COMP 330 - Fall 2010 - Assignment 2

Due 8:00 pm Oct 15, 2010

General rules: In solving this you may consult books and you may also consult with each other, but you must each find and write your own solution. In each problem list the people you consulted. This list will not affect your grade. There are in total 115 points, but your grade will be considered out of 100. You should drop your solutions in the assignment drop-off box located in the Trottier Building on the 3rd floor left of the elevators. Check the website of the course, "http://www.cs.mcgill.ca/~hatami/comp330" for possible corrections.

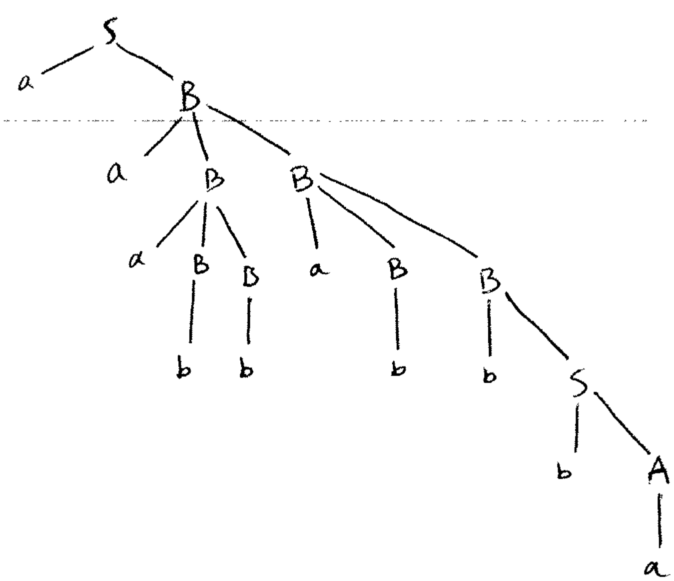
1. (a) (5 points) Find a left-most derivation for *aaabbabbba* in the following context-free grammar:

$$\begin{aligned} S &\to aB \mid bA \\ A &\to a \mid aS \mid bAA \\ B &\to b \mid bS \mid aBB \end{aligned}$$

Answer:

$$\begin{aligned} S &\Rightarrow aB \Rightarrow aaBB \Rightarrow aaaBBB \Rightarrow aaabBB \Rightarrow aaabbB \Rightarrow aaabbaBB \\ &\Rightarrow aaabbabB \Rightarrow aaabbabbS \Rightarrow aaabbabbbA \Rightarrow aaabbabbba \end{aligned}$$

(b) (5 points) Draw the corresponding parse-tree of your left-most derivation.



2. (5 Points) Show that each prefix[1] of every word in the language of the following context-free grammar has at least as many 0's as 1's.

$$S \rightarrow 0S \mid 0S1S \mid \varepsilon$$

Answer: Let L be the corresponding language. We prove this by induction on m, the size of the word. The base of induction trivially holds for $m = 0$. Now assume that the statement holds for every word of size less than some $m > 0$ (Induction hypothesis). We want to prove the statement for m.

Let $w \in L$ be a word of length m, and consider a left-most derivation for w. That is

$$S \Rightarrow \ldots \Rightarrow w.$$

We consider the three cases:

- The first rule that is applied is $S \Rightarrow \varepsilon$: Then $w = \varepsilon$, and we can verify it immediately.
- The first rule that is applied is $S \Rightarrow 0S$: Then $w = 0u$, where $u \in L$. By induction hypothesis, every prefix of u has at least as many 0's as 1's. Adding a 0 to the beginning of u certainly will not violate this condition. So every prefix of $w = 0u$ also has as many 0's as 1's.
- The first rule that is applied is $S \Rightarrow 0S1S$: Then $w = 0u1v$, where $u, v \in L$. By induction hypothesis, prefixes of u and v have at least as many 0's as 1's. Consider a prefix of w. The last letter of this prefix is either the first 0 in $0u1v$, some letter in the u part, the 1 that appears immediately after u, or a letter in v. In each case, since prefixes of u and v have at least as many 0's as 1's (by induction hypothesis), we can easily see that the same statement holds for this prefix of w too.

3. (30 points) For each one of the following languages give a proof that it is or is not regular.

(a)

$$\{0^m 1^n \mid m \geq 5 \text{ and } n \geq 0\}.$$

Answer: It is regular: We can express it as $0^5 0^* 1^*$.

(b)

$$\{0^m 1^n \mid m \geq n^2\}.$$

Answer: It is not regular. Suppose to the contrary that it is regular. Consider the pumping constant $p > 0$ and set $w = 0^{p^2} 1^p$. By pumping lemma we can decompose it as $w = xyz$, where

- $|xy| \leq p$ which here implies that $y \in 0^*$ and $|y| \leq p$;
- $|y| > 0$;
- $xy^i z$ is in the language for every $i \geq 0$.

Set $i = 0$. Note that $xy^i z = xz = 0^{p^2 - |y|} 1^p \notin \{0^m 1^n \mid m \geq n^2\}$ which is a contradiction.

