VC-Dimension of Visibility on Terrains

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Outline

Guarding problems

- No time to explain set systems, but...
- We all love guarding problems!
  - The *art gallery problem* (many flavours!)
  - Terrain guarding problems
  - *etc.*

Guarding problem = set cover

- In a guarding problem, we have
  - Points that need to be guarded,
  - Potential sites for guards,
  - Obstacles – a guard only sees a point if the line of sight is unobstructed.
- Guards define sets of points that they see.
- We need to cover the points with the minimum number of these sets.

VC-dimension of a guarding problem

- VC-dimension is one way to quantify the complexity of a set system, but I'll just define it for guarding problems.

Definition

A set of points P is *shattered* if, for every possible subset $P' \subseteq P$, there is a guard that sees everything in P' and nothing in $P \setminus P'$. The *VC-dimension* of the guarding problem is the size of the largest such set P that is shattered.

VC-dimension of guarding problems

- Each instance of a guarding problem has a well-defined VC-dimension.
- We define the VC-dimension of a guarding problem to be the maximum VC-dimension over all instances of the problem.

VC-dimension of guarding problems

- Previously known bounds for VC-dimension of guarding problems:
  - For guarding polygons without holes (*i.e.* simple polygons) is at least 6 and at most 23 [Valtr '98].
  - For guarding polygons with holes the VC-dimension is unbounded [Eidenbenz *et al.*'01].
  - Tight constant bounds are known for guarding the exterior of polygons (different bounds for different variations) [Isler *et al.*'04].
  - For guarding the exterior of polyhedra (in $\mathbb{R}^d$, $d \geq 3$) the VC-dimension is unbounded [Isler *et al.*'04].

Motivation

- Low VC-dimension means simplicity!
- VC-dimension has consequences w.r.t. approximability, small ϵ-nets.
- Bounding the VC-dimension is mainly of theoretical interest.

Our results

- For guarding 1.5-dimensional terrains, the VC-dimension is 4.
  - The lower bound comes from an example.
  - The upper bound comes from application of a simple structural property.
- For guarding 2.5-dimensional terrains, the VC-dimension is unbounded.
  - This comes from a very simple reduction from polygons with holes.

Outline

What is a 1.5-D terrain?

- Also known as an x-monotone chain.
- The terrain intersects any vertical line at most once.
- No caves or overhangs.

- Points on the terrain 'see' each other if the line segment connecting them is never below the terrain.

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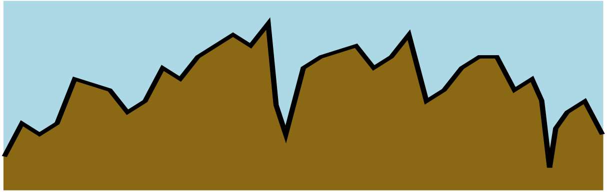

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The 1.5-D terrain guarding problem

- We want a minimum set of guards on the terrain that see the entire terrain.

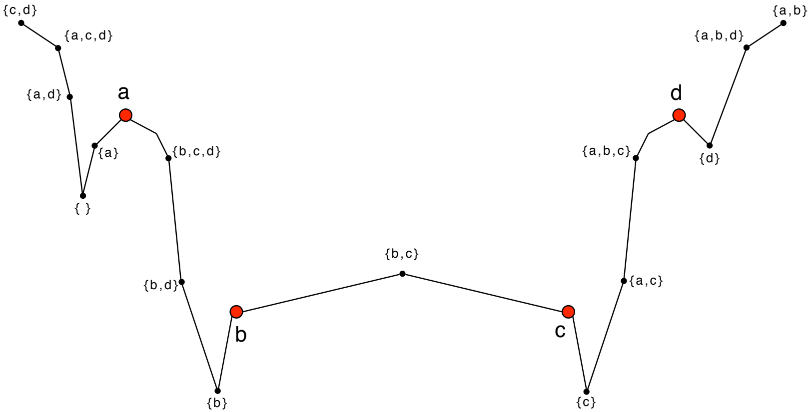
Complexity of 1.5-D terrain guarding

- It is unknown whether or not the problem is NP-complete!
- We know constant factor approximation algorithms exist.
  - The best approximation factor so far is 5.
- Knowing the VC-dimension does not improve this, but is of theoretical interest.

Finding a lower bound of 4

- We will give a lower bound of 4 for the VC-dimension.
- To find a lower bound for the VC-dimension of terrains all we need is an example.
- We need 4 points that are shattered by 16 guards.
- Not shown in the example (next slide) is a guard on the left side, on a spike high enough that it sees the whole terrain.

Lower bound of 4

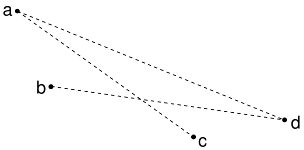


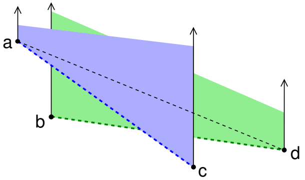
Upper bounding the VC-dimension

- Finding an upper bound is only slightly more complicated.
- We use the *Order Claim*, which captures the simplicity of 1.5-D terrains.

