Effective resistance of random trees

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Abstract

We investigate the effective resistance R_n and conductance C_n between the root and leaves a binary tree of height n. In this electrical network, the resistance of each an edge e at distance d from the root is defined by $r_e = 2^d X_e$ where the X_e are i.i.d. positive random variables bounded away from zero and infinity. It is shown that $\mathbf{E}R_n = n\mathbf{E}X_e - (\mathbf{Var}(X_e)/\mathbf{E}X_e)\ln n + O(1)$ and $\mathbf{Var}(R_n) = O(1)$. Some of the results are extended to the case when the underlying tree is a supercritical Galton–Watson tree. (In this case the correct scale for r_e is $b^d X_e$ where b is the branching number of the tree.)

1 Introduction

Electric networks have been known to be closely related to random walks and their investigation often offers an elegant and effective way of studying properties of random walks. See Doyle and Snell [7] and Lyons and Peres [15] for very nice introductions to the subject. For the better understanding of certain random walks in random environments, it is natural to study random electric networks, that is, electric networks in which edges are equipped with independent random resistances. This model was studied by Benjamini and Rossignol [3] who considered the case of the cubic lattice $\mathbb{Z}^d$ where the resistance of each edge is an independent copy of a Bernoulli random variable. Using an inequality of Falik and Samorodnitsky [10], they proved that the effective resistance to the boundary of a cubic box of side n has submean variance in $\mathbb{Z}^2$, whereas the variance is of the order of the mean when $d \geq 3$. In this paper, we study the corresponding problem for binary trees and for supercritical Galton–Watson trees.

An electric network is a locally finite connected graph $G = (V, E)$ with vertex set V and edge set E such that each edge $e \in E$ is equipped with a number $r_e \geq 0$ called *resistance*. (In this paper we only consider finite graphs.) Alternatively, an edge is associated with a conductance $c_e = 1/r_e$. The *effective resistance* between two disjoint sets of vertices $A, B \subset V$ is defined as follows: assign "voltage" (or potential) $U(u) = 1$ to each vertex $u \in A$ and $U(v) = 0$ for all $v \in B$. Then the function U can be extended, in a unique way, to all vertices in V according to two basic laws given by *Ohm's law* and *Kirchhoff's node law*. In order to describe these laws, we need the notion of *current*. Given two vertices $u, v \in V$ joined by and edge $e \in E$, the current flowing from u to v is a real number $i(u, v)$. Ohm's law states that for each edge of the graph, $i(u, v)r_e = U(u) - U(v)$. Kirchhoff's node law postulates that for any vertex $u \notin A \cup B$, $\sum_{v:v\sim u} i(u, v) = 0$. (For the proof that these two laws uniquely determine the function $U : V \to [0, 1]$, see [7] or [15].) Now the *effective conductance* between the vertex sets A and B is defined as the total current flowing into the network, that is,

$$C(A \leftrightarrow B) = \sum_{u\in A} \sum_{v:v\sim u} i(u, v) \ .$$

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The effective resistance between A and B is $R(A \leftrightarrow B) = 1/C(A \leftrightarrow B)$.

Several useful tricks of network reduction are known that help simplify resistance calculations. Since in this paper we focus on trees, it suffices to recall two of the simplest rules. One of them states that two resistors in series are equivalent to a single resistor whose resistance is the sum of the original resistances. The other rule states that two conductors in parallel are equivalent to a single conductor whose conductance is the sum of the original conductances. Apart from these two simple rules, a formula called *Thomson's principle* will be useful for our purposes. Thomson's principle gives an explicit expression for the effective resistance. It states that

$$R(A \leftrightarrow B) = \inf_{\Theta \in F} \sum_{e \in E} r_e \Theta(e)^2$$

where the infimum is taken over the set F of all *unit flows*. A unit flow is a function Θ over the set of oriented edges $\{(u,v) : u \sim v\}$ which is antisymmetric (i.e., $\Theta(u,v) = -\Theta(v,u)$), satisfies $\sum_{v:v\sim u} \Theta(u,v) = 0$ for any vertex $u \notin A \cup B$, and has

$$\sum_{u \in A} \sum_{v \notin A : v \sim u} \Theta(u,v) = \sum_{v \in B} \sum_{u \notin B : u \sim v} \Theta(u,v) = 1 \ .$$

It can be shown that the unique flow Θ^* which attains the above infimum is just the current $i(u,v)$ (see, e.g., Doyle and Snell [7], page 50).

In the first part of the paper we consider the case of a complete infinite binary tree T with root r. The *depth* of a node v in T is the number of edges on the path from the root to v. We say that an edge e has depth d if there are d edges on the path starting with edge e and ending at the root. The resistance of an edge e at depth d is defined by $2^d X_e$ where the X_e are independent copies of some strictly positive random variable with finite mean. This exponential weighting corresponds to the "critical" (with respect to transience/recurrence) case of the biased random walk in random environment obtained by traversing an edge e, starting from either endpoint, with probability proportional to its conductance (the inverse of its resistance). This type of exponential scaling of resistances was considered, for example, by Lyons [13]. He showed that in an infinite rooted tree with branching number b, if the (deterministic) resistance of an edge equals λ^d then the effective resistance between the root and "infinity" is infinite if $\lambda > b$ and finite if $\lambda < b$. Thus, our choice of scaling corresponds to the critical case. Similar biased random walks have been studied in depth by Lyons and Pemantle [13, 14, 17], who beautifully characterize the type of such random walks in many situations. (However, our model does not quite fit within their framework, as the transition probabilities fail to satisfy a certain independence requirement.) For more background on the connection between effective resistance of networks and random walks, see Lyons and Peres [15], Peres [18], [7] or [19].

For a random network such as that described in the previous paragraph, interesting and non-trivial behavior emerges. Let R_n be the effective resistance between the root r and the set of vertices at depth n, and let μ and σ^2 be the mean and variance of X_e, respectively. The primary results of our paper are that as long as X_e is bounded away from both zero and infinity,

$$\mathbf{E}R_n = \mu n - \frac{\sigma^2}{\mu} \ln n + O(1) \qquad \text{and} \qquad \mathbf{Var}\,[R_n] = O(1).$$

(These results are contained in Theorems 5 and 11, below.) We also derive correspondingly precise results about the conductance $C_n = 1/R_n$. Interestingly, in order to estimate the expected resistance, our main tool is a sharp upper bound for the variance of the conductance. Intuitively, concentration of the conductance implies that the behavior of the electric network is not very different from the one with deterministic resistances $2^d\mu$. Thus, Section 2 is devoted to the variance of the conductance C_n. In particular, we show that $\mathbf{Var}[C_n] = O(n^{-4})$. In Section 3 we derive the bounds for the expected resistance and conductance mentioned above. In Section 4 we prove that the variance of R_n is of constant order. The proof is based on Thomson's formula and relies on ideas of de Bruijn [6] to handle a certain type of recurrence relations.

In Section 5 we show how some of our results extend to supercritical Galton–Watson processes (in this case the appropriate scaling for the resistances is $[\mathbf{E}Z_1]^d$ for edges at depth d, where Z_1 is the number of offspring of the root). However, in this case we do not tie down either the expected resistance or its variance precisely. Furthermore, in this case our bounds are not deterministic, but hold almost surely conditional upon the Galton–Watson process. In Section 5 we shall also observe that if the random variable X_e is constant then the "scaled analogue" of Question 4.1 from Lyons et al. [16] is easily answered; motivated by this, we suggest a more general question.

From this point on, we assume X_e is any random variable taking values in some interval $[a, b]$ with $0 < a < b$. Most our arguments rely on a recursive decomposition of the tree. This decomposition is made easier by rooting the tree at an edge instead of at a vertex; this trick was also used in [16] to facilitate conductance computations. More precisely, we define the tree T_n as follows: the root has one single child whose subtree is a complete binary tree with $n-1$ levels. Then, T_n decomposes exactly into a single edge connected (in series) to two independent copies of T_{n-1} (in parallel) as shown in Figure 1. We let R_n be the effective resistance of T_n taken between the root and the leaves. Let $C_n = 1/R_n$ be the corresponding effective conductance, so in particular R_1 is distributed as X and C_1 is distributed as $1/X$. The difference between R_n and the effective resistance of the complete binary tree of height $n-1$ is at most b, so bounds on the moments of the former immediately imply corresponding bounds for the latter.

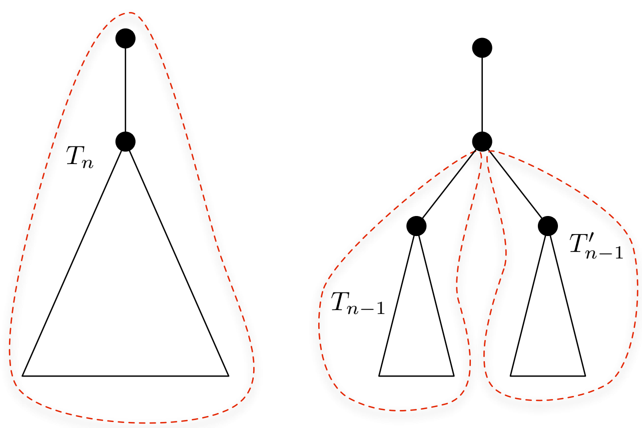

Figure 1. Rooting the binary tree rooted at an edge instead of a vertex simplifies the recursive decomposition.

We close this introduction by noting that the results of Benjamini and Rossignol [3] are proved by adapting an argument first used by Benjamini et al. [4] to prove submean variance bounds for first-passage percolation on $\mathbb{Z}^2$. Addario-Berry and Reed [1] have studied first-passage percolation on supercritical Galton–Watson processes; though their approach is entirely different from ours, their result is strikingly similar: under suitable assumptions on the edge lengths (which in their case are i.i.d.), the height of the first-passage percolation cluster of (weighted) diameter n has expected value $\alpha n - \beta \ln n + O(1)$, for computable constants a and b, and has bounded variance. We are not sure whether this similarity is more than a coincidence.

2 The variance of the conductance

The purpose of this section is to derive an upper bound for the variance of the conductance C_n. We start by noticing that R_n and C_n admit the following scalings.

Lemma 1. *When $a \le X \le b$, we have $an \le R_n \le bn$ and $1/b \le nC_n \le 1/a$, for all $n \ge 1$.*

The lemma follows from Rayleigh's monotonicity law [see 7, page 53] by bounding the resistance of T_n between that of two deterministic networks in which the random variables either always take their minimum value a or their maximum value b. We first derive a bound on $\mathbf{Var}[C_n]$. Using Chebyshev's inequality; this bound yields a quadratically decaying tail bound for R_n.

