A Hierarchy of Equivalence Relations for Partially Observable Markov Decision Processes

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Abstract

We discuss the problem of comparing the behavioural equivalence of partially observable systems with observations. We examine different types of equivalence relations on states, and show that branching equivalence relations are stronger than linear ones. Finally, we discuss how this hierarchy can be used in duality theory.

1 Introduction

Systems with hidden states are one of the main topics of modern AI research, and much work has been devoted to planning and learning under uncertainty. One of the most studied models in the field is that of Partially Observable Markov Decision Processes (POMDPs). The model is described in terms of a set of discrete states, actions and observations. Actions cause stochastic transitions, which in turn generate stochastic observations. This paper addresses the problem of behavioural equivalences, and constructs a hierarchy of equivalence relations on states.

The study is motivated by a recent interest in duality theory[HPP06], that shows a double dual construction which gives a minimal representation of the original system. The basic idea of duality is that one starts from a system with states, transitions labelled by input symbols and observations associated with the states. One can formulate a notion of experiment on such systems and can, in a natural way, define a dual system where the role of state and observation are interchanged. In the case of POMDPs, the double dual is a deterministic version of the primal, with stochastic observations. The absence of the hidden state in this representation promises better planning and learning algorithms than those currently available. However, the states of the double dual representation hold very little resemblance to those in the primal. Rather than thinking of them as states of the physical system, they should be thought of as predictions for experiments, but that cannot be directly measured. This paper discusses possible relationships between the states of the primal and those of the double dual, and more specifically, attempts to determine the equivalence relation used in collapsing the states of the primal.

Kupferman et. al. have distinguished between linear and branching equivalence relations between states[KPV00]. Linear relations are required to agree on linear behaviours, i.e. all linear paths between a start and an end state, while branching equivalence relations are required to agree on all possible behaviour trees between the two states. We will consider bisimulation as the main branching equivalence relation, and in addition to the usual linear equivalence relation that is trace equivalence, introduce the experiment and test based equivalences.

The purpose of this paper is to study the expressive power of both types of relations, in capturing the behaviour of a system. We show that the branching approach is stronger than the linear one, in the sense that bisimulation implies linear equivalence relations, but none of the linear relations mentioned above are enough to imply bisimulation. We discuss the implication of this result in the context of duality theory, by conjecturing that the states of the primal and those of the double dual are trace equivalent, but not bisimilar.

2 Background

We begin by introducing the notion of probabilistic automata with hidden state, enriched with a notion of observations.

Definition 1 *A* ***Partially Observable Markov Decision Process*** *(POMDP) is a quintuple*

$$\mathcal{M} = (\mathcal{S}, \mathcal{A}, \mathcal{O}, \tau : \mathcal{S} \times \mathcal{S} \times \mathcal{A} \to [0,1], \gamma : \mathcal{S} \times \mathcal{A} \times \mathcal{O} \to [0,1])$$

where $\mathcal{S}$ is a set of states, $\mathcal{A}$ is a set of actions, $\mathcal{O}$ is a set of observations, τ is a transition function and γ is an observation function. We will often write $\tau_a(s, s')$ for $\tau(s, s', a)$, and $\tau_a(s, X)$ for $\sum_{s' \in X} \tau_a(s, s')$, where $X \subseteq \mathcal{S}$. Similarly, we will write $\gamma_a(s', \omega)$ for $\gamma(s', a, \omega)$. Finally, we will write $P(s', \omega|s, a)$ for $\tau_a(s, s')\gamma_a(s', \omega)$.

It is important to note that the observations are associated with the transitions rather than with the states; the number $\gamma(s, a, \omega)$ is the probability of observing ω, given that the system takes the action a and ends up in s.

Generally, POMDPs also have a set of rewards associated with each action-state pair, which are crucial in learning optimal policies for a system. However, for the purpose of this paper, we will ignore the reward function, as it does not affect the equivalence relations discussed.

Using the elements of $\mathcal{O}$ as basic observations triggered by actions in $\mathcal{A}$, we can define a family of tests and experiments on states as follows:

Definition 2 *A* ***test*** *t is a sequence of actions followed by a single observation and is defined by the following grammar:*

$$t = a\omega | a \cdot t', a \in \mathcal{A}, \omega \in \mathcal{O}$$

For example, $a_1 a_2 a_3 o$ is a test.