Order Claim

- The fundamental property of 1.5D terrains that we exploit.





- Consider $a < b < c < d$.
- If a sees c and b sees d then a sees d.

Giving an upper bound of 4

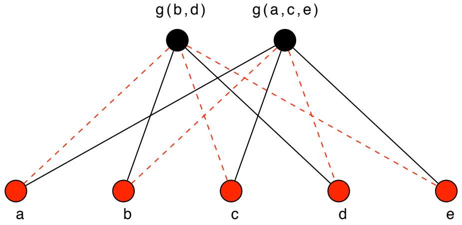
- We prove that the VC-dimension is at most 4 as follows:
  - Assume we have a set $P = \{a, b, c, d, e\}$ of points on a terrain, with $a < b < c < d < e$.
  - State that P can be shattered.
  - Prove, using the order claim, that we must arrive at a contradiction.

Proving the upper bound

- If P is shattered there must be 32 shattering guards, of which we consider 3:
  - $g(a, c, e)$ that sees a, c, and e but not b or d.
  - $g(b, d)$ that sees b and d but not a, c, or e.
  - $g(b, d, e)$ that sees b, d, and e but not a or c.

Proving the upper bound

- Assume without loss of generality that $g(b, d) < g(a, c, e)$.
  - If this is not true we can flip everything horizontally and relabel a, b, c, d, e in the opposite order.
- The order claim now tells us that $g(b, d) < c < d < g(a, c, e)$.



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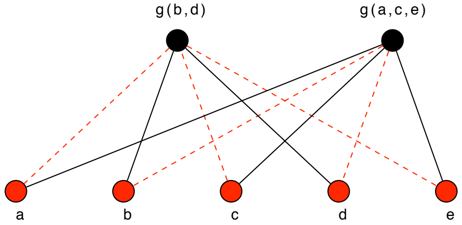
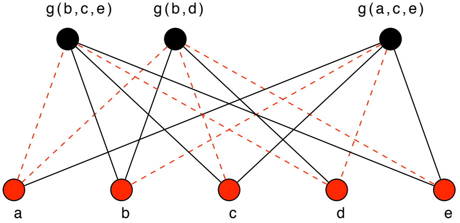


Proving the upper bound

- Now $g(b, d, e)$ must fit in one of the following ranges:
  - left of $g(b, d)$
  - between $g(b, d)$ and d
  - between d and $g(a, c, e)$
  - right of $g(a, c, e)$.
- Each range contradicts the order claim.



Outline

What is a 2.5-D terrain?

- Like a 1.5-D terrain with an extra horizontal dimension.
- More formally, a polygonal mesh in $\mathbb{R}^3$ with no holes that intersects any vertical line at at most 1 point.
- Again, no caves and no overhangs.

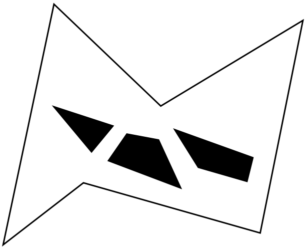
VC-dimension of polygons with holes

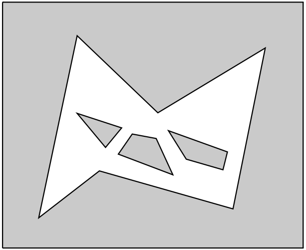
- Polygons with holes are much more troublesome than simple polygons.
  - Guarding them is as hard as SET COVER in general.
  - Approximation is therefore hard.
  - Guarding polygons with holes has unbounded VC-dimension.
  - In particular this is true when guarding the perimeter with guards on the perimeter.

A simple reduction

- Assume we have a polygon with holes and a set A of points on the perimeter that is shattered by guards on the perimeter.
  - This exists for any specified size of A.
- We can build a 2.5-D terrain with a corresponding point set of the same size that is shattered by a corresponding set of guards.

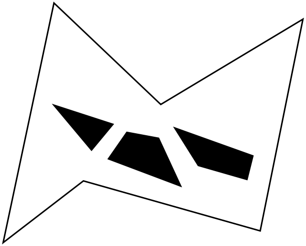
A simple reduction

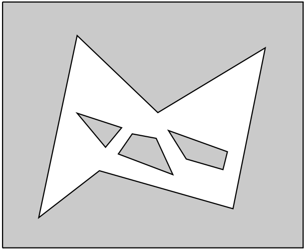




- Start with a flat rectangular terrain – a 'bounding box'.
- Draw the polygon's perimeter on the terrain.
- Keep the perimeter fixed while raising the 'exterior' (including holes) and lowering the 'interior'.

A simple reduction





- Lines of sight between perimeter points on the polygon are now lines of sight on the terrain at altitude 0.
- They can only be broken by the 'mountains' made by raising the 'exterior', so visibility is preserved by the reduction.

Thank you!