Theorem 2. *There exists a constant K depending only on a and b such that $\mathbf{Var}[C_1] \le K$ and for all integers $n \ge 2$,*

$$\mathbf{Var}[C_n] \le K \sum_{i=1}^{n-1} \frac{2^{-i}}{(n-i)^4} \le \frac{2^9 K}{n^4}.$$

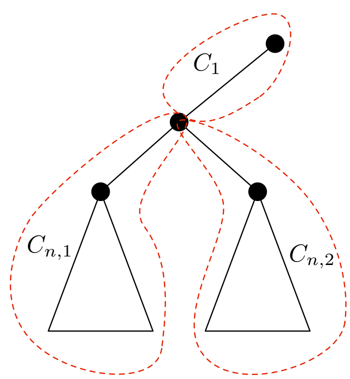

Figure 2. The decomposition of T_n into three conductors C_1, $C_{n,1}$ and $C_{n,2}$ at the origin of our recurrence relations.

Our main tool in proving Theorem 2 is the Efron–Stein inequality, which provides an upper bound on the variance of functions of independent random variables.

Theorem 3 (8, 20)**.** *Let Y_i, $i \ge 1$, be independent random variables, and let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a measurable function of n variables. Then,*

$$\mathbf{Var}[f(Y_1, \ldots, Y_n)] \le \frac{1}{2} \cdot \sum_{i=1}^{n} \mathbf{E}\left[\left(f(Y_1, \ldots, Y_i, \ldots, Y_n) - f(Y_1, \ldots, Y_i', \ldots, Y_n)\right)^2\right],$$

where Y_i', $i \ge 0$, are independent copies of Y_i, $i \ge 0$.

Proof of Theorem 2. It clearly suffices to treat the case $n \ge 2$. We decompose T_n into three independent conductors C_1, $C_{n,1}$ and $C_{n,2}$ as shown in Figure 2. Then, C_n is a function of these three independent random variables:

$$C_n = \frac{C_1 \cdot (C_{n,1} + C_{n,2})}{C_1 + C_{n,1} + C_{n,2}}. \tag{1}$$

By the Efron–Stein inequality and the symmetry of $C_{n,1}$ and $C_{n,2}$, we have

$$\begin{aligned}\mathbf{Var}\left[C_n\right] \;\leq\; & \mathbf{E}\left[\left(\frac{C_1\cdot(C_{n,1}+C_{n,2})}{C_1+C_{n,1}+C_{n,2}}-\frac{C_1\cdot(C'_{n,1}+C_{n,2})}{C_1+C'_{n,1}+C_{n,2}}\right)^2\right] \\ & +\frac{1}{2}\cdot\mathbf{E}\left[\left(\frac{C_1\cdot(C_{n,1}+C_{n,2})}{C_1+C_{n,1}+C_{n,2}}-\frac{C'_1\cdot(C_{n,1}+C_{n,2})}{C'_1+C_{n,1}+C_{n,2}}\right)^2\right]\end{aligned} \tag{2}$$

where $C'_{n,1}$ and $C'_{n,2}$ are independent copies of $C_{n,1}$ and $C_{n,2}$. Letting $\alpha = C_{n,1}+C_{n,2}$, the second term of (2) reduces to

$$\mathbf{E}\left[\left(\frac{C_1\alpha}{C_1+\alpha}-\frac{C'_1\alpha}{C'_1+\alpha}\right)^2\right].$$

Observe that, since $1/b \leq C_1, C'_1 \leq 1/a$, we have

$$\left|\frac{C_1\alpha}{C_1+\alpha}-\frac{C'_1\alpha}{C'_1+\alpha}\right| \;=\; \left|\frac{\alpha^2(C_1-C'_1)}{(C_1+\alpha)(C'_1+\alpha)}\right| \;\leq\; b^2\alpha^2\left(\frac{1}{a}-\frac{1}{b}\right).$$

Hence

$$\frac{1}{2}\mathbf{E}\left[\left(\frac{C_1\alpha}{C_1+\alpha}-\frac{C'_1\alpha}{C'_1+\alpha}\right)^2\right] \;\leq\; b^4\left(\frac{1}{b}-\frac{1}{a}\right)^2\frac{1}{2}\mathbf{E}\left[(C_{n,1}+C_{n,2})^4\right].$$

Since both $C_{n,1}$ and $C_{n,2}$ are distributed as $C_{n-1}/2$, by Lemma 1, we have $C_{n,1}+C_{n,2} \leq 1/(a(n-1))$, and this yields

$$\frac{1}{2}\mathbf{E}\left[\left(\frac{C_1\alpha}{C_1+\alpha}-\frac{C'_1\alpha}{C'_1+\alpha}\right)^2\right] \;\leq\; \frac{1}{2}\left(\frac{b}{a}\right)^4\left(\frac{1}{b}-\frac{1}{a}\right)^2\frac{1}{(n-1)^4} \stackrel{\text{def}}{=} \frac{K}{(n-1)^4}. \tag{3}$$

We now use the first term on the right-hand side of (2) to devise a recurrence. We have

$$\begin{aligned}\left|\frac{C_1(C_{n,1}+C_{n,2})}{C_1+C_{n,1}+C_{n,2}}-\frac{C_1(C'_{n,1}+C_{n,2})}{C_1+C'_{n,1}+C_{n,2}}\right| \;&=\; \frac{C_1^2\left|C_{n,1}-C'_{n,1}\right|}{(C_1+C_{n,1}+C_{n,2})(C_1+C'_{n,1}+C_{n,2})} \\ &\leq\; \left|C_{n,1}-C'_{n,1}\right|.\end{aligned}$$

Accordingly,

$$\begin{aligned}\mathbf{E}\left[\left(\frac{C_1(C_{n,1}+C_{n,2})}{C_1+C_{n,1}+C_{n,2}}-\frac{C_1(C'_{n,1}+C_{n,2})}{C_1+C'_{n,1}+C_{n,2}}\right)^2\right] \;&\leq\; \mathbf{E}\left[(C_{n,1}-C'_{n,1})^2\right] \\ &=\; 2\cdot\mathbf{Var}\left[C_{n,1}\right] \\ &=\; \frac{1}{2}\cdot\mathbf{Var}\left[C_{n-1}\right].\end{aligned}$$

Therefore, recalling (3) we have the following recurrence relation:

$$\mathbf{Var}\left[C_n\right] \leq \frac{K}{(n-1)^4}+\frac{1}{2}\cdot\mathbf{Var}\left[C_{n-1}\right].$$

Expanding the recurrence yields readily

$$\mathbf{Var}\left[C_n\right] \leq K\cdot\sum_{i=1}^{n-1}\frac{2^{-i}}{(n-i)^4}.$$

Since

$$\begin{aligned}\sum_{i=1}^{n-1} \frac{2^{-i}}{(n-i)^4} &\leq \sum_{i=1}^{\lfloor n/2 \rfloor} \frac{2^{-i}}{(n-i)^4} + \sum_{i=\lfloor n/2 \rfloor}^{n-1} \frac{2^{-i}}{(n-i)^4} \\ &\leq \frac{2^4}{n^4} \sum_{i \geq 1} 2^{-i} + 2^{-n/2} \sum_{i \geq 1} 2^{-i} \leq \frac{2^9}{n^4},\end{aligned}$$

the proof is complete. □

We remark that this theorem is tight up to a constant factor unless X is deterministic (in which case $\mathbf{Var}[C_n] = 0$). Indeed, by considering equation (1), since $C_{n,1}$ and $C_{n,2}$ are both of order n^{-1}, we see that fluctuations of constant size in the value of C_1 change C_n by order n^{-2}. Such fluctuations occur with positive probability, so we must have that $\mathbf{Var}[C_n] \geq \epsilon n^4$ for some $\epsilon > 0$ depending on X.

Theorem 2 also immediately yields tail bounds on the probability that the resistance R_n deviates from $1/\mathbf{E}C_n$.

Lemma 4. *There exists a constant $A \geq 1$ depending on a and b only, such that, for all $t > 0$ and all $n \geq 1$, we have*

$$\mathbf{P}\left\{\left|R_n - \frac{1}{\mathbf{E}C_n}\right| > t\right\} \leq \frac{A}{t^2}.$$

In particular, $\mathbf{E}|R_n - 1/\mathbf{E}C_n| \leq 1 + A$.

Proof. Manipulating the desired tail bound, we observe that

$$\begin{aligned}\mathbf{P}\left\{\left|R_n - \frac{1}{\mathbf{E}C_n}\right| > t\right\} &= \mathbf{P}\left\{\left|\frac{1}{C_n} - \frac{1}{\mathbf{E}C_n}\right| > t\right\} \\ &= \mathbf{P}\left\{\left|\frac{\mathbf{E}C_n}{C_n} - 1\right| > t\mathbf{E}C_n\right\} \\ &= \mathbf{P}\left\{|C_n - \mathbf{E}C_n| > t\mathbf{E}[C_n]\, C_n\right\} \\ &\leq \mathbf{P}\left\{|C_n - \mathbf{E}C_n| > \frac{t}{b^2 n^2}\right\},\end{aligned}$$

by Lemma 1. By Chebyshev's inequality, and Theorem 2, we obtain the required bound

$$\mathbf{P}\left\{\left|R_n - \frac{1}{\mathbf{E}C_n}\right| > t\right\} \leq \frac{n^4 b^4 \mathbf{Var}[C_n]}{t^2} \leq \frac{A}{t^2},$$

with $A = \max\{2^9 b^4 K, 1\}$, where K is the constant from Theorem 2. Moreover, for all $n \geq 1$, we have

$$\mathbf{E}|R_n - 1/\mathbf{E}C_n| \leq 1 + \int_1^\infty \mathbf{P}\left\{|R_n - 1/\mathbf{E}C_n| \geq x\right\} dx \leq 1 + A \int_1^\infty \frac{dx}{x^2} = 1 + A,$$

which finishes the proof. □

3 The expected resistance and conductance

In this section we give precise locations for the expected values $\mathbf{E}C_n$ and $\mathbf{E}R_n$, respectively. Let $\sigma^2 = \mathbf{Var}[X]$ and let $\mu = \mathbf{E}X$.

Theorem 5. *There exist constants M_1 and M_2 depending only on a and b such that for all integers $n \geq 2$,*

$$\left| \mathbf{E} R_n - \mu n - \frac{\sigma^2}{\mu} \ln n \right| \leq M_1 \qquad \text{and} \qquad \left| \mathbf{E} C_n - \frac{1}{\mu n} - \frac{\sigma^2 \ln n}{\mu^2 n^2} \right| \leq \frac{M_2}{n^2}.$$

We remark that since $\mathbf{Var}[R_n]$ is certainly bounded from below by a positive constant (unless X is deterministic, in which case we know R_n precisely), we have determined the value of $\mathbf{E} R_n$ up to the order of its standard deviation. In the next section, we shall show that $\mathbf{Var}[R_n] = O(1)$. Furthermore, since $\mathbf{Var}[C_n]$ is of order n^{-4}, we have likewise determined $\mathbf{E} C_n$ up to the order of its standard deviation.

