arity.

A *first-order structure* of type $\mathcal{L} = (\Omega, \mathcal{R})$ is a tuple $\mathcal{M} = (M, \Omega^{\mathcal{M}}, \mathcal{R}^{\mathcal{M}})$ where $(M, \Omega^{\mathcal{M}})$ is an Ω-algebra and for any relation $R : i \in \mathcal{R}$, $R^{\mathcal{M}} \subseteq M^i$.

A *morphism of first-order structures* of type $\mathcal{L} = (\Omega, \mathcal{R})$ is a map $f : (M, \Omega^{\mathcal{M}}, \mathcal{R}^{\mathcal{M}}) \to (N, \Omega^{\mathcal{N}}, \mathcal{R}^{\mathcal{N}})$ that is a homomorphism of Ω-algebras such that for any relation $R : i \in \mathcal{R}$ and $m_1, ..m_i \in M$,

$$f(\mathcal{R}^{\mathcal{M}}(m_1, .., m_i)) = \mathcal{R}^{\mathcal{N}}(f(m_1), .., f(m_i)).$$

$\mathcal{M} = (M, \Omega^{\mathcal{M}}, \mathcal{R}^{\mathcal{M}})$ is a *subobject* of $\mathcal{N} = (N, \Omega^{\mathcal{N}}, \mathcal{R}^{\mathcal{N}})$ if $(M, \Omega^{\mathcal{M}})$ is an Ω-subalgebra of $(N, \Omega^{\mathcal{N}})$ and for any $R : i \in \mathcal{R}$ and $m_1, ..m_i \in M$, $R^{\mathcal{M}}(m_1, .., m_i)$ iff $R^{\mathcal{N}}(m_1, .., m_i)$. We write $\mathcal{M} \leq \mathcal{N}$.

Equational First-Order Logic. Given a first-order structure $\mathcal{L} = (\Omega, \mathcal{R})$ and a set X of variables, let $\mathbb{T}X$ be the set of terms induced by X over Ω. The *atomic formulas* of type $\mathcal{L} = (\Omega, \mathcal{R})$ over X are expressions of the form

- $s = t$ for $s, t \in \mathbb{T}X$;
- $R(s_1, .., s_k)$ for $R : k \in \mathcal{R}$ and $s_1, .., s_k \in \mathbb{T}X$.

The set $\mathcal{L}X$ of first-order formulas of type $\mathcal{L}$ over X is the smallest collection of formulas containing the atomic formulas and closed under conjunction, negation and universal quantification $\forall x$ for $x \in X$. In addition we consider all the Boolean operators and the existential quantification.

If $\mathcal{M}$ is a structure of type $\mathcal{L}$, let $\mathcal{L}_{\mathcal{M}}$ be the first-order language obtained by adding to $\mathcal{L}$ the elements of $\mathcal{M}$ as constants.

Given a first-order formula $\phi(x_1,..,x_i,..x_k)$ in which $x_1,..,x_k \in X$ are all the free variables, we denote by $\phi(x_1,..,x_{i-1},m,x_{i+1},..x_k)$, as usual, the formula obtained by replacing all the free occurrences of x_i by $m \in \mathcal{M}$.

Satisfiability. For a closed formula $\phi \in \mathcal{L}_{\mathcal{M}}$, we define $\mathcal{M} \models \phi$ inductively on the structure of formulas as follows.

- $\mathcal{M} \models s = t$ for $s,t \in TX$ containing no variables iff $s^{\mathcal{M}} = t^{\mathcal{M}}$.
- $\mathcal{M} \models R(s_1,..,s_k)$ for $R : k \in \mathcal{R}$ and $s_1,..,s_k \in \mathbb{T}X$ containing no variables iff $\mathcal{R}^{\mathcal{M}}(s_1^{\mathcal{M}},..,s_k^{\mathcal{M}})$;
- $\mathcal{M} \models \phi \wedge \psi$ iff $\mathcal{M} \models \phi$ and $\mathcal{M} \models \psi$;
- $\mathcal{M} \models \neg\phi$ iff $\mathcal{M} \not\models \phi$;
- $\mathcal{M} \models \forall x \phi(x)$ iff $\mathcal{M} \models \phi(m)$ for any $m \in \mathcal{M}$.