Definition 3 *An* ***experiment*** *e is a non-empty sequence of tests, $e = t_1 t_2 \dots t_n$.*

We can describe the behaviour of a system as a prediction for a set of tests or experiments that can be done on it. Thus, a prediction for a test (or experiment), is just the probability of a set of observations occuring if a set of actions is executed.

Definition 4 *We use $\langle s_1|t|s_2\rangle$ to denote the probability that the system starts in state s_1, is subject to the test t and ends up in state s_2. This can be defined by induction on t as follows:*

$$\begin{aligned} t = a\omega : & \\ \langle s_1|a\omega|s_2\rangle &= \tau_a(s_1, s_2)\gamma_a(s_2, \omega) \\ t = at' : & \\ \langle s_1|at'|s_2\rangle &= \sum_{s'} \tau_a(s_1, s')\langle s'|t'|s_2\rangle \end{aligned}$$

Definition 5 *A similar notation, $\langle s_1|e|s_2\rangle$, can be used for experiments to express the probability that we see the observations in the experiment at the appropriate points of the action sequence:*

$$\begin{aligned} e = t : & \\ \langle s_1|t|s_2\rangle & \quad \text{as defined above} \\ e = te' : & \\ \langle s_1|te'|s_2\rangle &= \sum_{s'} \langle s_1|t|s'\rangle\langle s'|e'|s_2\rangle \end{aligned}$$

Note: $\langle s|e\rangle = \sum_{s'} \langle s|e|s'\rangle$

It should now become clear that tests and experiments can be seen as measurements on the system that entirely capture its behaviour.

3 Behavioural equivalences for POMDPs

We now proceed by identifying an equivalence relation on experiments based on the states that satisfy them. While the result might not appear to be of importance in this paper, it is necessary in understanding the dual and double dual construction outlined by Hundt et. al.[HPP06]

Definition 6 *We say two experiments, e_1 and e_2 are equivalent ($\sim$) if $\forall s \in \mathcal{S}\langle s|e_1\rangle = \langle s|e_2\rangle$*

The same definition can be used for tests, as they are just simpler experiments.

Intuitively, two experiments are considered the same if they express the same information about the system. However, this equivalence is not enough to reason about the entire system behaviour. Instead, we must look at when two different states of a system are identical.

Definition 7 *We say two states, $s_1, s_2 \in \mathcal{S}$ are **t-equivalent** ($\approx$) if*

$$\forall \text{ tests } t,\ \langle s_1|t\rangle = \langle s_2|t\rangle$$

*Similarly, $s_1, s_2 \in \mathcal{S}$ are **e-equivalent** ($\simeq$) if*

$$\forall \text{ experiments } e,\ \langle s_1|e\rangle = \langle s_2|e\rangle$$

From this perspective, two states are the same if they have the same behaviour: the experiments taken in one are satisfied with the same probability by the other.

Both t- and e-equivalence relations are linear ones, as they do not take branching into consideration at any stage. A similar linear equivalence relation can be defined by considering a sequence of actions, and its resulting sequence of observations.

Definition 8 *We say two states $s_1, s_2 \in \mathcal{S}$ are* ***trace equivalent*** *if:*

$$P(\omega_1 \dots \omega_n | s_1, a_1 \dots a_n) = P(\omega_1 \dots \omega_n | s_2, a_1 \dots a_n)$$

The central theorem of this paper will show that in fact, trace equivalence is identical to the previously introduced notion of e-equivalence.

An even stronger notion of state equivalence is bisimulation. Bisimulation for discrete systems was proposed and studied by Larsen and Skou[LS91], but can also be found in earlier works by Park and Milner[Mil80].

Definition 9 *We define a* ***bisimulation*** *relation R to be an equivalence relation on $\mathcal{S}$ such that:*

$$s_1 R s_2 \Rightarrow \forall a \in \mathcal{A}, \forall \omega \in \mathcal{O}, \forall C \in \mathcal{S}/R, P(C, \omega | s_1, a) = P(C, \omega | s_2, a)$$

where $P(C, \omega | s_1, a) = \sum_{s' \in C} P(s', \omega | s_2, a)$.