The techniques we use to handle the recurrence relation has been used by de Bruijn [6] and Flajolet and Odlyzko [11] in the context of heights of simple trees.

Proof of Theorem 5. We focus on $\mathbf{E} C_n$; by Lemma 4, bounds on $\mathbf{E} C_n$ immediately yield bounds on $\mathbf{E} R_n$. As in the proof of Theorem 2, we decompose T_{n+1} into three independent conductors C_1, $C_{n+1,1}$, and $C_{n+1,2}$ (see Figure 2). Let C_n and C_n' be independent copies of the conductance between the root and level n. Since

$$C_{n+1} = \frac{C_1 \cdot (C_{n+1,1} + C_{n+1,2})}{C_1 + C_{n+1,1} + C_{n+1,2}},$$

$C_{n+1,1}$ and $C_{n+1,2}$ are both distributed as $C_n/2$, and C_1 is distributed as $1/X$, we have, in distribution,

$$C_{n+1} = \frac{C_n + C_n'}{2} \cdot \frac{1}{1 + X\left(\frac{C_n + C_n'}{2}\right)}, \tag{4}$$

where X is independent of all the other random variables appearing in (4). The second factor in (4) can be rewritten as

$$\begin{aligned} \frac{1}{1 + X\left(\frac{C_n + C_n'}{2}\right)} &= 1 - X\left(\frac{C_n + C_n'}{2}\right) + X^2 \cdot \left(\frac{C_n + C_n'}{2}\right)^2 \\ &\quad - X^3 \cdot \left(\frac{C_n + C_n'}{2}\right)^3 \cdot \frac{1}{1 + X\left(\frac{C_n + C_n'}{2}\right)}. \end{aligned} \tag{5}$$

Using the equality (5) to replace the term $1/(1 + X(C_n + C_n')/2)$ in (4) and taking expectations, we obtain

$$\begin{aligned} \mathbf{E} C_{n+1} &= \mathbf{E} C_n - \frac{\mathbf{E} X}{2} \cdot \left(\mathbf{E}\left[C_n^2\right] + [\mathbf{E} C_n]^2\right) + \frac{\mathbf{E}\left[X^2\right]}{4} \cdot \left(\mathbf{E}\left[C_n^3\right] + 3\mathbf{E}\left[C_n^2\right] \mathbf{E} C_n\right) \\ &\quad - \mathbf{E}\left[\frac{X^3 (C_n + C_n')^4}{16\left(1 + X\left(\frac{C_n + C_n'}{2}\right)\right)}\right], \end{aligned} \tag{6}$$

where we have used the equalities $\mathbf{E}\left[(C_n + C_n')^2\right] = 2(\mathbf{E}\left[C_n^2\right] + [\mathbf{E} C_n]^2)$ and $\mathbf{E}\left[(C_n + C_n')^3\right] = 2(\mathbf{E}\left[C_n^3\right] + 3\mathbf{E}\left[C_n^2\right] \mathbf{E} C_n)$. By Lemma 1, we have deterministically

$$\frac{a^3}{16 b^4 n^4} \cdot \frac{1}{1 + \frac{b}{an}} \leq \frac{X^3 (C_n + C_n')^4}{16\left(1 + X\left(\frac{C_n + C_n'}{2}\right)\right)} \leq \frac{b^3}{16 a^4 n^4},$$

so (6) yields

$$\mathbf{E}[C_{n+1}] = \mathbf{E}[C_n] - \frac{\mathbf{E} X}{2} \cdot \left(\mathbf{E}\left[C_n^2\right] + [\mathbf{E} C_n]^2\right) + \frac{\mathbf{E}\left[X^2\right]}{4} \cdot \left(\mathbf{E}\left[C_n^3\right] + 3\mathbf{E}\left[C_n^2\right] \mathbf{E} C_n\right) + O(n^{-4}), \tag{7}$$

where the order notation $O(\cdot)$ depends only on a and b. We observe that, by Theorem 2,

$$\mathbf{E}\left[C_n^2\right] + [\mathbf{E}C_n]^2 = \mathbf{Var}\left[C_n\right] + 2[\mathbf{E}C_n]^2 = 2[\mathbf{E}C_n]^2 + O(n^{-4}). \tag{8}$$

Furthermore, since $\mathbf{E}\left[(C_n - \mathbf{E}C_n)^3\right] = O(n^{-1}) \cdot \mathbf{Var}\left[C_n\right] = O(n^{-5})$ by Theorem 2, we have

$$\begin{aligned}
\mathbf{E}\left[C_n^3\right] &= \mathbf{E}\left[(C_n - \mathbf{E}C_n)^3\right] + 3\mathbf{E}\left[C_n^2\right]\mathbf{E}C_n - 3[\mathbf{E}C_n]^3 + [\mathbf{E}C_n]^3 \\
&= O(n^{-5}) + 3\left(\mathbf{Var}\left[C_n\right] + [\mathbf{E}C_n]^2\right)\mathbf{E}C_n - 2[\mathbf{E}C_n]^3 \\
&= 3\mathbf{Var}\left[C_n\right]\mathbf{E}C_n + [\mathbf{E}C_n]^3 + O(n^{-5}),
\end{aligned}$$

so

$$\mathbf{E}\left[C_n^3\right] + 3\mathbf{E}\left[C_n^2\right]\mathbf{E}C_n = 4[\mathbf{E}C_n]^3 + 6\mathbf{Var}\left[C_n\right]\mathbf{E}C_n = 4[\mathbf{E}C_n]^3 + O(n^{-5}). \tag{9}$$

Combining (7), (8), and (9), we obtain

$$\mathbf{E}C_{n+1} = \mathbf{E}C_n - \mathbf{E}X[\mathbf{E}C_n]^2 + \mathbf{E}X^2[\mathbf{E}C_n]^3 + O(n^{-4}).$$

Dividing through by $\mathbf{E}C_{n+1}\mathbf{E}C_n$ and letting $x_n = 1/\mathbf{E}C_n$ gives

$$x_n = x_{n+1} - \mathbf{E}X \cdot \frac{x_{n+1}}{x_n} + \mathbf{E}\left[X^2\right] \cdot \frac{x_{n+1}}{x_n^2} + O(n^{-2}). \tag{10}$$

We let $\delta_n = x_{n+1}/x_n - 1$ and let $\epsilon_n = x_{n+1}/x_n^2$, and remark that δ_n and ϵ_n are both $O(n^{-1})$. Summing (10) gives

$$x_{n+1} = n\mathbf{E}X + \mathbf{E}X \cdot \sum_{i=1}^{n} \delta_i - \mathbf{E}\left[X^2\right] \sum_{i=1}^{n} \epsilon_i + O(1). \tag{11}$$

Since both δ_i and ϵ_i are $O(i^{-1})$, (11) immediately yields the bound

$$x_{n+1} = n\mathbf{E}X + O(\ln n) = (n+1)\mathbf{E}X + O(\ln n), \tag{12}$$

a bound we will bootstrap to prove the theorem. From (10) we have

$$\frac{x_{n+1}}{x_n} = 1 + \frac{\mathbf{E}X + \mathbf{E}X \cdot \delta_n - \mathbf{E}\left[X^2\right] \cdot \epsilon_n}{x_n} + O(n^{-3}),$$

so since x_{n+1}/x_n also equals $1 + \delta_n$, solving for δ_n we obtain

$$\delta_n = \frac{\mathbf{E}X - \mathbf{E}\left[X^2\right]\epsilon_n}{x_n - \mathbf{E}X} + O(n^{-3}) = \frac{\mathbf{E}X}{x_n - \mathbf{E}X} + O(n^{-2}), \tag{13}$$

as long as n is large enough to ensure that $x_n - \mathbf{E}X$ does not happen to be zero (say $n \geq n_0$ for some fixed n_0 depending only on a and b). Similarly,

$$\epsilon_n = \frac{1}{x_n} \cdot \frac{x_{n+1}}{x_n} = \frac{1}{x_n} + \frac{\delta_n}{x_n} = \frac{1}{x_n} + O(n^{-2}), \tag{14}$$

for all $n \geq 1$. Combining (11), (13), and (14) gives the identity

$$\begin{aligned}
x_{n+1} - n\mathbf{E}X &= [\mathbf{E}X]^2 \sum_{i=n_0}^{n} \frac{1}{x_i - \mathbf{E}X} - \mathbf{E}\left[X^2\right] \sum_{i=n_0}^{n} \frac{1}{x_i} + O(1) \\
&= [\mathbf{E}X]^3 \sum_{i=n_0}^{n} \frac{1}{x_i(x_i - \mathbf{E}X)} - \mathbf{Var}\left[X\right] \sum_{i=n_0}^{n} \frac{1}{x_i} + O(1) \\
&= -\mathbf{Var}\left[X\right] \sum_{i=n_0}^{n} \frac{1}{x_i} + O(1).
\end{aligned} \tag{15}$$

Since $x_i = i\mathbf{E}X + O(\ln i)$ by (12), we have

$$\sum_{i=n_0}^{n} \frac{1}{x_i} = \sum_{i=n_0}^{n} \frac{1}{i\mathbf{E}X + O(\ln i)} = \sum_{i=n_0}^{n} \left(\frac{1}{i\mathbf{E}X} + \frac{O(\ln i)}{(i\mathbf{E}X)^2} \right) = \frac{\ln n}{\mathbf{E}X} + O(1),$$

so (15) yields

$$x_{n+1} - n\mathbf{E}X = \frac{\mathbf{Var}\,[X]}{\mathbf{E}X} \ln n + O(1).$$

The first assertion of the theorem follows immediately, and second assertion of the theorem follows since $x_{n+1} = \mathbf{E}R_{n+1} + O(1)$ by Lemma 4. □

We finish this section by stating an immediate corollary which will be useful in when proving that the variance of R_n is bounded.