The semantics of the derived operators is standard. The de Morgan laws give us semantically-equivalent prenex forms for any first-order formula.

A first-order formula is an *universal formula* if it is in prenex form and all the quantifiers are universal.

A *Horn formula* has the following prenex form

$$Q_1x_1..Q_kx_k(\phi_1(x_1,..,x_k)\wedge..\wedge\phi_j(x_1,..,x_k) \longrightarrow \phi(x_1,..,x_k)),$$

where each Q_l is a quantifier and each ϕ_l and ϕ is an atomic formula with (a subset of) the set $\{x_1,..,x_k\}$ of free variables[5].

A *universal Horn formula* is a Horn formula which is also an universal formula.

Direct Products. Given a nonempty indexed family $(\mathcal{M}_i)_{i\in I}$ of first-order structures of type $\mathcal{L} = (\Omega,\mathcal{R})$, where $\mathcal{M}_i = (M_i, \Omega^{\mathcal{M}_i}, \mathcal{R}^{\mathcal{M}_i})$, the *direct product* $\prod_{i\in I}\mathcal{M}_i$ is the $\mathcal{L}$-structure whose universe is the product set $\prod_{i\in I} M_i$ and its functions and relations are defined as follows, where $\pi_i : \prod_{i\in I}\mathcal{M}_i \longrightarrow \mathcal{A}_i$ denotes the i-th projection.

- for $f : k \in \Omega$,
  $\pi_i(f(m_1,..,m_k)) = f^{\mathcal{M}_i}(\pi_i(m_1),..,\pi_i(m_k))$;
- for $R : k \in \mathcal{R}$,
  $R(m_1,..,m_k)$ iff $R^{\mathcal{M}_i}(\pi_i(m_1),..,\pi_i(m_k))$ for all $i \in I$.

Reduced Products. Let $(\mathcal{M}_i)_{i\in I}$ be an indexed family of first-order structures of type $\mathcal{L} = (\Omega,\mathcal{R})$ and F a proper filter over I.

Consider the relation $\sim_F \subseteq \prod_{i\in I}\mathcal{M}_i \times \prod_{i\in I}\mathcal{M}_i$ s.t.

$$m \sim_F n \text{ iff } \{i \in I \mid \pi_i(m) = \pi_i(n)\} \in F.$$

[5]Some authors define a Horn formula as a conjunction of such constructs, or allow $\phi = \top$; none of these choices affect our development here.

It is known that when F is a proper filter of I, $\sim_F$ is a congruence relation with respect to the algebraic structure of $\prod_{i\in I}\mathcal{M}_i$ (see, e.g., [19, Lemma 2.2]). This allows us to define the *reduced product induced by a proper filter* F, written $\prod_{i\in I}\mathcal{M}_i|_F$, as the $\mathcal{L}$ first-order structure such that

- its universe is the set $\prod_{i\in I} M_i|_{\sim_F}$, which is the quotient of $\prod_{i\in I} M_i$ with respect to $\sim_F$; we denote by m_F the $\sim_F$-congruence class of $m \in \prod_{i\in I} M_i$;
- for $f : k \in \Omega$, $f(m_F^1,..,m_F^k) = f(m^1,..,m^k)_F$;
- for $R : k \in \mathcal{R}$, $R(m_F^1,..,m_F^k)$ iff
  $\{i \in I \mid R(\pi_i(m^1),..,\pi_i(m^k)\} \in F$.