We are now ready to state and prove our main result:

Theorem 1

bisimulation $\Rightarrow$ e-equivalence $\Rightarrow$ trace equivalence $\Rightarrow$ t-equivalence

bisimulation $\not\Leftarrow$ e-equivalence $\Leftarrow$ trace equivalence $\not\Leftarrow$ t-equivalence

This result should not be surprising, given the difference between linear and branching relations. Bisimulation is stronger than the equivalences presented before in the sense that it is dependent on the branching structure, requiring not only the two starting states to exhibit the same behaviour, but each of their successors as well.

Before proving any of the equivalences stated above, we will show that the probability of observing any test or experiment can be expressed in terms of the probability of observing some experiment of the form $(a\omega)^*$. We will do this in two steps: first by showing that any test can be expressed in this form, and then by generalizing it to any experiment.

Lemma 1 *Let s, s' be states in $\mathcal{S}$, $a_1, \dots, a_n$ actions in $\mathcal{A}$ and ω an observation in $\mathcal{O}$. Then, $\forall n$, $\langle s | a_1 a_2 \dots a_n \omega | s' \rangle = \sum_{\omega_1, \dots, \omega_{n-1} \in \mathcal{O}} \langle s | a_1 \omega_1 a_2 \omega_2 \dots a_n \omega | s' \rangle$.*

Proof . The proof is by induction on n. The base case, when $n = 1$ is trivial.

Consider now taking another action a_{n+1},before observing ω. Then,

$$\begin{aligned}
\langle s|a_1a_2\dots a_na_{n+1}\omega|s'\rangle &= \sum_{s_1\in\mathcal{S}} \tau_{a_1}(s,s_1)\langle s_1|a_2a_3\dots a_{n+1}\omega|s'\rangle \\
&= \sum_{s_1\in\mathcal{S}} \tau_{a_1}(s,s_1) \sum_{\omega_2\dots\omega_n\in\mathcal{O}} \langle s|a_2\omega_2\dots a_n\omega_n a_{n+1}\omega|s'\rangle \text{ by I.H} \\
&= \sum_{s_1\in\mathcal{S}} \tau_{a_1}(s,s_1) \sum_{\omega_1\in\mathcal{O}} \gamma_{a_1}(s_1,\omega_1) \sum_{\omega_2\dots\omega_n\in\mathcal{O}} \langle s|a_2\omega_2\dots a_{n+1}\omega|s'\rangle \\
&= \sum_{\omega_1\dots\omega_n\in\mathcal{O}} \sum_{s_1\in\mathcal{S}} \langle s|a_1\omega_1|s_1\rangle\langle s_1|a_2\omega_2\dots a_{n+1}\omega|s'\rangle \\
&= \sum_{\omega_1\dots\omega_n\in\mathcal{O}} \langle s|a_1\omega_1a_2\omega_2\dots a_{n+1}\omega|s'\rangle
\end{aligned}$$

∎

We now extend this result for any experiment.

Theorem 2 *Let s, s' be states in $\mathcal{S}$ and $t_1,\dots,t_n$ tests, with $t_1 = a_1\dots a_k\omega_k$, $t_2 = a_{k+1}\dots$ etc. Then, $\forall n$, $\langle s|t_1\dots t_n|s'\rangle = \sum_{\omega1,\dots,\omega_{m-1}\in\mathcal{O}}\langle s|a_1\omega_1a_2\omega_2\dots a_m\omega|s'\rangle$ for some m (where m is the total number of actions in all the n tests).*