Corollary 6. *There exists a constant B depending only on a and b such that for all $n \geq 2$, $\mathbf{E}|R_n - R_{n-1}| \leq B$.*

4 The variance of the resistance

In order to show that the variance of the resistance is bounded, we bootstrap our weaker tail bound on the resistance (Lemma 4). We again use the Efron–Stein inequality, together with Thomson's formula. This flow-based formulation of the resistance was used by Benjamini and Rossignol [3] to show submean variance bounds for the random resistance in $\mathbb{Z}^2$. Given a graph G, let $E(G)$ be the set of edges of G. Recall that if F denotes the set of unit flows from the root r to depth n in T_n, then

$$R_n = \inf_{\Theta \in F} \left\{ \sum_{e \in E(T_n)} r_e \Theta(e)^2 \right\}. \tag{16}$$

Furthermore, there is a unique flow Θ^* (the current) which attains the above infimum. As observed by Benjamini and Rossignol [3], page 4, it is a straightforward consequence of the Efron–Stein inequality that

$$\mathbf{Var}\,[R_n] \leq \frac{(b-a)^2}{2} \sum_{e \in E(T_n)} \mathbf{E}\left[\Theta^*(e)^4\right]. \tag{17}$$

Let e_i be the edge along the leftmost (principal) branch of T connecting the nodes at depth i and $i+1$. By symmetry we may rewrite the bound (17) as

$$\mathbf{Var}\,[R_n] \leq \frac{(b-a)^2}{2} \sum_{i=0}^{n-1} 2^i \mathbf{E}\left[\Theta^*(e_i)^4\right], \tag{18}$$

so, by this approach, proving bounds on $\mathbf{Var}\,[R_n]$ amounts to studying the behavior of the flow along the principal branch of T. Denote by $v_0, v_1, \ldots, v_n$ the nodes of T_n along the principal branch; so $r = v_0$ and $e_i = v_i v_{i+1}$. For all $i = 1, \ldots, n-1$, write $T^\ell_{n,i}$ and $T^r_{n,i}$ for the two copies of T_{n-i} rooted at v_i (with resistances multiplied by 2^i), and write $R^\ell_{n,i}$ (resp. $R^r_{n,i}$) for the effective resistance in $T^\ell_{n,i}$ (resp. $T^r_{n,i}$) between v_i and the descendants of v_i in $T^\ell_{n,i}$ (resp. $T^r_{n,i}$) at depth n (See Figure 3). By the definition of Θ^*, we must have

$$\Theta^*(e_i) = \Theta^*(e_{i-1}) \cdot \frac{R^r_{n,i}}{R^\ell_{n,i} + R^r_{n,i}}.$$

Since Θ is a unit flow, $\Theta^*(e_0) = 1$, it follows immediately that for $i \geq 1$

$$\Theta^*(e_i) = \prod_{j=1}^{i} A_j \qquad \text{where} \qquad A_i = \frac{R^r_{n,i}}{R^\ell_{n,i} + R^r_{n,i}}. \tag{19}$$

We will use the product representation (19) to bound $\mathbf{E}\left[\Theta^*(e_i)^4\right]$. To do so, we first bound $\prod_{j=1}^{i} A_j$ by another product of random variables whose dependencies are easier to handle. Before explaining the details of our bound, we briefly motivate our approach.

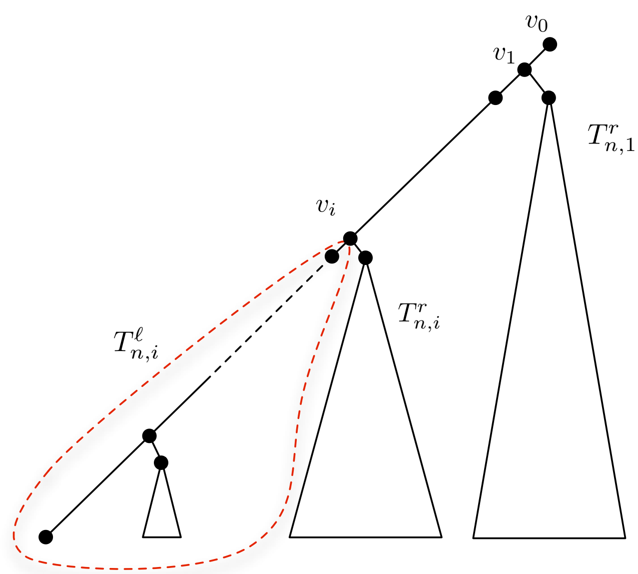

Figure 3. The decomposition we use along the leftmost branch in order to bound $\mathbf{Var}\left[R_n\right]$.

First observe that while A_1 depends on the whole tree, A_j depends only on the edge labels in $T^\ell_{n,j}$ and $T^r_{n,j}$, which makes it easier to control its influence on the other elements of the product. We shall thus bound the terms in (19) starting from A_i rather than from A_1.

Given some k with $1 < k \leq j$, if A_k is far from $1/2$, the resistances of the left and right subtrees are far apart, so it is much more likely that A_{k-1} is also far from $1/2$. However, we are able to show that the effect of A_k on $A_{k'}$ on A_k decays exponentially in $k' - k$. Each time we come across such a "lopsided" A_k, we will thus ignore (bound by 1) each $A_{k'}$ with $k' - k$ at most some large constant M. We now turn to the details.

We first let $\epsilon = 1/80$ and choose an integer M large enough that the following conditions hold:

1. $(6b/a)(b/(a+b))^M < \epsilon/6$;
2. $M \geq 1600B/a$, where B is the constant from Corollary 6; and
3. for all $n \geq M$, $\epsilon \mathbf{E} R_n/6 \geq A + 2$ where A is the constant from Lemma 4.

These conditions may seem opaque at the moment but will arise in the course of the proof. We have chosen to list them at the start to make all dependencies clear. We remark that M depends only on a and b. For fixed i with $M + 2 \leq i \leq n/2$, we now define an integer sequence $k_1, \ldots, k_\tau$ (where τ is random and will be specified momentarily) which, intuitively speaking, is the set of indices k for which we do not ignore the contribution of A_k to the product (19). We define the sequence as follows:

1. Let $k_1 = i$.

2. Given k_j, if $|R^\ell_{n,k_j}/\mathbf{E}R^\ell_{n,k_j} - 1| \le \epsilon/3$, $|R^r_{n,k_j}/\mathbf{E}R^r_{n,k_j} - 1| \le \epsilon/3$, and $k_j \ge 2$ then set $k_{j+1} = k_j - 1$. Otherwise, if $k_j \ge M+2$ then set $k_{j+1} = k_j - (M+1)$. Otherwise set $\tau = j$ and stop.

So τ is the number of factors we do not ignore. For $j = 1, \ldots, \tau$, we let

$$B_j = \begin{cases} \frac{1}{2} + \epsilon & \text{if } k_{j+1} = k_j - 1 \\ 1 & \text{otherwise.} \end{cases} \tag{20}$$

(The variables B_j in fact depend on our choice of i, but we suppress this dependence for readability as i will always be clear from context.) We remark that for a given j, if $k_{j+1} = k_j - 1$ then due to our choice of ϵ and the bounds on R^ℓ_{n,k_j} and on R^r_{n,k_j} , we have $1/2 - \epsilon \le A_{k_j} \le 1/2 + \epsilon$. It follows immediately that

$$\prod_{j=1}^{\tau} B_j \ge \prod_{j=1}^{\tau} A_{k_j} \ge \prod_{j=1}^{i} A_j.$$

We further remark that k_{j+1} is a measurable function of $\{X_e : e \in E(T^\ell_{n,k_j}) \cup E(T^r_{n,k_j})\}$. As alluded to above, the key property of the random variables B_j is that for $j' < j$, knowledge about $B_{j'}$ has very little effect on B_j, which allows us to prove a stochastic domination result for the B_j.

Lemma 7. *There is a positive real number D such that given integers i, j, n with $M + 2 \le i \le n/2$ and $2 \le j \le i$, there is a sequence of independent Bernoulli random variables $B'_m, m \ge 1$, with mean $\min\{D/n^2, 1\}$ such that for all $j \le \tau$, $B_j \le (\frac{1}{2} + \epsilon) + \left(\frac{1}{2} - \epsilon\right) \cdot B'_j$.*

In proving Lemma 7 we will use the following fact about how changing the resistances in a subtree of T_n can affect the effective resistance of T_n. Given any edge $e = uv$ of T_n, let $\tilde{T}_n(e)$ be the network given by replacing all edge labels in the subtree rooted at e (i.e., the subtree induced by u, v, and all nodes which are descendants of both u and v) by independent copies of the same variables. Let $\tilde{R}_n(e)$ be the effective resistance between the root and leaves of $\tilde{T}_n(e)$ (see Figure 4). We say edge $e = uv$ has depth k if one of $\{u, v\}$ has depth k and the other has depth $k + 1$.

Lemma 8. *For any integer k with $0 \le k \le n-1$ and any edge e at depth k, it is deterministically the case that*

$$|\tilde{R}_n(e) - R_n| \le 3bn \left(\frac{b}{a+b}\right)^k .$$

Proof. Fix $k \in \{0, \ldots, n-1\}$. By symmetry, it suffices to prove the claim with $e = e_k$. The claim clearly holds with $k = 0$ (in fact, in this case $|\tilde{R}_n(e) - R_n| \le (b-a)n$) so we presume $k \ge 1$. Letting $\tilde{R}_{n,k} = \tilde{R}_n(e_k)$ and letting $\tilde{T}_n = \tilde{T}_n(e_k)$, we may equivalently view $\tilde{T}_n$ as a copy of T_n in which $T^\ell_{n,k}$ is replaced by a single resistor $\tilde{e}_k$ that has resistance $\tilde{R}_{n,k}$ independent from and distributed as $R_{n,k}$ (see Figure 4), and we hereafter do so. Without loss of generality we may assume $\tilde{R}_{n,k} > R_{n,k}$ (and so $\tilde{R}_n \ge R_n$).

Observe first that, since both $R_{n,k}$ and $\tilde{R}_{n,k}$ are between $a2^k(n-k)$ and $b2^k(n-k)$ we have $\tilde{R}_{n,k} - R_{n,k} \le (b-a)2^k n$. Recall that Θ^* denotes the current in T_n. Also,

$$\Theta^*(e_k) = \prod_{j=1}^{k} A_j \le \left(\frac{b}{a+b}\right)^k ,$$

by Lemma 1.

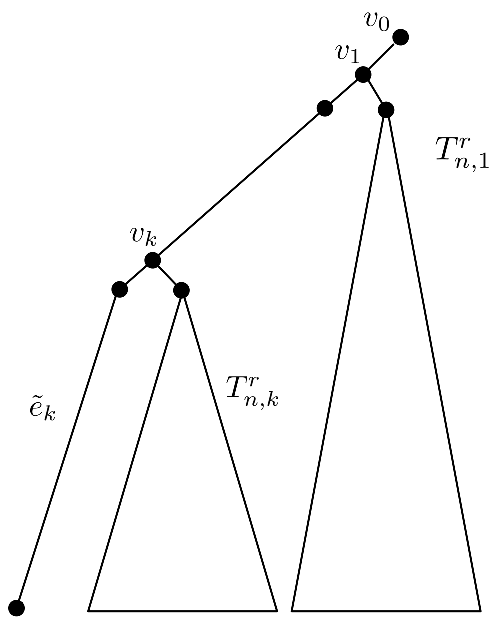

Figure 4. $\tilde{R}_n(e_k)$ is the effective resistance of the tree $\tilde{T}_n(e_k)$, the network T_n with $T^\ell_{n,k}$ replaced by the single resistor $\tilde{e}_k$.