Quasivariety Theorem. A class $\mathfrak{M}$ of $\mathcal{L}$-structures is an *elementary class* if there exists a set Φ of first-order $\mathcal{L}$-formulas such that for any $\mathcal{L}$-structure $\mathcal{M}$,

$$\mathcal{M} \in \mathfrak{M} \text{ iff } \mathcal{M} \models \Phi.$$

An elementary class is an *universal class* if it can be axiomatized by universal formulas; it is an *universal Horn class* if it can be axiomatized by universal Horn formulas.

We conclude this section with the quasivariety theorem (see, e.g., [19, Theorem 2.23]). To state it, we define a few closure operators on classes of $\mathcal{L}$-structures.

Let $\mathfrak{M}$ be an arbitrary class of $\mathcal{L}$-structures.

- $\mathbb{I}(\mathfrak{M})$ denotes the closure of $\mathfrak{M}$ under isomorphisms;
- $\mathbb{S}(\mathfrak{M})$ denotes the closure of $\mathfrak{M}$ under subobjects;
- $\mathbb{P}(\mathfrak{M})$ denotes the closure of $\mathfrak{M}$ under direct products;
- $\mathbb{P}_R(\mathfrak{M})$ denotes the closure of $\mathfrak{M}$ under reduced products.

Theorem 4.1 (Quasivariety Theorem): Let $\mathfrak{M}$ be a class of $\mathcal{L}$-structures. The following statements are equivalent.

1) $\mathfrak{M}$ is a universal Horn class;
2) $\mathfrak{M}$ is closed under $\mathbb{I}$, $\mathbb{S}$ and $\mathbb{P}_R$;
3) $\mathfrak{M} = \mathbb{ISP}_R(\mathfrak{M}')$ for some class $\mathfrak{M}'$ of $\mathcal{L}$-structures.

B. Quantitative First-Order Structures

In this subsection we identify a class of first-order structures, the quantitative first-order structures (QFOs), which are the first-order counterparts of the quantitative algebras.

Given a first-order structure $\mathcal{M} = (M,\Omega^{\mathcal{M}},\mathcal{R}^{\mathcal{M}})$ of type $(\Omega,\mathcal{R})$, $f : k \in \Omega$ and $R : l \in \mathcal{R}$, let $f(R^{\mathcal{M}}) \subseteq M^l$ be the set of the tuples $(f(m_1^1,..,m_k^1),..,f(m_1^l,..,m_k^l))$ such that for each $i = 1,..,k$, $(m_i^1,..,m_i^l) \in R^{\mathcal{M}}$.

Definition 4.2: An *Ω-quantitative first-order structure* for a signature Ω is a first-order structure $\mathcal{M} = (M, \Omega^{\mathcal{M}}, \equiv^{\mathcal{M}})$ of type $(\Omega, \equiv)$, where $\equiv = \{=_\epsilon | \ \epsilon \in \mathbb{Q}_+\}$, that satisfies the following axioms for any $\epsilon, \delta \in \mathbb{Q}_+$

1) $=_0^{\mathcal{M}}$ is the identity on $\mathcal{M}$;
2) $=_\epsilon^{\mathcal{M}}$ is symmetric;
3) $=_\epsilon^{\mathcal{M}} \circ =_\delta^{\mathcal{M}} \subseteq =_{\epsilon+\delta}^{\mathcal{M}}$;
4) $=_\epsilon^{\mathcal{M}} \subseteq =_{\epsilon+\delta}^{\mathcal{M}}$;
5) for any $f : k \in \Omega$, $f(=_\epsilon^{\mathcal{M}}) \subseteq =_\epsilon^{\mathcal{M}}$;
6) for any δ, $\bigcap_{\epsilon>\delta} =_\epsilon \subseteq =_\delta$;

Theorem 4.3: (i) Any quantitative algebra $\mathcal{A} = (A, \Omega, d)$ defines uniquely a quantitative first-order structure by

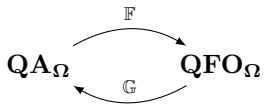
$$a =_\epsilon b \ \text{ iff } \ d(a,b) \leq \epsilon.$$

(ii) Any quantitative first-order structure $\mathcal{M} = (M, \Omega^{\mathcal{M}}, \equiv^{\mathcal{M}})$ defines uniquely a quantitative algebra by letting

$$d(m,n) = \inf\{\epsilon \in \mathbb{Q}_+ \mid m =_\epsilon n\}.$$

These define an isomorphism between the category of Ω-quantitative algebras and Ω-quantitative first-order structures.