Proof . The proof is by induction on the size on n. The base case, when $n = 1$ has been proved in Lemma 1. Using the induction hypothesis states, consider the case of $n + 1$:

$$\begin{aligned}
\langle s|t_1t_2\dots t_{n+1}\omega|s'\rangle &= \sum_{s_1\in\mathcal{S}} \langle s|t_1|s_1\rangle\langle s_1|t_2\dots t_{n+1}|s'\rangle \\
&= \sum_{s_1\in\mathcal{S}} \sum_{\omega_1\dots\omega_{k-1}\in\mathcal{O}} \langle s|a_1\omega_1\dots a_k\omega_k|s_1\rangle\langle s_1|t_2\dots t_{n+1}|s'\rangle \text{ by Lemma 1} \\
&= \sum_{s_1\in\mathcal{S}} \sum_{\omega_1\dots\omega_{k-1}\in\mathcal{O}} \langle s|a_1\omega_1\dots a_k\omega_k|s_1\rangle \sum_{\omega_k,\dots,\omega_{p-1}\in\mathcal{O}} \langle s_1|a_k\omega_k\dots a_p\omega_p|s'\rangle \text{by I.H} \\
&= \sum_{\omega_1,\dots,\omega_{p-1}\in\mathcal{O}} \langle s_1|a_1\omega_1\dots a_p\omega_p|s'\rangle
\end{aligned}$$

∎

We proceed by showing that e-equivalence and trace equivalence are identical. In other words, neither equivalence relation is more expressive regarding the behaviour of the system.

Lemma 2 *Let s be any state in $\mathcal{S}$, $a_1,\dots,a_n$ actions in $\mathcal{A}$ and $\omega_1,\dots,\omega_n$ observations in $\mathcal{O}$. Then, $\forall n$, $P(\omega_1,\dots,\omega_n|s,a_1,\dots,a_n) = \langle s|a_1\omega_1\dots a_n\omega_n\rangle$.*

Proof . The proof is by induction on n. Let us first consider the base case of $n = 1$. Then

$$\begin{aligned}
P(\omega_1|s,a_1) &= \sum_{s'} \tau_{a_1}(s,s')\gamma_{a_1}(s',\omega_1) \\
&= \sum_{s'} \langle s|a_1\omega_1|s'\rangle \\
&= \langle s|a_1\omega_1\rangle
\end{aligned}$$

The inductive hypothesis is that $\forall n, P(\omega_1, \ldots, \omega_n | s, a_1, \ldots, a_n) = \langle s | a_1\omega_1 \ldots a_n\omega_n \rangle$.

Now suppose another action a_{n+1} is taken, resulting in the observation ω_{n+1}. Then,

$$\begin{aligned}
P(\omega_1, \ldots, \omega_{n+1} | s, a_1, \ldots, a_{n+1}) &= \sum_{s'} \tau_{a_1}(s, s')\gamma_{a_1}(s', \omega_1) P(\omega_2, \ldots, \omega_{n+1} | s', a_2, \ldots, a_{n+1}) \\
&= \sum_{s'} \tau_{a_1}(s, s')\gamma_{a_1}(s', \omega_1) \langle s' | a_2\omega_2 \ldots a_{n+1}\omega_{n+1} \rangle \quad \text{by I.H} \\
&= \sum_{s'} \langle s | a_1\omega_1 | s' \rangle \langle s' | a_2\omega_2 \ldots a_{n+1}\omega_{n+1} \rangle \\
&= \langle s | a_1\omega_1 a_2\omega_2 \ldots a_{n+1}\omega_{n+1} \rangle
\end{aligned}$$

■

Theorem 3 *E-equivalence is equivalent to trace equivalence.*

Proof . Let s_1 and s_2 be two states in $\mathcal{S}$, $a_1 \ldots a_n$ be actions in $\mathcal{A}$ and $\omega_1 \ldots \omega_n$ observations in $\mathcal{O}$. We begin by proving the forward direction, namely that e-equivalence implies trace equivalence by considering experiments of the form $(a\omega)^*$. By definition,

$$\langle s_1 | a_1\omega_1 \ldots a_n\omega_n \rangle = \langle s_2 | a_1\omega_1 \ldots a_n\omega_n \rangle$$

But by Lemma 2,

$$\begin{aligned}
P(\omega_1 \ldots \omega_n | s_1, a_1 \ldots a_n) &= \langle s_1 | a_1\omega_1 \ldots a_n\omega_n \rangle \quad \text{and} \\
P(\omega_1 \ldots \omega_n | s_2, a_1 \ldots a_n) &= \langle s_2 | a_1\omega_1 \ldots a_n\omega_n \rangle
\end{aligned}$$