For $m = 1, \ldots, n$, let $\mathscr{L}_m$ be the set of edges e at depth m in $\tilde{T}_n$ such that $\Theta^*(e) \leq 1/2^m$. Since Θ^* is a unit flow, $\mathscr{L}_m$ is non-empty for $m = 1, \ldots, k$, and if $\Theta^*(e_k) > 1/2^k$ then $\mathscr{L}_m$ is also non-empty for $m = k+1, \ldots, n$. We bound $\tilde{R}_n$ by considering a new flow $\tilde{\Theta}$, which we now define. First, if $\Theta^*(e_k) \leq 1/2^k$ then set $\tilde{\Theta}(e) = \Theta^*(e)$ for all $e \notin E(T^\ell_{n,k})$, and set $\tilde{\Theta}(\tilde{e}_k) = \Theta^*(e_k)$. Otherwise, let $\delta = \Theta^*(e_k) - 1/2^k > 0$, let $E = E(\tilde{T}_n) \setminus (\bigcup_{m=1}^n \mathscr{L}_m \cup \{\tilde{e}_k\})$, and let $\tilde{\Theta}$ be a flow satisfying the following three properties.

1. $\tilde{\Theta}(\tilde{e}_k) = 1/2^k = \Theta^*(e_k) - \delta$.
2. For each $e \in E$, $\tilde{\Theta}(e) = \Theta^*(e)$.
3. For each $m = 1, \ldots, n$ and each $e \in \mathscr{L}_m$, $\tilde{\Theta}(e) \leq 1/2^m$.

It is easy to see that such a flow always exists (redistribute the excess flow δ along the edges in $\bigcup_{m\geq 1} \mathscr{L}_m$). We first observe that if $\Theta^*(e_k) \leq 1/2^k$ then we have the bound

$$
\begin{aligned}
\tilde{R}_n &\leq \sum_{e \in E(\tilde{T}_n)\setminus\{\tilde{e}_k\}} \tilde{\Theta}(e)^2 r_e + \tilde{\Theta}(\tilde{e}_k)^2 \tilde{R}_{n,k} \\
&= \sum_{e \in E(\tilde{T}_n)\setminus\{\tilde{e}_k\}} \Theta^*(e)^2 r_e + \Theta^*(e_k)^2 R_{n,k} + \Theta^*(e_k)^2 (\tilde{R}_{n,k} - R_{n,k}) \\
&= R_n + \Theta^*(e_k)^2 (\tilde{R}_{n,k} - R_{n,k}) \\
&\leq R_n + n \cdot \frac{b-a}{2^k}. \qquad (21)
\end{aligned}
$$

On the other hand, if $\Theta^*(e_k) > 1/2^k$ then $\tilde{\Theta}(\tilde{e}_k) = 2^{-k}$, and we have

$$\begin{aligned}
\tilde{R}_n &\le \sum_{e\in E}\tilde{\Theta}(e)^2 r_e + \sum_{m=1}^{n}\sum_{e\in\mathscr{L}_m}\tilde{\Theta}(e)^2 r_e + 2^{-2k}\tilde{R}_{n,k} \\
&= \sum_{e\in E}\Theta^*(e)^2 r_e + \sum_{m=1}^{n}\sum_{e\in\mathscr{L}_m}\tilde{\Theta}(e)^2 r_e + 2^{-2k}\tilde{R}_{n,k} \\
&= R_n + \sum_{m=1}^{n}\sum_{e\in\mathscr{L}_m}(\tilde{\Theta}(e)^2 - \Theta^*(e)^2) r_e + 2^{-2k}\tilde{R}_{n,k} - \Theta^*(e_k)^2 R_{n,k} \\
&\le R_n + \sum_{m=1}^{n}\sum_{e\in\mathscr{L}_m}(\tilde{\Theta}(e)^2 - \Theta^*(e)^2) r_e + 2^{-2k}(\tilde{R}_{n,k} - R_{n,k}) \\
&\le R_n + \sum_{m=1}^{n}\sum_{e\in\mathscr{L}_m}(\tilde{\Theta}(e) + \Theta^*(e))(\tilde{\Theta}(e) - \Theta^*(e)) r_e + \frac{(b-a)n}{2^k}. \qquad (22)
\end{aligned}$$

Letting $\delta_e = \tilde{\Theta}(e) - \Theta^*(e)$, since $\Theta^*(e) \le \tilde{\Theta}(e) \le 1/2^m$ for all $m = 1, \ldots, n$ and all $e \in \mathscr{L}_m$, (22) yields

$$\begin{aligned}
\tilde{R}_n &\le R_n + \frac{(b-a)n}{2^k} + \sum_{m=1}^{n}\sum_{e\in\mathscr{L}_m}\frac{\delta_e r_e}{2^{m-1}} \\
&\le R_n + \frac{(b-a)n}{2^k} + \sum_{m=1}^{n}\sum_{e\in\mathscr{L}_m} 2b\delta_e. \qquad (23)
\end{aligned}$$

Since we have distributed the excess flow δ, for each $m = 1, \ldots, n$, we have

$$\sum_{e\in\mathscr{L}_m}\delta_e \le \delta < \left(\frac{b}{a+b}\right)^k,$$

so from (23) we obtain

$$\tilde{R}_n \le R_n + \frac{(b-a)n}{2^k} + 2bn\left(\frac{b}{a+b}\right)^k < R_n + 3bn\left(\frac{b}{a+b}\right)^k, \qquad (24)$$

completing the proof. □

Proof of Lemma 7. We let $D = 1152A(A+1)^2/(\epsilon^2 a^2)$, where A is the constant from Lemma 4. The lemma clearly holds if $n^2 \le D$, so we hereafter assume that $n^2 > D$. Given i, j, and n as above, for simplicity we temporarily extend the sequence $B_1, \ldots, B_\tau$ by a sequence of i.i.d. random variables with

$$B_m = \begin{cases} \frac{1}{2}+\epsilon & \text{with probability } \frac{D}{n^2} \\ 1 & \text{with probability } 1-\frac{D}{n^2}, \end{cases}$$

for integers $m \ge \tau$, and let $k_m = 1$ for integers $m \ge \tau$. To prove the lemma it suffices (see [12, page 179]) to prove that given any event Γ with $\mathbf{P}\{\Gamma\} > 0$ which is measurable with respect to $B_1, \ldots, B_{j-1}$,

$$\mathbf{P}\{B_j = 1 \mid \Gamma\} \le \frac{D}{n^2}. \qquad (25)$$

This clearly holds if $j > \tau$, so we may assume that $\Gamma \subseteq \{j \le \tau\}$. Our proof of (25) has two cases, depending on whether or not $B_{j-1} = 1$. Accordingly, we make the following definitions:

$$\Gamma_1 = \Gamma \cap \{B_{j-1} = 1\} \qquad \text{and} \qquad \Gamma_2 = \Gamma \setminus \Gamma_1.$$

If $B_{j-1}=1$ then $k_{j-1}=k_j+(M+1)$, so the event Γ_1 is measurable with respect to $\{X_e : e \in E(T^\ell_{n,k_j+M+1}) \cup E(T^r_{n,k_j+M+1})\}$ and thus with respect to $\{X_e : e \in E(T^\ell_{n,k_j+M})\}$. Let $\tilde{R}^\ell_{n,k_j}$ be the resistance of T^ℓ_{n,k_j} after replacing T^ℓ_{n,k_j+M} with an independent resistor $\tilde{R}_{n,k_j+M}$ distributed as R^ℓ_{n,k_j+M}. Since T_{n,k_j} is distributed as T_{n,k_j} with all resistors multiplied by 2^{k_j}, we may apply Lemma 8 to bound $\tilde{R}^\ell_{n,k_j}$. Doing so, we obtain that

$$|\tilde{R}^\ell_{n,k_j} - R^\ell_{n,k_j}| \le 2^{k_j} 3b(n-k_j)\left(\frac{b}{a+b}\right)^M < 2^{k_j} 3bn \left(\frac{b}{a+b}\right)^M .$$

Therefore, since also $\mathbf{E}R_{n,k_j} \ge 2^{k_j} a(n-k_j) \ge 2^{k_j-1}an$, by our choice of M (on page 10) we have

$$|\tilde{R}^\ell_{n,k_j} - R^\ell_{n,k_j}| \le \mathbf{E}R_{n,k_j} \cdot \frac{6b}{a}\left(\frac{b}{a+b}\right)^M \le \frac{\epsilon \cdot \mathbf{E}R_{n,k_j}}{6}.$$

As $\mathbf{E}R_{n,k_j} = \mathbf{E}\tilde{R}_{n,k_j}$ and $\tilde{R}_{n,k_j}$ is independent of Γ_1, it follows that

$$\begin{aligned}
\mathbf{P}\left\{ \left| \frac{R^\ell_{n,k_j}}{\mathbf{E}R^\ell_{n,k_j}} - 1 \right| > \frac{\epsilon}{3} \,\middle|\, \Gamma_1 \right\} &= \mathbf{P}\left\{ \left| R^\ell_{n,k_j} - \mathbf{E}R^\ell_{n,k_j} \right| > \frac{\epsilon \cdot \mathbf{E}R^\ell_{n,k_j}}{3} \,\middle|\, \Gamma_1 \right\} \\
&\le \mathbf{P}\left\{ \left| \tilde{R}^\ell_{n,k_j} - \mathbf{E}\tilde{R}^\ell_{n,k_j} \right| > \frac{\epsilon \cdot \mathbf{E}\tilde{R}^\ell_{n,k_j}}{6} \,\middle|\, \Gamma_1 \right\} \\
&= \mathbf{P}\left\{ \left| \tilde{R}^\ell_{n,k_j} - \mathbf{E}\tilde{R}^\ell_{n,k_j} \right| > \frac{\epsilon \cdot \mathbf{E}\tilde{R}^\ell_{n,k_j}}{6} \right\}.
\end{aligned}$$

