COMP 330 - Fall 2010 - Assignment 4

Due 8:00 pm Nov 3, 2010

General rules: In solving these questions you may consult books but you may not consult with each other. There are in total 110 points, but your grade will be considered out of 100. You should drop your solutions in the assignment drop-off box located in the Trottier Building on the 3rd floor left of the elevators.

1. (25 Points) Let M_1 and M_2 be two Turing machines. Consider the following Turing machine:

   On input w:

   - Step 1: Run M_1 on w. If M_1 accepts w, then accept.
   - Step 2: Run M_2 on w. If M_2 accepts w, then accept.

   What is the language of this Turing Machine? Explain.

2. (25 Points) Is the following language decidable[1]?

$$L = \{\langle M \rangle \mid M = (\{1, 2, \ldots, 100\}, \{0, 1\}, \{0, 1, \sqcup\}, \delta, 1, 2, 3) \text{ is a decider}\}.$$

3. A polynomial is called *non-negative*, if it is non-negative for every assignment of real numbers to its variables. For example $x_1^2 + x_2^2 - 2x_1x_2$ is non-negative because we can express it as $(x_1 - x_2)^2$.

   (a) (15 Points) Show that if a multi-variate polynomial p is <u>not</u> *non-negative*, then there is an assignment of *rational numbers* to its variables that makes it negative.

   (b) (15 Points) Use the previous part to show that the following language is recursively enumerable (Turing recognizable):

$$L = \{\langle p \rangle \mid p \text{ is a multi-variate polynomial with integer coefficients, and } p \text{ is } \underline{\text{not}} \text{ non-negative.}\}$$

   (c) (30) Consider a multi-variate polynomial p with integer coefficients (e.g. $x_1^5 x_2 + x_3 x_4 - x_1^2$). It is known that if p is *non-negative*, then there exists an integer k and polynomials $q_1, \ldots, q_k$ and r with integer coefficients such that $p = \left(\frac{q_1}{r}\right)^2 + \ldots + \left(\frac{q_k}{r}\right)^2$. Use this fact (and the previous part) to show that the following language is <u>decidable</u>:

$$\{\langle p \rangle \mid p \text{ is a } \underline{\text{non-negative}} \text{ multi-variate polynomial with integer coefficients}\}.$$

[1] Recall that a TM is formally defined as $(Q, \Sigma, \Gamma, \delta, q_{\text{start}}, q_{\text{accept}}, q_{\text{reject}})$. A Turing Machine is called a decider if it halts on every input.