Thus,

$$P(\omega_1 \ldots \omega_n | s_1, a_1 \ldots a_n) = P(\omega_1 \ldots \omega_n | s_2, a_1 \ldots a_n)$$

The same argument in reverse order shows that trace equivalence implies e-equivalence. ■

The result should not come as a surprise. In Theorem 2, we have shown that any experiment can be observed with the same probability as some action-observation sequence, thus, it should be expected that the probability of observing a particular experiment is the same as the probability of observing each of its observations in sequence, if its actions are taken in sequence.

Lemma 3 *If R is a bisimulation relation, and C is any R-equivalence class, then, for any two states s and s', and $\forall \omega_1, \ldots, \omega_n \in \mathcal{O}$, and $\forall a_1, \ldots, a_n \in \mathcal{A}$:*

$$P(\omega_1, \ldots, \omega_n, C | s, a_1, \ldots, a_n) = P(\omega_1, \ldots, \omega_n, C | s', a_1, \ldots, a_n)$$

Proof . The proof is by induction on n. The base case of $n = 1$, namely $P(\omega, C | s, a) = P(\omega, C | s', a)$, follows from the definition of bisimulation.

The inductive hypothesis is $P(\omega_1, \ldots, \omega_n, C | s, a_1, \ldots, a_n) = P(\omega_1, \ldots, \omega_n, C | s', a_1, \ldots, a_{n-1})$

Now suppose another action a_{n+1} is taken, resulting in the observation ω_{n+1}. Then,

$$
\begin{aligned}
& P(\omega_1, \ldots, \omega_{n+1}, C | s, a_1, \ldots, a_{n+1}) \\
= & \sum_{s_1 \in S} \tau_{a_1}(s, s_1) \gamma_{a_1}(s_1, \omega_1) \cdots \sum_{s_{n+1} \in S} \tau_{a_{n+1}}(s_n, s_{n+1}) \gamma_{a_{n+1}}(s_{n+1}, \omega_{n+1}) \\
= & \sum_{s_1 \ldots s_n \in \mathcal{S}} \left(\prod_{i=1}^{n} \tau_{a_i}(s_{i-1}, s_i) \gamma_{a_i}(s_i, \omega_i) \right) \sum_{s_{n+1} \in C} \tau_{a_{n+1}}(s_n, s_{n+1}) \gamma_{a_{n+1}}(s_{n+1}, \omega_{n+1}) \\
= & \sum_{s_1 \ldots s_n \in \mathcal{S}} \left(\prod_{i=1}^{n} \tau_{a_i}(s_{i-1}, s_i) \gamma_{a_i}(s_i, \omega_i) \right) \sum_{C} P(\omega_{n+1}, C | s_n, a_{n+1}),
\end{aligned}
$$

where $s_0 = s$.

Replacing $P(\omega_{n+1}, C | s_n, a_{n+1})$ with K_C, where K_C only depends on the equivalence class of s_n, and summing over all equivalence classes:

$$
\begin{aligned}
& P(\omega_1, \ldots, \omega_{n+1}, C | s, a_1, \ldots, a_{n+1}) \\
= & \sum_{s_1 \ldots s_{n-1} \in S} \left(\prod_{i=1}^{n-1} \tau_{a_i}(s_{i-1}, s_i) \gamma_{a_i}(s_i, \omega_i) \right) \sum_{C} K_C \sum_{D} \sum_{s_n \in D} \tau_{a_n}(s_{n-1}, s_n) \gamma_{a_n}(s_n, \omega_n) \\
= & \sum_{C} K_C \sum_{D} \sum_{s_1 \ldots s_n \in S} \prod_{i=1}^{n} \tau_{a_i}(s_{i-1}, s_i) \gamma_{a_i}(s_i, \omega_i) \\
= & \sum_{C} K_C \sum_{D} P(\omega_1, \ldots, \omega_n, D | s, a_1, \ldots, a_n) \\
= & \sum_{C} K_C \sum_{D} P(\omega_1, \ldots, \omega_n, D | s', a_1, \ldots, a_n) \quad \text{by I.H} \\
= & \sum_{C} K_C \sum_{D} \sum_{s'_1 \ldots s'_n \in S} \prod_{i=1}^{n} \tau_{a_i}(s'_{i-1}, s'_i) \gamma_{a_i}(s'_i, \omega_i),
\end{aligned}
$$

where $s'_0 = s'$.