Since T^ℓ_{n,k_j} is a copy of T_{n-k_j} with weights multiplied by 2^{k_j}, by Lemma 4 we therefore have

$$\begin{aligned}
\mathbf{P}\left\{ \left| \frac{R^\ell_{n,k_j}}{\mathbf{E}R^\ell_{n,k_j}} - 1 \right| > \frac{\epsilon}{3} \,\middle|\, \Gamma_1 \right\} &= \mathbf{P}\left\{ \left| R_{n-k_j} - \mathbf{E}R_{n-k_j} \right| \ge \frac{\epsilon \cdot \mathbf{E}R_{n-k_j}}{6} \right\} \\
&\le \mathbf{P}\left\{ \left| R_{n-k_j} - \frac{1}{\mathbf{E}C_{n-k_j}} \right| \ge \frac{\epsilon \cdot \mathbf{E}R_{n-k_j}}{6} - (A+1) \right\} \\
&\le \frac{A}{t^2}, \qquad (26)
\end{aligned}$$

where $A \ge 1$ is the constant from Lemma 4 and $t = \epsilon \mathbf{E}R_{n-k_j}/6 - (A+1)$. By the third condition on M (p. 10), since $n-k_j \ge n/2 > M$ we have $\epsilon \mathbf{E}R_{n-k_j}/6 \ge A+2$, so $t \ge 1$ and

$$\frac{1}{t^2} \le \frac{4(A+1)^2}{(t+A+1)^2} = \frac{144(A+1)^2}{\epsilon^2[\mathbf{E}R_{n-k_j}]^2}.$$

From (26) and the fact that $R_{n-k_j} \ge a(n-k_j) \ge an/2$, we therefore obtain

$$\begin{aligned}
\mathbf{P}\left\{ \left| \frac{R^\ell_{n,k_j}}{\mathbf{E}R^\ell_{n,k_j}} - 1 \right| > \frac{\epsilon}{3} \,\middle|\, \Gamma_1 \right\} &\le \frac{144A(A+1)^2}{\epsilon^2(\mathbf{E}R_{n-k_j})^2} \\
&\le \frac{576A(A+1)^2}{\epsilon^2 a^2 n^2}. \qquad (27)
\end{aligned}$$

Furthermore, R^r_{n,k_j} is independent of Γ_1, so the bound (27) immediately holds with R^ℓ_{n,k_j} replaced by R^r_{n,k_j}. On the other hand, for B_j to occur we must have either

$$\left| \frac{R^\ell_{n,k_j}}{\mathbf{E}R^\ell_{n,k_j}} - 1 \right| > \frac{\epsilon}{3} \quad \text{or} \quad \left| \frac{R^r_{n,k_j}}{\mathbf{E}R^r_{n,k_j}} - 1 \right| > \frac{\epsilon}{3},$$

and it follows by (27) and a union bound that

$$\begin{aligned}
\mathbf{P}\{B_j = 1 \mid \Gamma_1\} &\le \mathbf{P}\left\{\left|\frac{R^\ell_{n,k_j}}{\mathbf{E}R^\ell_{n,k_j}} - 1\right| > \frac{\epsilon}{3} \;\middle|\; \Gamma_1\right\} + \mathbf{P}\left\{\left|\frac{R^r_{n,k_j}}{\mathbf{E}R^r_{n,k_j}} - 1\right| > \frac{\epsilon}{3} \;\middle|\; \Gamma_1\right\} \\
&\le \frac{1152A(A+1)^2}{\epsilon^2 a^2 n^2} = \frac{D}{n^2}. \qquad (28)
\end{aligned}$$

We argue similarly for Γ_2. If Γ_2 occurs then $B_{j-1} = 1/2 + \epsilon$, so $k_{j-1} = k_j + 1$, so

$$\left|\frac{R^\ell_{n,k_j-1}}{\mathbf{E}R^\ell_{n,k_j-1}} - 1\right| \le \frac{\epsilon}{3} \quad \text{and} \quad \left|\frac{R^r_{n,k_j-1}}{\mathbf{E}R^r_{n,k_j-1}} - 1\right| \le \frac{\epsilon}{3}.$$

Let Γ^r be the event that $|R^r_{n,k_{j-1}}/\mathbf{E}R^r_{n,k_{j-1}} - 1| > \epsilon/6$. As R^r_{n,k_j-1} is independent from T^ℓ_{n,k_j-1}, we have

$$\mathbf{P}\{\Gamma^r \mid \Gamma_2\} = \mathbf{P}\left\{\Gamma^r \;\middle|\; \left|\frac{R^r_{n,k_{j-1}}}{\mathbf{E}R^r_{n,k_{j-1}}} - 1\right| \le \frac{\epsilon}{3}\right\} \le \mathbf{P}\{\Gamma^r\}. \qquad (29)$$

Arguing just as we did to obtain (27), it thus follows that

$$\mathbf{P}\{\Gamma^r \mid \Gamma_2\} \le \frac{576A(A+1)^2}{\epsilon^2 a^2 n^2}. \qquad (30)$$

If Γ^r does not occur but Γ_2 does, then since for any positive numbers r_1 and r_2 the function $r_1 r_2/(r_1 + r_2)$ is increasing in both r_1 and r_2, we have

$$\begin{aligned}
R^\ell_{n,k_j} &= \frac{R^\ell_{n,k_{j-1}} R^r_{n,k_{j-1}}}{R^\ell_{n,k_{j-1}} + R^r_{n,k_{j-1}}} \\
&\le \mathbf{E}R^\ell_{n,k_{j-1}} \frac{(1+\frac{\epsilon}{3})(1+\frac{\epsilon}{6})}{1+\frac{\epsilon}{3}+1+\frac{\epsilon}{6}},
\end{aligned}$$

Since $k_{j-1} = k_j + 1$, this yields

$$R^\ell_{n,k_j} = \mathbf{E}R^\ell_{n,k_j+1} \frac{(1+\frac{\epsilon}{3})(1+\frac{\epsilon}{6})}{2+\frac{\epsilon}{2}} \qquad (31)$$

By the definition of R_{n,k_j+1} and R_{n,k_j} and by Corollary 6, we have

$$\mathbf{E}R^\ell_{n,k_j+1} = 2^{k_j+1}\mathbf{E}R_{n-(k_j+1)} \le 2^{k_j+1}(\mathbf{E}R_{n-k_j} + B), \qquad (32)$$

where B is the constant from Corollary 6. Since $n \ge 2M = 3200B/a$ and $\mathbf{E}R_{n-k_j} \ge a(n-k_j) \ge an/2 \ge 1600B$, (32) yields that

$$\mathbf{E}R^\ell_{n,k_j+1} \le 2^{k_j+1}(\mathbf{E}R_{n-k_j})\left(1+\frac{\epsilon}{20}\right) = 2\mathbf{E}R^\ell_{n,k_j}\left(1+\frac{\epsilon}{20}\right) \qquad (33)$$

(we remind the reader that $\epsilon = 1/80$). It is a simple calculation to check that

$$\left(1+\frac{\epsilon}{20}\right)\left(1+\frac{\epsilon}{2}+\frac{\epsilon^2}{18}\right)\left(1+\frac{\epsilon}{4}\right)^{-1} < 1+\frac{\epsilon}{3},$$

so (31) and (33) yield that if Γ^r does not occur and Γ_2 does then $R^\ell_{n,k_j} \le (1+\epsilon/3)\mathbf{E}R^\ell_{n,k_j}$. An essentially identical argument shows that if Γ^r does not occur and Γ_2 does then $R^\ell_{n,k_j} \ge \mathbf{E}R^\ell_{n,k_j}(1-\epsilon/3)$. It follows immediately from these bounds and from (30) that

$$\mathbf{P}\left\{\left|\frac{R^\ell_{n,k_j}}{\mathbf{E}R^\ell_{n,k_j}} - 1\right| > \frac{\epsilon}{3} \;\middle|\; \Gamma_2\right\} \le \frac{576A(A+1)^2}{\epsilon^2 a^2 n^2}. \qquad (34)$$

Furthermore, R^r_{n,k_j} is independent of Γ_2, so the bound (34) immediately holds with R^ℓ_{n,k_j} replaced with R^r_{n,k_j}. It follows from these two bounds that

$$\mathbf{P}\{B_j = 1 \mid \Gamma_2\} \le \frac{1152A(A+1)^2}{\epsilon^2 n^2} = \frac{D}{n^2}. \tag{35}$$

The lemma follows immediately from (28) and (35). □

We are now ready for our bounds on $\prod_{j=1}^{\tau} B_j$.

Lemma 9. *There is a constant $c_1 > 0$ such that for all integers i, n with $n/2 > i > 20M$ and $n > \sqrt{D}$,*

$$\mathbf{P}\left\{\prod_{j=1}^{\tau} B_j \ge \left(\frac{1}{2}+\epsilon\right)^{4i/5}\right\} \le 2e^{-c_1 i n^2}.$$

Proof. We shall omit ceiling notation to preserve readability. We first remark that for a given $j \le \tau$, if $k_{j+1} = k_j - (M+1)$ then $B_j = 1$ so $B'_j = 1$. (This follows from the definition of k_j, B_j, and B'_j on page 11.) It follows that if $M\sum_{k=1}^{j} B'_k < i - j$ then k_{j+1} is defined and is at least $i - j - M\sum_{k=1}^{j} B'_k \ge 1$, so in particular $\tau > j$. Thus, by a Chernoff bound ([5]) and the assumption that $i > 20M$, we have

$$\begin{aligned}
\mathbf{P}\left\{\tau \le \frac{9i}{10}\right\} &\le \mathbf{P}\left\{\sum_{k=1}^{9i/10} B'_k \ge \frac{i}{10M}\right\} \\
&= \mathbf{P}\left\{\sum_{k=1}^{9i/10} B'_k \ge \frac{9i}{10}\mathbf{E}B'_1 + \left(\frac{i}{10M} - \frac{9i}{10}\mathbf{E}B'_1\right)\right\} \\
&\le \mathbf{P}\left\{\sum_{k=1}^{9i/10} B'_k \ge \frac{9i}{10}\mathbf{E}B'_1 + \left(\frac{i}{10M} - 1\right)\right\} \\
&\le \mathbf{P}\left\{\sum_{k=1}^{9i/10} B'_k \ge \frac{9i}{10}\mathbf{E}B'_1 + \frac{i}{20M}\right\} \\
&\le \exp\left\{-\frac{n^2 i}{360M^2 D}\right\}. \qquad (36)
\end{aligned}$$

Furthermore, if $\tau > 9i/10$ then

$$\prod_{j=1}^{\tau} B_j \le (1/2+\epsilon)^{9i/10 - \sum_{j=1}^{9i/10} B'_j},$$

and a calculation just as that leading to (36) shows that

$$\mathbf{P}\left\{\sum_{j=1}^{9i/10} B'_j \ge \frac{i}{10}\right\} \le \exp\left\{-\frac{n^2 i}{360D}\right\}. \tag{37}$$