Let $\mathbf{QA_\Omega}$ be the category of Ω-quantitative algebras and $\mathbf{QFO_\Omega}$ the category of Ω-quantitative first-order structures. Theorem 4.3 defines two functors $\mathbb{F}$ and $\mathbb{G}$ that act as identities on morphisms, which define an isomorphism of categories as in the figure below.



We already know that the subobjects and the direct products of quantitative first-order structures are first-order structures. However, since the isomorphisms of categories preserve limits and colimits, we can prove that the subobjects and the direct products of quantitative first-order structures are, in fact, quantitative first-order structures, i.e., they satisfy the axioms (1)-(6) of Definition 4.2, as the next lemma establishes.

Lemma 4.4: I. If $\mathcal{M}, \mathcal{N}$ are two Ω-QFOs s.t. $\mathcal{M} \leq \mathcal{N}$, then

$$\mathbb{G}\mathcal{M} \leq \mathbb{G}\mathcal{N}.$$

II. If $(\mathcal{M}_i)_{i\in I}$ is a family of Ω-QFOs, then

$$\mathbb{G}(\prod_{i\in I} \mathcal{M}_i) = \prod_{i\in I} \mathbb{G}\mathcal{M}_i.$$

III. If $\mathcal{A}, \mathcal{B}$ are two Ω-QAs such that $\mathcal{A} \leq \mathcal{B}$, then

$$\mathbb{F}(\mathcal{A}) \leq \mathbb{F}\mathcal{B}.$$

IV. If $(\mathcal{A}_i)_{i\in I}$ is a family of Ω-QAs, then

$$\mathbb{F}(\prod_{i\in I} \mathcal{A}_i) = \prod_{i\in I} \mathbb{F}\mathcal{A}_i.$$

C. Subreduced Products of QFOs

Given an indexed family $(\mathcal{M}_i)_{i\in I}$ of Ω-QFOs and a proper filter F on I, we can construct, as before, the reduced product $(\mathcal{M}_i)_{i\in I}|_F$ of first-order structures, which is a first-order structure. But it is not guaranteed that it satisfies the axioms in Definition 4.2. From the definition of the reduced product we obtain a first-order structure $(\mathcal{M}_i)_{i\in I}|_F$ that enjoys the following property for any $\epsilon \in \mathbb{Q}_+$.

$$m_F =_\epsilon n_F \ \text{ iff } \ \{i \in I \mid \pi_i(m) =_\epsilon \pi_i(n)\} \in F.$$

Note that if for all $i \in I$, $\mathcal{M}_i$ satisfies the axioms (1)-(5) from Definition 4.2, then $(\mathcal{M}_i)_{i\in I}|_F$ satisfies them as well.

For instance, we can verify the condition (3): suppose that $m_F =_\epsilon n_F$ and $n_F =_\delta u_F$. Hence,

$$\{i \in I \mid \pi_i(m) =_\epsilon \pi_i(n)\}, \{i \in I \mid \pi_i(n) =_\delta \pi_i(u)\} \in F.$$

Since F is a filter, it is closed under intersection, so

$$\{i \in I \mid \pi_i(m) =_\epsilon \pi_i(n) \text{ and } \pi_i(n) =_\delta \pi_i(u)\} \in F.$$

