Solution to Q5

[Question 5: 10 points] Prove by induction that the insertion sort program shown in class works correctly. You can assume that the insert program works correctly but you need to write carefully what it means for this program to be correct before you proceed with your main proof.

Solution Let us represent a list as

\[ [x_1; x_2; x_3; \ldots ; x_n] \]

and the empty list as \[]. The insertion sort program is the following:

```ocaml
let rec insert n lst =
  match lst with
  | [] -> [n]
  | x :: xs -> if (n < x) then n:: lst else x::(insert n xs)

let rec isort lst =
  match lst with
  | [] -> []
  | x :: xs -> insert x (isort xs)
```

We get to assume that insert works correctly. This means that if \( l \) is a sorted list and if \( x \) is an item then \( \text{insert}x,l \) is sorted and will contain \( x \) and all the elements that were in \( l \) and nothing else.

Now we claim that if \( l \) is a list then \( \text{isort}(l) \) will contain all the elements of \( l \) and nothing else and will be sorted. We prove this by induction on the length of the list.

Base case: \( l = [] \). The result is \( [] \) and clearly satisfies the statement.

Inductive assumption: The claim is true for all lists of length \( n \). Induction step: we prove the claim for any list of length \( n+1 \). Let us write a list of length \( n+1 \) as \( [x_0; x_1; \ldots ; x_n] \). 


According to the program we have $x = x_0$ and $xs = [x_1; \ldots, x_n]$. The length of $xs$ is $n$ hence $isort$ will correctly produce a sorted version of the list with no extraneous elements. By our assumption $insert_{x_0}isort(xs)$ will produce a list with $x_0$ and with all the elements in $xs$ and the result will be sorted and there will be no extraneous elements. Thus $isort$ works correctly on lists of length $n + 1$.

Remarks for the graders: I have written this more carefully than I expect the students to. The carefully stated assumption about insert will be a common omission. Take 1 point off for not stating that that the result of insert is sorted and 1 point off for not stating that all the elements of the original list are in the result. Do not penalize them for failing to say that there are no extraneous elements. If they say nothing about insert they lose 2 marks of course. If they “unwind” the definition of isort and write .... give them no more than a total of 2 points. They have completely missed the point of an induction proof.