Finally, if $\prod_{j=1}^{\tau} B_j \ge (1/2+\epsilon)^{4i/5}$ then either $\tau \le 9i/10$ or $\sum_{j=1}^{9i/10} B'_j \ge i/10$, so the lemma follows from (36) and (37) by letting $c_1 = 1/360M^2D$. □

Corollary 10. *For all integers n, i with $n \ge \sqrt{(4\ln 2)/c_1}$, with $n \ge \sqrt{D}$, with $i \ge 35$, and with $20MD < i < n/2$, $\mathbf{E}\left[\Theta^*(e_i)^4\right] \le 2^{-3i}$ where c_1 is the constant of Lemma 9.*

Proof. Since $\Theta^*(e_i) \le \prod_{j=\tau}^{i} B'_j \le 1$, by Lemma 9 we have

$$\begin{aligned}
\mathbf{E}\left[\Theta^*(e_i)^4\right] &\le \left(\frac{1}{2}+\epsilon\right)^{16i/5} + \mathbf{P}\left\{\prod_{j=1}^{\tau} B'_j > \left(\frac{1}{2}+\epsilon\right)^{4i/5}\right\} \\
&\le \left(\frac{1}{2}+\epsilon\right)^{16i/5} + 2e^{-c_1 i n^2}. \qquad (38)
\end{aligned}$$

Since $\epsilon = 1/80$, we have $(1/2+\epsilon)^{16i/5} < (41/80)^{16i/5} < 2^{-3i-1}$ for $i \ge 35$. Furthermore, $2e^{-c_1 i n^2} \le 2^{-3i-1}$ when $c_1 n^2 \ge 4\ln 2$; the corollary thus follows from (38). □

Finally, we have

Theorem 11. *There exist two constants n_0 and C such that, for all $n \ge n_0$, $\mathbf{Var}[R_n] \le C$.*

Proof. Let $n_0 = \max\{\sqrt{(4\ln 2)/c_1}, \sqrt{D}\}$, and assume that $n \ge n_0$. Equation (18) states that

$$\mathbf{Var}[R_n] \le \frac{(b-a)^2}{2} \sum_{i=0}^{n-1} 2^i \mathbf{E}\left[\Theta^*(e_i)^4\right].$$

Let $C' = \lceil \max\{35, 20MD\} \rceil$. For all $i \le C'$, we have $2^i\mathbf{E}\left[\Theta^*(e_i)^4\right] \le 2^{C'}$. For all i with $C' < i < n/2$, by Corollary 10, we have $2^i\mathbf{E}\left[\Theta^*(e_i)^4\right] \le 2^{-2i}$. Finally, for all $i \ge n/2$, again by Corollary 10 we have

$$2^i\mathbf{E}\left[\Theta^*(e_i)^4\right] \le 2^n\mathbf{E}\left[\Theta^*(e_{\lfloor n/2 \rfloor})^4\right] \le 2^n 2^{-3\lfloor n/2 \rfloor} \le 2^{1-n/2}.$$

Combining these bounds with (18), we obtain that

$$\mathbf{Var}[R_n] \le \frac{(b-a)^2}{2}\left(C'2^{C'} + \sum_{i=1}^{\infty} 2^{-2i} + n \cdot 2^{1-n/2}\right) < \frac{(b-a)^2}{2}(C'2^{C'}+4).$$

Letting C be the right-hand side above finishes the proof. □

5 Branching random networks

As mentioned in the introduction, when extending our results from binary (or d-ary) trees to branching processes, we are no longer able to prove deterministic bounds and must settle for almost sure bounds. In stating and proving our results, the following formulation of branching processes is useful. We start from a single "root edge" uv and let v be the root of a supercritical branching process with branching distribution B that satisfies $\mathbf{P}\{B = 0\} = 0$. We use $\mathcal{T}$ to refer this edge-rooted branching process. We say that a node $w \ne u$ has depth i if there are i edges on the path from v to w (v has depth 0). For $i \ge 0$, we let Z_i be the number of nodes of $\mathcal{T}$ at depth i – so in particular $Z_0 = 1$.

Also for $i \ge 0$, we fix an ordering of the nodes at depth i and let $B_{i,j}$ be the number of offspring of the j'th node at depth i – so $\sum_{j=1}^{Z_i} B_{i,j} = Z_{i+1}$. We assume this ordering is "consistent" in the sense that if w and w' are the j'th and $(j')'th$ nodes at depth i, respectively, and $j < j'$, then the children of w precede the children of j' in the ordering of the depth $i+1$ nodes. Given this condition, we can reconstruct $\mathcal{T}$ from the ordered lists of branch variables.

A branching random network is simply an edge-rooted branching process $\mathcal{T}$ as above. To each edge e connecting $B_{i,j}$ to its parent, we assign a random resistance $r_e = [\mathbf{E}B]^i X_{i,j}$ where the $X_{i,j}$ are independent, identically distributed positive random variables as in the binary case. (So $X_{1,1}$ is attached to root edge uv.) As in the binary case, this scaling most naturally corresponds to the critical case of a random walk in a random environment on branching processes with push-back. In particular, if B is deterministically 2 then we recover the model of the previous sections.

Given random functions $f(n)$ and $g(n)$, we say that $f(n) = O(g(n))$ $\mathcal{T}$*-almost surely* (or $\mathcal{T}$*-a.s.*) if with probability 1 (with respect to $\mathcal{T}$) there exists a constant C such that for all n sufficiently large, $f(n) \le C \cdot g(n)$. (Since C is allowed to depend on $\mathcal{T}$, we may equivalently say that there is an a.s. finite *random variable* C which is measurable with respect to $\mathcal{T}$ and such that for all $\mathcal{T}$, for all n sufficiently large, $f(n) \le C \cdot g(n)$. We are now prepared to state our results for branching processes.

Theorem 12. *There is a constant $K = K(a, b) > 0$ such that for all integers $n \ge 2$,*

$$\mathbf{Var}\left[\, C_n \mid \mathcal{T} \,\right] \le \sum_{i=1}^{n-1} \sum_{j=1}^{Z_{i-1}} \frac{K \cdot B_{i,j}^4}{[\mathbf{E}B]^{2i}(n-i)^4}$$

By making an assumption about the moments of the branch distribution, we can transform Theorem 12 into a cleaner, almost sure statement about the variance of C_n.

Corollary 13. *If* $\mathbf{E}\left[B^8\right] < \infty$ *then* $\mathbf{Var}\left[\, C_n \mid \mathcal{T} \,\right] = O(n^{-4})$ $\mathcal{T}$*-a.s.*

We first prove this corollary, then return to the proof of Theorem 12.

Proof. Doob's limit law [2, Theorem I.6.1] states that there is an almost surely finite random variable W such that

$$\frac{Z_n}{[\mathbf{E}B]^n} \xrightarrow[n\to\infty]{a.s.} W.$$

It follows in particular that for any $\epsilon > 0$,

$$\mathbf{P}\left\{Z_i \ge (\mathbf{E}B + \epsilon)^i \text{ infinitely often}\right\} = 0,$$

so letting τ_1^ϵ be the last time which $Z_i \ge (\mathbf{E}B + \epsilon)^i$, τ_1^ϵ is a.s. finite. Also, Erdős [9] has proved that the following two conditions are equivalent for any random variable Y:

1. $\mathbf{E}\left[Y^2\right] < \infty$.
2. For all $\epsilon > 0$, $\sum_{i=1}^{\infty} \mathbf{P}\left\{\left|\sum_{j=1}^{i} Y_{i,j} - i\mathbf{E}Y\right| > \epsilon \cdot i\right\} < \infty$, where $\{Y_{i,j}\}_{i,j=1}^{\infty}$ are independent copies of Y.

Since $\mathbf{P}\{B = 0\} = 0$, $W > 0$ a.s. and it is easily seen that the sequence $Z_0, Z_1, \ldots$ is a.s. eventually strictly increasing, and it easily follows from this fact, the fact that $\mathbf{E}\left[B^8\right] = \mathbf{E}\left[(B^4)^2\right] < \infty$, and Erdős's result that for all $\epsilon > 0$,

$$\sum_{i=1}^{\infty} \mathbf{P}\left\{\left|\sum_{j=1}^{Z_{i-1}} B_{i,j}^4 - Z_{i-1} \cdot \mathbf{E}\left[B^4\right]\right| > \epsilon \cdot Z_{i-1}\right\} < \infty.$$

So by the first Borel-Cantelli lemma,

$$\mathbf{P}\left\{\left|\sum_{j=1}^{Z_{i-1}} B_{i,j}^4 - Z_{i-1} \cdot \mathbf{E}\left[B^4\right]\right| > \epsilon \cdot Z_{i-1} \text{ infinitely often}\right\} = 0,$$

Let τ_2^ϵ be the last time the above $\left|\sum_{j=1}^{Z_{i-1}} B_{i,j}^4 - Z_{i-1}\cdot \mathbf{E}\left[B^4\right]\right| > \epsilon \cdot Z_{i-1}$. Then τ_2^ϵ is a.s. finite. Choose ϵ small enough that $(\mathbf{E}B+\epsilon)/[\mathbf{E}B]^2 < (1-\epsilon)$ and let $\tau = \max\{\tau_1^\epsilon, \tau_2^\epsilon\}$. For all n we have

$$\begin{aligned}\mathbf{Var}\left[\, C_n \mid \mathcal{T}, \{\tau < \infty\}\,\right] &\le \sum_{i=1}^{\tau}\sum_{j=1}^{Z^{i-1}} \frac{K\cdot B_{i,j}^4}{[\mathbf{E}B]^{2i}(n-i)^4} + \sum_{i=\tau+1}^{n-1}\left(\frac{\mathbf{E}B+\epsilon}{[\mathbf{E}B]^2}\right)^i \cdot \frac{K\cdot(\mathbf{E}\left[B^4\right]+\epsilon)}{(n-i)^4} \\ &= O(n^{-4}) \qquad \text{a.s.,}\end{aligned}$$

where the second sum is interpreted to be zero if $n \le \tau$. Since τ is a.s. finite, this completes the proof. □

Proof of Theorem 12. We follow the lines of the proof of Theorem 2, but in this case we shall explicitly apply induction on n. To this end, let

$$K = \max\left\{\frac{(b\cdot\mathbf{E}B)^2}{4}, \frac{1}{2}\cdot\left(\frac{b}{a\cdot\mathbf{E}B}\right)^4\cdot\left(\frac{1}{b}-\frac{1}{a}\right)^2\right\}.$$

The first term in the above maximum will take care of the base case of the induction, while the second will be used for the inductive step. When $n=2$, if C_1 is the conductance of the root edge then $0 \le C_n \le C_1 \le b$, so $\mathbf{Var}\left[\, C_n \mid \mathcal{T}\,\right] \le b^2/4$ so the base case is proved as $B_{1,1} \ge 1$.