Now, axiom (3) guarantees that

$$\{i \in I \mid \pi_i(m) =_\epsilon \pi_i(n) \text{ and } \pi_i(n) =_\delta \pi_i(u)\}$$
$$\subseteq \{i \in I \mid \pi_i(m) =_{\epsilon+\delta} \pi_i(u)\}$$

and since F is closed under supersets,

$$\{i \in I \mid \pi_i(m) =_{\epsilon+\delta} \pi_i(u)\} \in F.$$

Similarly, one can verify each of the axioms but (6). This is because axiom (6) requires that any reduced product has the property that for any $\delta \in \mathbb{Q}_+$,

$$\{i \in I \mid \pi_i(m) =_\epsilon \pi_i(n)\} \in F \text{ for all } \epsilon > \delta$$

implies

$$\{i \in I \mid \pi_i(m) =_\delta \pi_i(n)\} \in F.$$

This is a very strong condition not necessarily satisfied by a filter or an ultrafilter. It is, for instance, satisfied by the filters and ultrafilters closed under countable intersections, but the existence of such filters requires measurable cardinals (see for instance [20] for a detailed discussion).

Hence, while the reduced products of quantitative first-order structures can always be defined as first-order structures, they are not always quantitative first-order structures, since they might not satisfy axiom (6) in Definition 4.2. Therefore, taking reduced products and ultraproducts are not internal operations over the class of quantitative first-order structures of the same type, even if they are internal operations over the larger class of first-order structures of the same type. This observation motivates our next definition.

Definition 4.5 (Subreduced Products): Given an indexed family $(\mathcal{M}_i)_{i\in I}$ of quantitative first-order structures and a proper filter F on I, a *subreduced product* of this family induced

by F is a subobject $\mathcal{M}$ of the first-order structure $\prod_{i\in I}\mathcal{M}_i|_F$ such that $\mathcal{M}$ is a quantitative first-order structure.

Given a class $\mathfrak{M}$ of quantitative first-order structures of the same type, the closure of $\mathfrak{M}$ under subreduced products is denoted by $\mathbb{P}_{SR}(\mathfrak{M})$.

With this concept in hand we can generalize the quasivariety theorem for first-order structures to get a similar result for classes of QFOs that can be properly axiomatized.

Theorem 4.6 (Quasivariety Theorem for QFOs): Let $\mathfrak{M}$ be a class of Ω-quantitative first-order structures. Then, the following statements are equivalent.

1) $\mathfrak{M}$ is an universal Horn class;
2) $\mathfrak{M}$ is closed under $\mathbb{I}, \mathbb{S}$ and $\mathbb{P}_{SR}$;
3) $\mathfrak{M} = \mathbb{ISP}_{SR}(\mathfrak{M}_0)$ for some class $\mathfrak{M}_0$ of Ω-quantitative first-order structures.

Proof. $(1) \Longrightarrow (2)$: let $\mathfrak{M}$ be an universal Horn class of Ω-QFOs. Then there exists an universal Horn class of Ω-first-order structures $\mathfrak{M}'$ that satisfies the same first-order theory $\mathcal{T}$ that $\mathfrak{M}$ does. If we denote the class of Ω-quantitative first-order theories by $\mathbf{QFO_\Omega}$, we have

$$\mathfrak{M} = \mathfrak{M}' \cap \mathbf{QFO_\Omega}.$$

Applying Theorem 4.1, $\mathfrak{M}'$ is closed under $\mathbb{I}$, $\mathbb{S}$ and $\mathbb{P}_R$.

Obviously, $\mathfrak{M}$ is closed under $\mathbb{I}$, since isomorphic first-order structures satisfy the same first-order sentences. $\mathfrak{M}$ is also closed under $\mathbb{S}$, as Lemma 4.4 guarantees.

