Realization of Degree 10 Minimum Spanning Trees in 3-Space

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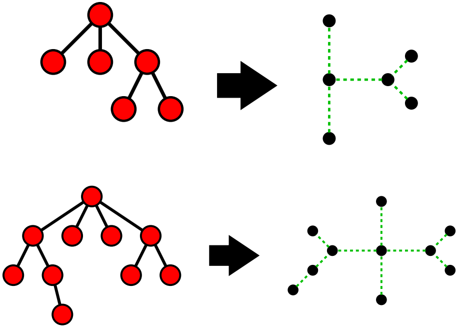
Outline

What's the Realization Problem?

- Given a tree T, build a point set P such that MST(P) is isomorphic to T.
- Essentially the inverse of the MST problem.
- Originally studied in Euclidean 2-space.
- Current interest is in the 3-space version of the problem.

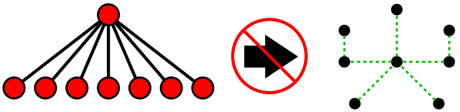
Some 2D Examples

- Fairly straightforward if all nodes have low degree.



Some 2D Examples

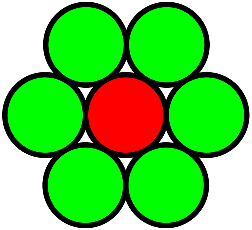
- What if we have a node with high degree like 7?
- How can we pack its neighbours around it?



- Unfortunately there are limits as to how many neighbours we can pack around a point.

Restriction on Maximum Degree

- In 2-space, no MST has a node of degree greater than 6.
- In 3-space, no MST has a node of degree greater than 12.



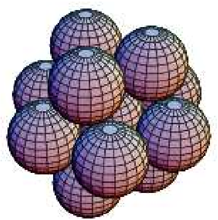


Image courtesy of MathWorld.

- These numbers are directly related to kissing numbers for circles and spheres.

Outline

Previously Known Results

- Let Δ be the maximum node degree in the tree we are trying to realize.
- The 2-space realization problem is:
  - Never possible when $\Delta \geq 7$.
  - Sometimes possible when $\Delta = 6$ (decision is NP-complete).
  - Always possible when $\Delta \leq 5$.
- The 3-space realization problem is:
  - Never possible when $\Delta \geq 13$.
  - Always possible when $\Delta \leq 9$.
  - Previously unknown when $10 \leq \Delta \leq 12$.
  - Our contribution: always possible when $\Delta = 10$.

Outline

Introducing the Problem

Previous Work

The Algorithm

Future Work

Our Algorithm

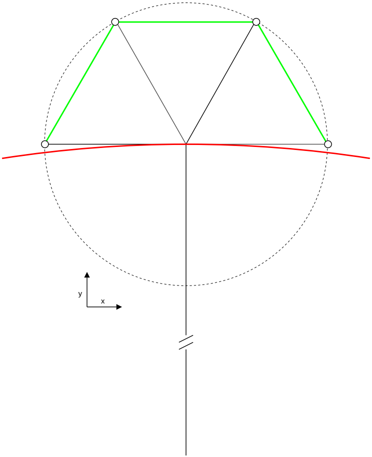
- Our algorithm for $\Delta = 10$ in 3-space is basically an extension of Monma & Suri's algorithm for $\Delta = 5$ in 2-space.
- I'll present an interpretation of their algorithm first for simplicity.

Monma & Suri's 2-Space Algorithm

- Label a degree 1 node as the root and place the corresponding point at the origin.
  - This ensures every node has at most 4 children.
- While there is a vertex p that has been placed but whose children have not been placed, place each child q_i of p.
- Be careful not to violate the relationship between p and its parent!
- We can rotate and dilate/contract the point set as we please to standardize the placement of a node's children.

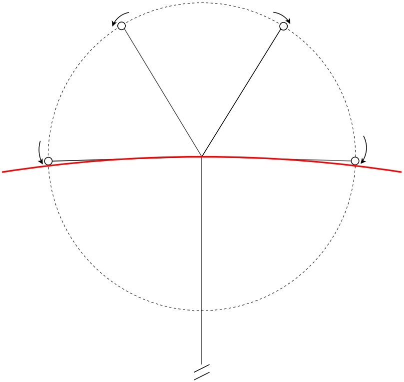
Layout of the Children

- Layout starts at 4 points of a regular hexagon centred at the parent.
- Children are at distance exactly 1 from the parent and from their nearest siblings.
- Problem: no room to recurse in!
  - Need to reserve room for subtrees rooted at these children.
- Let's look more closely at the region we have to work with.