$$
\begin{aligned}
& P(\omega_1, \ldots, \omega_{n+1}, C | s, a_1, \ldots, a_{n+1}) \\
= & \sum_{s'_1 \ldots s'_n \in S} \left(\prod_{i=1}^{n} \tau_{a_i}(s'_{i-1}, s'_i) \gamma_{a_i}(t_i, \omega_i) \right) \sum_{C} P(\omega_{n+1}, C | s'_n, a_{n+1}) \\
= & \sum_{s'_1 \ldots s'_n \in S} \left(\prod_{i=1}^{n} \tau_{a_i}(s'_{i-1}, s'_i) \gamma_{a_i}(s'_i, \omega_i) \right) \sum_{s'_{n+1} \in C} \tau_{a_{n+1}}(s'_n, s'_{n+1}) \gamma_{a_{n+1}}(s'_{n+1}, \omega_{n+1}) \\
= & P(\omega_1, \ldots, \omega_{n+1}, C | s', a_1, \ldots, a_{n+1})
\end{aligned}
$$

■

We will now show that bisimulation is in fact a stronger relation than e-equivalence, by showing that, while bisimulation implies e-equivalence, the converse does not hold.

Theorem 4 *Bisimulation implies e-equivalence*

Proof . Since e-equivalence is equivalent to trace equivalence, it is enough to show that bisimulation implies trace equivalence. Let $a_1, \ldots, a_n$ be actions in $\mathcal{A}$ and $\omega_1, \ldots, \omega_n$ be observations in $\mathcal{O}$. If s and s' are states in $\mathcal{S}$ such that sRs' for a bisimulation relation R then $P(\omega_1, \ldots, \omega_n | s, a_1, \ldots, a_n) = P(\omega_1, \ldots, \omega_n | s', a_1, \ldots, a_n)$.

The proof is by induction on n. First suppose that ω is a simple observation. Then:

$$\begin{aligned} P(\omega|s,a) &= \sum_C P(\omega, C|s,a) \\ &= \sum_C P(\omega, C|s',a) \qquad \text{by Lemma 3} \\ &= P(\omega|s',a) \end{aligned}$$

By the induction hypothesis, $P(\omega_1, \ldots, \omega_n | s, a_1, \ldots, a_n) = P(\omega_1, \ldots, \omega_n | s', a_1, \ldots, a_n)$.

Now suppose the action a_{n+1} is taken, producing the observation ω_{n+1}. Then,

$$\begin{aligned} & P(\omega_1, \ldots, \omega_{n+1} | s, a_1, \ldots, a_{n+1}) \\ = \; & P(\omega_1|s,a_1)P(\omega_2|s,a_1\omega_1a_2)\ldots P(\omega_n|s,a_1\omega_1\ldots\omega_{n-1}a_n)P(\omega_{n+1}|s,a_1\omega_1\ldots\omega_n a_{n+1}) \\ = \; & P(\omega_1, \ldots, \omega_n|s,a_1,\ldots,a_n)P(\omega_{n+1}|s,a_1\omega_1\ldots\omega_n a_{n+1}) \\ = \; & P(\omega_1, \ldots, \omega_n|s',a_1,\ldots,a_n)P(\omega_{n+1}|s,a_1\omega_1\ldots\omega_n a_{n+1}) \quad \text{by I.H} \end{aligned}$$

Thus, we need to show that: $P(\omega_{n+1}|s,a_1\omega_1\ldots\omega_n a_{n+1}) = P(\omega_{n+1}|s',a_1\omega_1\ldots\omega_n a_{n+1})$.