For the inductive step, we first decompose T_n into independent conductors C_1 and $C_{n,1},\ldots,C_{n,B_{1,1}}$, so that

$$C_n = \frac{C_1(C_{n,1}+\ldots+C_{n,B_{1,1}})}{C_1+C_{n,1}+\ldots+C_{n,B_{1,1}}}. \tag{39}$$

For $i=1,\ldots,B_{1,1}$, write $C_n^{(i)}$ for the quantity

$$\frac{C_1(C_{n,1}+\ldots+C'_{n,i}+\ldots+C_{n,B_{1,1}})}{C_1+C_{n,1}+\ldots+C'_{n,i}+\ldots+C_{n,B_{1,1}}},$$

where $C'_{n,i}$ is an independent copy of $C_{n,i}$ conditioned on $\mathcal{T}$ (i.e., with the same branch variables as $C_{n,i}$ but with independent conductors on the edges), and write C_n^* for the equivalent quantity where C_1 is replaced by an independent copy C'_1. Applying the Efron–Stein inequality conditionally, we have

$$\mathbf{Var}\left[\, C_n \mid \mathcal{T}\,\right] \le \frac{1}{2}\mathbf{E}\left[\,(C_n - C_n^*)^2 \,\middle|\, \mathcal{T}\right] + \sum_{i=1}^{B_{1,1}} \frac{1}{2}\mathbf{E}\left[\left(C_n - C_n^{(i)}\right)^2 \,\middle|\, \mathcal{T}\right]. \tag{40}$$

Arguing as in Theorem 2, we see that

$$\frac{1}{2}\mathbf{E}\left[\,(C_n - C_n^*)^2 \,\middle|\, \mathcal{T}\right] \le \frac{b^4}{2}\left(\frac{1}{b}-\frac{1}{a}\right)^2 \mathbf{E}\left[\,(C_{n,1}+\ldots+C_{n,B_{1,1}})^4 \,\middle|\, \mathcal{T}\right].$$

Since $C_{n,1}+\ldots+C_{n,B_{1,1}}$ is deterministically at most $B_{1,1}\cdot(a(n-1)\mathbf{E}B)^{-1}$, we have

$$\frac{1}{2}\mathbf{E}\left[\,(C_n - C_n^*)^2 \,\middle|\, \mathcal{T}\right] \le \frac{1}{2}\left(\frac{b}{a}\right)^4\left(\frac{1}{b}-\frac{1}{a}\right)^2\cdot\left(\frac{B_{1,1}}{\mathbf{E}B}\right)^4\cdot\frac{1}{(n-1)^4}. \tag{41}$$

Also by the same arguments as Theorem 2, we obtain

$$\sum_{i=1}^{B_{1,1}} \frac{1}{2}\mathbf{E}\left[\left(C_n - C_n^{(i)}\right)^2 \,\middle|\, \mathcal{T}\right] \le \sum_{i=1}^{B_{1,1}} \frac{1}{2}\mathbf{E}\left[\,(C_{n,i} - C'_{n,i})^2 \,\middle|\, \mathcal{T}\right] = \sum_{i=1}^{Z_1} \mathbf{Var}\left[\, C_{n,i} \mid \mathcal{T}\,\right].$$

For each $i = 1, \ldots, B_{1,1}$, $C_{n,i}$ is a conditioned copy of C_{n-1}, all of whose weights have been divided by $\mathbf{E}B$. It follows by induction that

$$\sum_{i=1}^{B_{1,1}} \frac{1}{2}\mathbf{E}\left[\left(C_n - C_n^{(i)}\right)^2 \;\middle|\; \mathcal{T} \right] \leq \sum_{i=2}^{n-1} \sum_{j=1}^{Z_{i-1}} \frac{K \cdot B_{i,j}^4}{[\mathbf{E}B]^{2i}(n-i)^4}, \tag{42}$$

and combining (40), (41), and (42) proves the result. □

By an identical argument to that of Lemma 4, we obtain the following lemma, whose proof we omit.

Lemma 14. *For all $t > 0$, we have*

$$\mathbf{P}\left\{ \left| R_n - \frac{1}{\mathbf{E}[\, C_n \mid \mathcal{T}\,]} \right| > t \;\middle|\; \mathcal{T} \right\} = O(t^{-2}) \qquad \mathcal{T}\text{-a.s.},$$

and hence

$$\mathbf{E}\left[\left| R_n - \frac{1}{\mathbf{E}[\, C_n \mid \mathcal{T}\,]} \right| \;\middle|\; \mathcal{T} \right] = O(1) \qquad \mathcal{T}\text{-a.s.}$$

Although C_n is conditionally concentrated given $\mathcal{T}$, unconditionally it is not concentrated, as can be easily seen by considering (39). If C_n is concentrated, $C_{n,1} + \ldots + C_{n,B_{1,1}}$ is well-approximated by $B_{1,1} \cdot \mathbf{E}C_{n-1}/2$, so C_n is close to

$$\frac{C_1 \cdot B_{1,1} \cdot \mathbf{E}C_{n-1}/2}{C_1 + B_{1,1} + \mathbf{E}C_{n-1}/2} = \frac{\mathbf{E}C_{n-1}}{2} \cdot \left(\frac{1}{B_{1,1}} + \frac{\mathbf{E}C_{n-1}}{2C_1} \right)^{-1}.$$

But the latter expression is *not* concentrated – a change in $B_{1,1}$ changes this expression by a constant factor. This should not be surprising: if the root has many offspring, the conductance is likely to be much (a constant factor) higher than if the root has a single child. It seems likely that at least the first-order behavior of C_n and R_n is governed by $W = \lim_{n\to\infty} Z_n/[\mathbf{E}B]^n$. If the resistance random variable X is constant (say $X = 1$) then this is easily seen: the series-parallel laws give $R_n = \sum_{i=0}^{n} [\mathbf{E}B]^i/Z_i$, and $\mathbf{E}B^i/Z_i$ tends to $1/W$ $\mathcal{T}$-a.s., so R_n/n tends to $1/W$ $\mathcal{T}$-a.s.. If we additionally assume that B has finite variance then this convergence is also in expectation [2, Theorem I.6.2], and it immediately follows that $C_n/\mathbf{E}C_n$ tends to $W/\mathbf{E}W$ a.s. and in expectation. In particular, this implies that $\lim_{n\to\infty} C_n/\mathbf{E}C_n$ has absolutely continuous distribution (as long as B is not constant; see [2, Theorem I.10.4]), which is the "scaled analogue" of Question 4.1 from Lyons et al. [16] mentioned in the introduction.

We would expect that even when X is not constant, for *any* λ with $1 \leq \lambda \leq \mathbf{E}B$, in the network where the resistances of edges at depth i are scaled by λ^i, $\lim_{n\to\infty} C_n/\mathbf{E}C_n$ has an absolutely continuous distribution. In the special case that $\lambda = \mathbf{E}B$, we would venture that R_n/n tends to $\mathbf{E}X/W$, $\mathcal{T}$-a.s. and in expectation. We were unable to prove this, essentially because the arguments used to prove Theorem 5 do not extend to the branching process setting as readily as those of Theorem 2. In particular, once we condition on $\mathcal{T}$, our technique for manipulating (4) in order to devise a recurrence fails, most notably because in this setting the identical distribution of subtrees at equal depth is lost. It seems plausible that as in the case of binary branching $R_n - W \cdot (n\mathbf{E}X)$ is $O(\ln n)$, again $\mathcal{T}$-a.s. and in expectation, but the coefficient of $\ln n$ also seems likely to depend on $\mathcal{T}$ and the precise nature of this dependence is unclear to us.

References

[1] L. Addario-Berry and B. Reed. Minima in Branching Random Walks. submitted, 2007+.

[2] K. B. Athreya and P. E. Ney. *Branching Processes*. Springer, Berlin, 1972.

[3] I. Benjamini and R. Rossignol. Submean variance for effective resistance on random electric networks. to appear in Communications in Mathematical Physics, 2006. URL `http://arxiv.org/abs/math.PR/0610393`.

[4] I. Benjamini, G. Kalai, and O. Schramm. First passage percolation has sublinear distance variance. *Annals of Probability*, 31(4), 2003.

[5] H. Chernoff. A measure of the asymptotic efficiency for tests of a hypothesis based on the sum of observables. *Ann. Math. Statist.*, 2:493–509, 1952.

[6] N.G de Bruijn. *Asymptotic methods in analysis*. Dover Publications, 1981.

[7] P. G. Doyle and J. L. Snell. *Random Walks and Electric Networks*. The Mathematical Association of America, 1984.

[8] B. Efron and C. Stein. The jackknife estimate of variance. *Annals of Statistics*, 9: 586–596, 1981.

[9] P. Erdős. Remark on my paper "on a theorem of Hsu and Robbins". *The Annals of Mathematical Statistics*, 21(1):138, 1950.

[10] D. Falik and A. Samorodnitsky. Edge-isoperimetric inequalities and influences. to appear, 2007+.

[11] P. Flajolet and A. Odlyzko. The average height of binary trees and other simple trees. *Journal of Computer and System Sciences*, 25(171–213), 1982.

[12] G. R. Grimmett. *Percolation*, volume 321 of *A Series of Comprehensive Studies in Mathematics*. Springer Verlag, second edition, 1999.

[13] R. Lyons. Random Walks and Percolation on Trees. *The Annals of Probability*, 18(3): 931–958, July 1990.

[14] R. Lyons and R. Pemantle. Random walk in a random environment and first-passage percolation on trees. *Annals of Probability*, 20:125–136, 1992.

[15] R. Lyons and Y. Peres. *Probability on Trees and Networks*. Cambridge University Press, 2006.

[16] R. Lyons, R. Pemantle, and Y. Peres. Unsolved problems concerning random walks on trees. In *Classical and Modern Branching Processes*, volume 84 of *IMA Volumes in Mathematics and its Applications*, pages 223–237. Springer, 1997.

[17] R. Pemantle. Phase Transition in Reinforced Random Walk and RWRE on Trees. *The Annals of Probability*, 16(3):1229–1241, July 1988.

[18] Y. Peres. Probability on trees: an introductory climb. In *Lectures on probability theory and statistics (Saint-Flour, 1997)*, volume 1717, pages 193—280, Berlin, 1999. Springer.

[19] P. Soardi. *Potential theory on infinite networks*, volume 1590 of *Lecture Notes in Mathematics*. Springer, Berlin, 1994.

[20] J. M. Steele. An Efron-Stein inequality for nonsymmetric statistics. *Annals of Statistics*, 14(2):753–758, 1986.