Let $\{\mathcal{M}_i \mid i \in I\} \subseteq \mathfrak{M}$ and F a proper filter of I.
Let $\mathcal{M} \leq \prod_{i\in I}\mathcal{M}_i|_F$ such that $\mathcal{M} \in \mathbf{QFO_\Omega}$.
Since $\{\mathcal{M}_i \mid i \in I\} \subseteq \mathfrak{M}'$ and $\mathbb{P}_R(\mathfrak{M}') = \mathfrak{M}'$, we get that $\prod_{i\in I}\mathcal{M}_i|_F \in \mathfrak{M}'$. Hence, $\mathcal{M} \in \mathbb{S}(\mathfrak{M}') = \mathfrak{M}'$. And further, $\mathcal{M} \in \mathfrak{M}' \cap \mathbf{QFO_\Omega} = \mathfrak{M}$. In conclusion, $\mathfrak{M}$ is also closed under $\mathbb{P}_{SR}$.

$(2) \Longrightarrow (3)$: since $\mathfrak{M}$ is closed under $\mathbb{I}$, $\mathbb{S}$ and $\mathbb{P}_{SR}$,

$$\mathfrak{M} = \mathbb{ISP}_{SR}(\mathfrak{M}).$$

$(3) \Longrightarrow (1)$: suppose that $\mathfrak{M} = \mathbb{ISP}_{SR}(\mathfrak{M}_0)$ for some class $\mathfrak{M}_0$ of quantitative first-order structures.
Let $\mathfrak{M}' = \mathbb{ISP}_R(\mathfrak{M})$. Applying Theorem 4.1, $\mathfrak{M}'$ is a universal Horn class of first-order structures. We prove now that $\mathfrak{M} = \mathfrak{M}' \cap \mathbf{QFO_\Omega}$.

Let $\mathcal{M} \in \mathfrak{M}' \cap \mathbf{QFO_\Omega}$. Then, $\mathcal{M}$ is isomorphic to some $\mathcal{N} \leq \prod_{i\in I}\mathcal{M}_i|_F$ for some $(\mathcal{M}_i)_{i\in I} \subseteq \mathfrak{M}$ and a proper filter F of I, and $\mathcal{N} \in \mathbf{QFO_\Omega}$. Hence, $\mathcal{M} \in \mathbb{ISP}_{SR}(\mathfrak{M}) = \mathfrak{M}$. And this concludes that $\mathfrak{M}' \cap \mathbf{QFO_\Omega} \subseteq \mathfrak{M}$.

Since we have trivially $\mathfrak{M} \subseteq \mathfrak{M}' \cap \mathbf{QFO_\Omega}$ from the way we constructed $\mathfrak{M}'$, we get that $\mathfrak{M} = \mathfrak{M}' \cap \mathbf{QFO_\Omega}$.

Now, since $\mathfrak{M}'$ is a universal Horn class of first-order structures, we obtain that $\mathfrak{M}$ is a universal Horn class of quantitative first-order structures. ∎

D. Subreduced Products of Quantitative Algebras

Theorem 4.6 characterizes classes of Ω-QFOs as universal Horn classes. In this subsection we convert this result into a result regarding the axiomatizability of classes of QAs.

For the beginning, we note an equivalence between the conditional equations interpreted over the class of quantitative algebras and the universal Horn formulas interpreted over the class of quantitative first-order structures. This relies on the fact that a quantitative equation of type $s =_\epsilon t$ is also an atomic formula in the corresponding quantitative first-order language and vice versa. The following theorem establishes this correspondence.

Theorem 4.7: Let $\phi_1(x_1,..,x_k)\ldots,\phi_l(x_1,..,x_k)$ and $\psi(x_1,..,x_k)$ be Ω-quantitative first-order atomic formulas depending of the variables $x_1,..,x_k \in X$.