(c) The set of strings in $\{0,1\}^*$ which are not of the form ww for some $w \in \{0,1\}^*$.

Answer: It is not regular. Its complement is $\{ww \mid w \in \{0,1\}^*\}$ which (as we saw in the class) by applying pumping lemma to $0^p 1 0^p 1$ can be shown that is not regular. Since regular languages are closed under complementing, the language in the question is not regular.

[1] A prefix is a substring that starts from the beginning of the word.

(d)

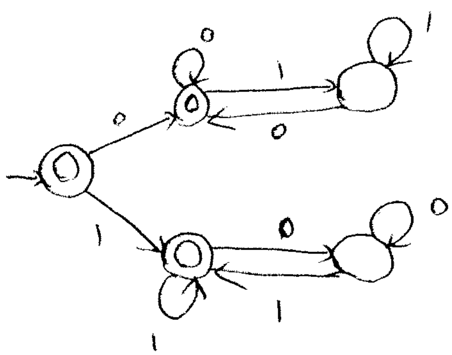
$$\left\{ 0^{\lfloor\sqrt{n}\rfloor} \,\middle|\, n = 0, 1, 2, \ldots \right\},$$

where for a real number x, $\lfloor x \rfloor$ denotes the largest integer that is less than or equal to x.
Answer: Since $\lfloor\sqrt{k^2}\rfloor = k$, we have that

$$\left\{ 0^{\lfloor\sqrt{n}\rfloor} \,\middle|\, n = 0, 1, 2, \ldots \right\} = \left\{ 0^k \,\middle|\, k = 0, 1, 2, \ldots \right\},$$

which is a regular language as it can be expressed as 0^*.

(e) The set of all strings $w \in \{0,1\}^*$ such that the number of occurrences of the substring 01 in w is equal to the number of occurrences of 10 as a substring.
Answer: It is regular as it can be accepted by the following NFA:



(f) The first two Fibonacci numbers are 0 and 1, and each subsequent number is the sum of the previous two: $0, 1, 1, 2, 3, 5, 8, 13, \ldots$. Now the language in question is

$$\{0^n | n \text{ is a Fibonacci number}\}.$$

Answer: It is not regular. Denote the k'th Fibonacci number by f_k. Suppose to the contrary that it is regular. Consider the pumping constant $p > 0$. Pick a number k so that $f_k > p$. Set $w = 0^{f_{k+1}}$ By pumping lemma we can decompose it as $w = xyz$, where

- $|xy| \leq p$;
- $|y| > 0$;
- xy^iz is in the language for every $i \geq 0$.

Set $i = 2$. Note that $xy^2z = xz = 0^{f_{k+1}+|y|}$. But $f_{k+1} < f_{k+1} + |y| \leq f_{k+1} + p < f_{k+1} + f_k \leq f_{k+2}$ which shows that $f_{k+1} + |y|$ is not a Fibonacci number and hence xy^2z does not belong to the language, and this is a contradiction.

4. (20 points) For each one of the following languages construct a context-free grammar that generates that language:

(a)

$$\{0,1\}^*.$$

Answer:

$$A \rightarrow 0A \mid 1A \mid \varepsilon$$

(b)

$$\{0^m 1^n \mid m \geq n \text{ and } m - n \text{ is even}\}.$$

Answer:

$$A \rightarrow 00A \mid 0A1 \mid \varepsilon$$

(c) The complement of $\{0^n 1^n \mid n \geq 0\}$ over the alphabet $\{0, 1\}$.
Answer:

$$\begin{aligned} A &\rightarrow B10B \mid 0A1 \mid C \mid D \mid \varepsilon \\ B &\rightarrow 0B \mid 1B \mid \varepsilon \\ C &\rightarrow 0C \mid 0 \\ D &\rightarrow 1D \mid 1 \end{aligned}$$

(d) The set of strings in $\{0, 1\}^*$ which are <u>not</u> palindromes:

$$\{w \in \{0, 1\}^* \mid w \neq w^R\}.$$

Answer:

$$\begin{aligned} A &\rightarrow 0A0 \mid 1A1 \mid 0B1 \mid 1B0 \\ B &\rightarrow 0B \mid 1B \mid \varepsilon \end{aligned}$$

5. (10 points) Show that the language of the grammar $S \rightarrow 0S1 \mid 1S0 \mid SS \mid \varepsilon$ is

$$\{w \in \{0, 1\}^* \mid w \text{ contains the same number of zeros and ones}\}.$$

Answer: Define an auxiliary function f as

$$f(w) = (\text{number of 0's in } w) - (\text{number of 1's in } w).$$

Note that the language in question is

$$\{w \in \{0, 1\}^* \mid f(w) = 0\}.$$

One direction of the question is easy. We only prove the difficult direction which says that if w contains the same number of zeros and ones, then it can be generated by $S \rightarrow 0S1 \mid 1S0 \mid SS \mid \varepsilon$.