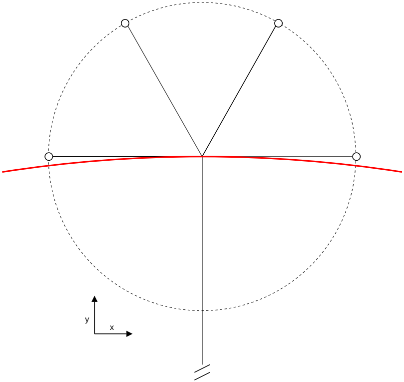
Spacing out the Children

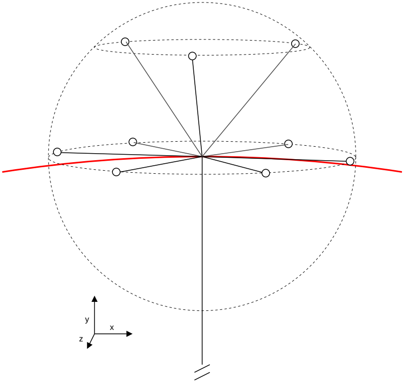
- We always have a bit more than a semicircle to work in.
- We space out the children a bit using this space.
- The children are now at distance strictly greater than 1 from each other.
- We use this extra room to place the subtrees in.
- The rest is recursion!



Extending to 3-Space

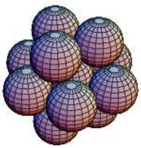
- To extend this algorithm to 3-space, we just change the way children of a node are laid out.



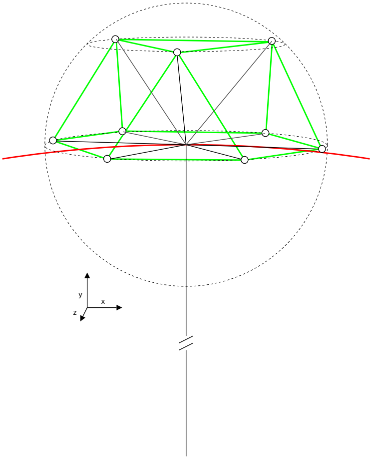


Extending to 3-Space

- Layout is based on hexagonal close sphere packing.

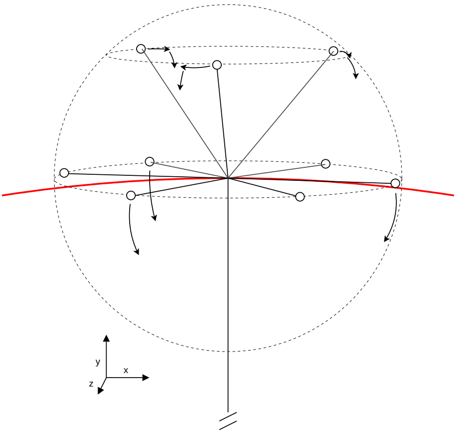


- Children are at distance exactly 1 from the parent and from their nearest siblings.
- Now we need to space them out to make room for recursion.



Spacing out the Children in 3-Space

- Lower every other equatorial child.
- Rotate the three polar children.
- Lower the three polar children.
- If we do this just right it actually works!



Outline

Closing the Gap for 3D

- We know that 3D realization is
  - always possible when $\Delta \leq 10$
  - never possible when $\Delta \geq 13$.
- Still unsolved for $\Delta = 11$ and $\Delta = 12$.
- Conjecture: impossible for certain trees with $\Delta = 11$.

Extending to Higher Dimensions

- What about d-space realization for $d > 3$?
- Let $H(d)$ be the maximum number of points on a d-dimensional hemisphere such that no two points are at distance less than 1 from each other.
- Conjecture: any tree with $\Delta \leq H(d) + 1$ can be realized in d-space.

Thank you!