I. If $\mathcal{M}$ is an Ω-quantitative first-order structure, then the following are equivalent

$$\mathcal{M} \models \forall x_1..\forall x_k(\phi_1(x_1,..x_k)\wedge..\wedge\phi_l(x_1,..x_k) \rightarrow \psi(x_1,..x_k)),$$
$$\{\phi_1(x_1,..,x_k) \wedge .. \wedge \phi_l(x_1,..,x_k)\} \models_{\mathbb{G}\mathcal{M}} \psi(x_1,..,x_k).$$

II. If $\mathcal{A}$ is an Ω-quantitative algebra, then the following are equivalent

$$\{\phi_1(x_1,..,x_k) \wedge .. \wedge \phi_l(x_1,..,x_k)\} \models_{\mathcal{A}} \psi(x_1,..,x_k),$$
$$\mathbb{F}\mathcal{A} \models \forall x_1..\forall x_k(\phi_1(x_1,..x_k)\wedge..\wedge\phi_n(x_1,..x_k) \rightarrow \psi(x_1,..x_k)).$$

As in the case of quantitative first-order structures, the concept of subdirect product of an indexed family of quantitative algebras for a given proper filter is not always defined. The following definition reflects this issue.

Definition 4.8: Let $(\mathcal{A}_i)_{i\in I}$ be an indexed family of quantitative algebras and F a proper filter of I. A *subreduced product* of this family induced by F is a quantitative algebra $\mathcal{A}$ s.t.

$$\mathbb{F}\mathcal{A} \leq \prod_{i\in I}(\mathbb{F}\mathcal{A}_i)|_F.$$

Let $\mathbb{P}_{SR}(\mathcal{K})$ be the closure of the class $\mathcal{K}$ of quantitative algebras under subreduced products. Now we can provide the analogue of Theorem 4.6 for quantitative algebras as a direct consequence of Theorems 4.6,4.3 and 4.7.

Theorem 4.9: Let $\mathcal{K}$ be a class of Ω-quantitative algebras. The following statements are equivalent.

1) $\mathcal{K}$ is a conditional equational class;
2) $\mathcal{K}$ is closed under $\mathbb{I}$, $\mathbb{S}$ and $\mathbb{P}_{SR}$;
3) $\mathcal{K} = \mathbb{ISP}_{SR}(\mathcal{K}_0)$ for some class $\mathcal{K}_0$ of Ω-quantitative algebras.

E. Going further: Complete Quantitative Algebras

The proof pattern that we developed to prove the quasivariety theorem for QFOs, Theorem 4.6, is actually more general and it could be used to provide similar theorems for other classes of quantitative algebras. In [1] we have shown that the class of quantitative algebras defined over complete metric spaces plays a central in the theory of quantitative algebras. For this reason we will briefly show how a quasivariety theorem could be done for complete metric spaces.

We call a quantitative algebra over a complete metric space a *complete quantitative algebra.*

If we follow the intuition behind Theorem 4.3, we will discover that we can define the concept of complete quantitative first-order structure as being a quantitative first-order structure for which the corresponding quantitative algebra through the functor $\mathbb{G}$ is a complete quantitative algebra. In fact, the completeness condition can be encoded by an infinitary axiom to be added to the conditions (1)-(6) in Definition 4.2, namely the axiom that requires that any Cauchy sequence has a limit. Let us call it the *Cauchy condition*.

We will be then able to prove that the category of Ω-complete quantitative algebras is isomorphic to the category of Ω-complete quantitative first-order structures.

Further we can define, given a class $\mathbb{M}$ of Ω-complete quantitative first-order structures, the concept of complete-subreduced product: given an indexed family $(\mathcal{M}_i)_{i\in I}$ of Ω-complete quantitative first-order structures, a complete-subreduced product is any Ω-complete quantitative first-order structure that is a subobject of the reduced product $\prod_{i\in I} \mathcal{M}_i|_F$ for some proper filter F of I.

With this in hand, one can redo the proof of Theorem 4.6 in these new settings and should obtain a quasivariety theorem for complete QFOs.