$$\begin{aligned} & P(\omega_{n+1}|s,a_1\omega_1\ldots\omega_n a_{n+1}) \\ = \; & \sum_{s_1 \in S} \tau_{a_1}(s,s_1)\gamma_{a_1}(s_1,\omega_1) \cdots \sum_{s_{n+1} \in S} \tau_{a_{n+1}}(s_n,s_{n+1})\gamma_{a_{n+1}}(s_{n+1},\omega_{n+1}) \\ = \; & \sum_{s_1 \ldots s_n \in \mathcal{S}} \left(\prod_{i=1}^{n} \tau_{a_i}(s_{i-1},s_i)\gamma_{a_i}(s_i,\omega_i) \right) \sum_{s_{n+1} \in C} \tau_{a_{n+1}}(s_n,s_{n+1})\gamma_{a_{n+1}}(s_{n+1},\omega_{n+1}) \\ = \; & \sum_{s_1 \ldots s_n \in \mathcal{S}} \left(\prod_{i=1}^{n} \tau_{a_i}(s_{i-1},s_i)\gamma_{a_i}(s_i,\omega_i) \right) \sum_C P(\omega_{n+1}, C|s_n, a_{n+1}) \end{aligned}$$

Where, as before, $s_0 = s$. Replacing $P(\omega_{n+1}, C|s_n, a_{n+1})$ by K_C,

$$\begin{aligned} & P(\omega_{n+1}|s,a_1\omega_1\ldots\omega_n a_{n+1}) \\ = \; & \sum_{s_1 \ldots s_{n-1} \in \mathcal{S}} \left(\prod_{i=1}^{n-1} \tau_{a_i}(s_{i-1},s_i)\gamma_{a_i}(s_i,\omega_i) \right) \sum_C K_C \sum_{s_n \in C} \tau_{a_n}(s_{n-1},s_n)\gamma_{a_n}(s_n,\omega_n) \\ = \; & \sum_C K_C \sum_{s_1 \ldots s_n \in \mathcal{S}} \left(\prod_{i=1}^{n} \tau_{a_i}(s_{i-1},s_i)\gamma_{a_i}(s_i,\omega_i) \right) \\ = \; & \sum_C P(\omega_1, \ldots, \omega_n, C|s, a_1, \ldots, a_n) \end{aligned}$$

Applying the result of Lemma 3,

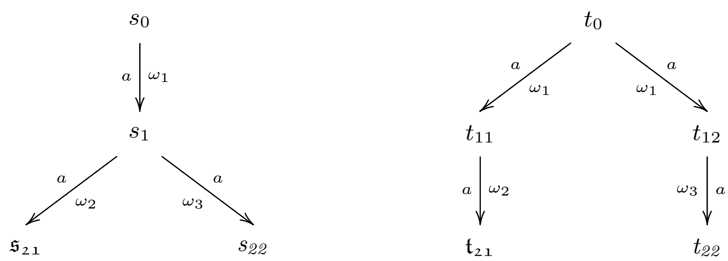
$$
\begin{aligned}
& P(\omega_{n+1}|s, a_1\omega_1 \ldots \omega_n a_{n+1}) \\
= \; & \sum_C P(\omega_1, \ldots, \omega_n, C|s', a_1, \ldots, a_n) \qquad \text{by Lemma 3} \\
= \; & \sum_C K_C \sum_{s'_1 \ldots s'_{n-1} \in \mathcal{S}} \left(\prod_{i=1}^{n-1} \tau_{a_i}(s'_{i-1}, t_i) \gamma_{a_i}(s'_i, \omega_i) \right) \sum_{s'_n \in C} \tau_{a_n}(s'_{n-1}, s'_n) \gamma_{a_n}(s'_n, \omega_n) \\
= \; & K_C \sum_{s'_1 \ldots s'_n \in \mathcal{S}} \left(\prod_{i=1}^{n} \tau_{a_i}(s'_{i-1}, 1_i) \gamma_{a_i}(s'_i, \omega_i) \right) \sum_{s'_{n+1} \in \mathcal{S}} \tau_{a_{n+1}}(s'_n, s'_{n+1}) \gamma_{a_{n+1}}(s'_{n+1}, \omega_{n+1}) \\
= \; & P(\omega_{n+1}|s', a_1\omega_1 \ldots \omega_n a_{n+1})
\end{aligned}
$$

$\blacksquare$

We will now show that bisimulation is the strongest of equivalence relations, as none of the linear relations presented are enough to imply it.