We prove this by induction on the size of w. The base case where $|w| = 0$ is trivial. *Induction hypothesis:* If a string has the same number of zeros and ones, and its length is less than m, then it can be generated by $S \rightarrow 0S1 \mid 1S0 \mid SS \mid \varepsilon$.

Consider w with the same number of zeros and ones, and $|w| = m$. Hence $f(w) = 0$. Consider all possible ways of splitting $w = xy$.

- **There is a splitting with** $f(x) = f(y) = 0$: In this case by induction hypothesis it is possible to generate both x and y by $S \to 0S1 \mid 1S0 \mid SS \mid \varepsilon$. In other words $S \Rightarrow^* x$ and $S \Rightarrow^* y$. Then we can use $S \Rightarrow SS$ to generate $w = xy$.
- $f(x) > 0$ **for every such splitting**: In this case w starts with a 0 (taking $x = w_1$ shows this) and ends with a 1 (taking $y = w_m$ shows this). So we can use $w \Rightarrow 0S1$ to generate w.
- $f(x) < 0$ **for every such splitting**: In this case w starts with a 1 (taking $x = w_1$ shows this) and ends with a 1 (taking $y = w_m$ shows this). So we can use $w \Rightarrow 1S0$ to generate w.
- Note that there are not other cases, as if f wants to change signs then we find an splitting $w = xy$ with $f(x) = f(y) = 0$. This is the first case that we considered.

6. (10 Points) Use the equivalence of context-free grammars and push-down automata to show that if A and B are regular languages, then $\{xy | x \in A, y \in B, |x| = |y|\}$ is context-free.

   Answer: Let M_1 and M_2 be NFA's accepting A and B respectively. We can modify M_1 and M_2, if necessary, and assume that each one of them has at most one accept state. Recall that in the class we constructed an NFA N which was accepting $\{xy | x \in A, y \in B\}$. Here we construct a PDA Which is similar to N but with the difference that in the M_1 part it pushes a symbol to the stack every time that it reads a letter from the input, and in the M_2 part it pops a letter from the input every time that it reads a letter. Before accepting we make sure that the stack is empty.

7. (10 Points) Let L be an *infinite* regular language over the single letter alphabet $\Sigma = \{0\}$. For every integer m, let
$$L_m = \{w \in L \mid |w| \leq m\}$$
   be the set of the strings of length at most m in L. Show that there is a real number $c > 0$ and an integer $M > 0$ such that for every $m \geq M$, we have $\frac{|L_m|}{m} > c$.

   Answer: Let p be the pumping constant of L. Since L is infinite, there is a word w with $|w| > p$. Then by pumping lemma we can split $w = xyz$ such that

   (1) $|y| \leq |xy| \leq p$.
   (2) $|y| > 0$.
   (3) $xy^iz \in L$ for every $i \geq 0$.

   Note that $xy^iz = 0^{|w|+(i-1)|y|}$. So the words $0^{|w|}, 0^{|w|+|y|}, 0^{|w|+2|y|}, 0^{|w|+3|y|}, \ldots$ all belong to L. Set $M = 2|w|$ and note that since $|y| \leq p$, for every $m \geq M$, we have
$$|L_m| \geq \frac{m - |w|}{p} \geq \frac{m - (m/2)}{p} \geq \frac{m}{2p}.$$

Hence we can take $c = 2p$.

8. For a positive integer m, a languages L over $\{0, 1\}$ is called *m-bounded*, if the length of every word in L is at most m.

   (a) (5 Points) How many m-bounded languages are there?

   Answer: For every $k \geq 0$, there are exactly 2^k words of length exactly k. Hence there are $1 + 2 + 4 + \ldots + 2^m = 2^{m+1} - 1$ words of length at most m. Now each one of these words either belong or do not belong to an m-bounded language. So there are $2^{2^{m+1}-1}$ such languages.

   (b) (15 Points) For $m \geq 100$, show that for more than half of the $2m$-bounded languages, there is no DFA with 2^m states that recognizes them. [Hint: Count the number of DFA's with 2^m states.]

   Answer: First we count the number of DFA's $M = (Q, \Sigma, \delta, q_0, F)$ with 2^m states. There are 2^m choices for picking the start state q_0. There are 2^{2^m} choices for picking the set of accept states F. Now we count the number of choices for picking the transition $\delta : Q \times \Sigma \longrightarrow Q$. For each one of the 2^{m+1} elements in $Q \times \Sigma$, we have to pick an element in Q. So there are $|Q|^{2^{m+1}} = 2^{m2^{m+1}}$ choices for δ. Multiplying these numbers, we get that the number of such DFA's is

$$2^m \times 2^{2^m} \times 2^{m2^{m+1}} \leq 2^{m+2m2^m} \leq 2^{4m2^m}.$$

   By Part (a) there are $2^{2^{2m+1}-1} \geq 2^{2^{2m}}$ languages that are $2m$-bounded. So it suffices to prove that for $m \geq 100$,

$$2 \times 2^{4m2^m} = 2^{1+4m2^m} < 2^{2^{2m}}.$$

   Equivalently

$$1 + m2^{m+2} < 2^{2m}$$

   which is straightforward to prove.