V. Conclusions

In this paper we have established the fundamental results on the axiomatizability of classes of quantitative algebras by equations, conditional equations and Horn clauses. These results required substantial new techniques. We have not put this work into a fully categorical framework such as described in [14], [9], [13], [15]. We are actively working on understanding these connections and also the connections with enriched Lawvere theories. There is also much to understand when looking at other approaches to quantitative reasoning, for example the work of Jacobs and his group [21].

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References

[1] R. Mardare, P. Panangaden, and G. Plotkin, "Quantitative algebraic reasoning," in *Proceedings of the 31st Annual ACM-IEEE Symposium on Logic in Computer Science*, 2016, pp. 700–709.

[2] A. Pfeffer, *Practical Probabilistic Programming*. Manning, 2016.

[3] D. Roy, "Probabilistic-programming.org," Web Site.

[4] M. H. Stone, "Postulates for the barycentric calculus," *Annali di Matematica Pura ed Applicata*, vol. 29, no. 1, pp. 25–30, December 1949.

[5] G. Bacci, G. Bacci, K. G. Larsen, and R. Mardare, "Complete axiomatization for the bisimilarity distance on markov chains," in *Proceedings of CONCUR*, 2016, pp. 21:1–21:14.

[6] F. van Breugel, C. Hermida, M. Makkai, and J. Worrell, "Recursively defined metric spaces without contraction," *Theoretical Computer Science*, vol. 380, no. 1/2, pp. 143–163, June 2007.

[7] J. Adámek, S. Milius, and L. S. Moss, "On finitary functors and their presentations," in *Coalgebraic Methods in Computer Science*. Springer, 2012, pp. 51–70.

[8] G. Birkhoff, "On the structure of abstract algebras," *Proc. Cambridge Philos. Soc.*, vol. 31, pp. 433–454, 1935.

[9] J. Adámek and H.-E. Porst, "Algebraic theories of quasivarieties," *Journal of Algebra*, vol. 208, no. 2, pp. 379–398, 1998.

[10] ——, "From varieties of algebras to covarieties of coalgebras," *Electronic Notes in Theoretical Computer Science*, vol. 44, no. 1, pp. 27–46, 2001.

[11] R. Goldblatt, "What is the coalgebraic analogue of Birkhoff's variety theorem?" *Theoretical Computer Science*, vol. 266, pp. 853–886, 2001.

[12] B. Banaschewski and H. Herrlich, "Subcategories defined by implications," *Houston Journal of Mathematics*, vol. 2(2), pp. 149–171, 1976.

[13] J. Adámek, J. Rosickỳ, and E. M. Vitale, *Algebraic theories: a categorical introduction to general algebra*. Cambridge University Press, 2010, vol. 184.

[14] M. Barr, "Functorial semantics and HSP type theorems," *algebra universalis*, vol. 31, no. 2, pp. 223–251, 1994.

[15] E. Manes, *Algebraic theories*. Springer-Verlag, 2012, vol. 26.

[16] E. Robinson, "Variations on algebra:monadicity and generalisations of equational theories," *Formal Aspects of Computing*, vol. 13, pp. 308–326, 2002.

[17] N. Weaver, "Quasi-varieties of metric algebras," *Algebra Universalis*, vol. 33(1), pp. 1–9, 1995.

[18] I. B. Yaacov, A. Berenstein, C. W. Henson, and A. Usvyatsov, *Model theory with applications to algebra and analysis. Vol. 2*, ser. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, 2008, no. 350, ch. Model theory for metric structures, pp. 315–427.

[19] S. Burris and H. P. Sankappanavar, *A course in universal algebra*, ser. Graduate Texts in Mathematics. Springer-Verlag, 1981, no. 78.

[20] C. C. Chang and H. J. Keisler, *Model Theory*. North-Holland, 1992.

[21] K. Cho, B. Jacobs, B. Westerbaan, and A. Westerbaan, "An introduction to effectus theory," Dec 2015, avialable on the Arxiv as 1512.05813.