Theorem 5 *E-equivalence does not imply bisimulation.*

Proof . Consider the example below, where $s_0, s_1, t_0, t_{11}, t_{12}$ emit the same observation, ω_1, s_{21} and t_{21} emit ω_2 and s_{22} and t_{22} emit ω_3:



Note that s_0 and t_0 are e-equivalent, as they satisfy all possible experiments with the same probability.

However, s_0 and t_0 are not bisimilar, as s_1, t_{11} and t_{12} are not in the same equivalence class (s_1 can transition to both s_{21} and s_{22}, thus observing both ω_2 and ω_3; on the other hand, t_{11} and t_{12} can only transition to t_{21} and t_{22} respectively, thus each observing only one of the two possible observations). $\blacksquare$

As we mentioned before, bisimulation is sensitive to branching. This means that it is not enough for two states to exhibit the same behaviour; they must transition to equivalent states as well. This is the most important difference between branching and linear equivalence relations, and shows that traces of the system are not enough to fully characterize its behaviour.

We will now show that trace equivalence is strictly stronger than t-equivalence.

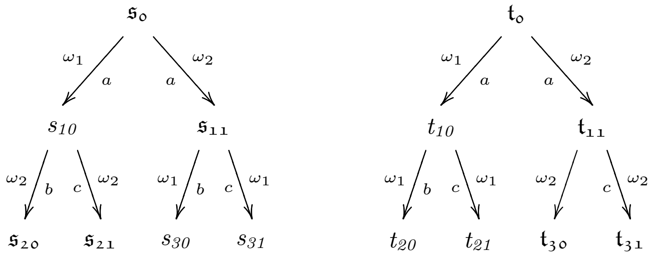
Theorem 6 *Trace equivalence implies t-equivalence.*

Proof . Since tests are just a specific case of experiments, it follows from the definition that e-equivalence implies t-equivalence. By Theorem 3, trace equivalence and e-equivalence are equivalent, and thus trace equivalence also implies t-equivalence. ∎

Finally, we will conclude by showing that t-equivalence is the weakest of linear equivalence relations.

Theorem 7 *T-equivalence does not imply trace equivalence.*

Proof . Consider the example below, where $s_0, s_{10}, s_{20}, s_{21}, t_0, t_{11}, t_{30}, s_{31}$ emit the same observation ω_1, and the remaining states emit ω_2:



Note that s_0 and t_0 are t-equivalent, as they satisfy the same tests. However, they are not e-equivalent, as $\langle s_0|a\omega_1 b\omega_2\rangle \neq \langle t_0|a\omega_1 b\omega_2\rangle$ (because t_0 does not satisfy the test $a\omega_1 b\omega_2$). Thus, t-equivalence does not imply e-equivalence. By Theorem 3, trace equivalence is equivalent to e-equivalence, so it follows that also t-equivalence does not imply trace equivalence either. ∎

This result should also look familiar. T-equivalence can only express very general similarities in the behaviour of two states, as tests are a sequence of actions followed by a single observation. Thus, t-equivalence says nothing about the observations interspersed in the sequence of actions. On the other hand, trace equivalence exactly pins down the intermediary observations, thus proving to be a stronger equivalence relation.

4 Conclusions and Future Work[1]

We have shown that state equivalence can be expressed in two different ways, namely through linear and branching equivalence relations. Furthemore, we have shown that a very clear hierarchy of relations exists, ranging from bisimulation to the rather weaker t-equivalence.

As we mentioned before, the double dual of a POMDP is a deterministic machine; there is no uncertainty associated with state transitions. Thus, ignoring the action choice, in between any two states, there is a single path rather than an entire computation tree. Thus, the double dual's states must be related by a linear equivalence relation. This means that a linear equivalence relation is enough to capture the behaviour of the double dual, and thus of the primal as well. We believe that the states of the primal are collapsed according to the strongest of linear equivalence relations presented, namely trace equivalence. This remains to be proven.

[1] Affectionately known as Outrageous Conjectures for All

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