COMP251: Network flows (1)

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Based on slides from M. Langer (McGill) & (Cormen et al., 2009)
Flow Network

$G = (V, E)$ directed.

Each edge $(u, v)$ has a capacity $c(u, v) \geq 0$.

If $(u,v) \notin E$, then $c(u,v) = 0$.

*Source* vertex $s$, *sink* vertex $t$, assume $s \leadsto v \leadsto t$ for all $v \in V$. 
Definitions

**Positive flow:** A function \( p : V \times V \rightarrow R \) satisfying.

**Capacity constraint:** For all \( u, v \in V \), \( 0 \leq p(u, v) \leq c(u, v) \),

**Flow conservation:** For all \( u \in V - \{s, t\} \), \[ \sum_{v \in V} p(v, u) = \sum_{v \in V} p(u, v) \]

Flow into \( u \) \quad Flow out of \( u \)

Flow in: \( 0 + 2 + 1 = 3 \)
Flow out: \( 2 + 1 = 3 \)
Example
Cancellation with positive flows

- Without loss of generality, can say positive flow goes either from $u$ to $v$ or from $v$ to $u$, but not both.
- In the above example, we can “cancel” 1 unit of flow in each direction between $x$ and $z$.
- Capacity constraint is still satisfied.
- Flow conservation is still satisfied.
Net flow

A function $f : V \times V \rightarrow R$ satisfying:

- **Capacity constraint:** For all $u, v \in V$, $f(u, v) \leq c(u, v)$,
- **Skew symmetry:** For all $u, v \in V$, $f(u, v) = -f(v, u)$, $v \in V$
- **Flow conservation:** For all $u \in V - \{s, t\}$, \( \sum_{v \in V} f(u, v) = 0 \)

\[
\sum_{v \in V ; f(v, u) > 0} f(v, u) = \sum_{v \in V ; f(u, v) > 0} f(u, v)
\]

Total positive flow entering $u$

Total positive flow leaving $u$
Positive vs. Net flows

Define net flow in terms of positive flow:

\[ f(u,v) = p(u,v) - p(v,u). \]

The differences between positive flow \( p \) and net flow \( f \) :

- \( p(u,v) \geq 0 \),
- \( f \) satisfies skew symmetry.
Values of flows

Definition: \( f = |f| = \sum_{v \in V} f(s, v) = \text{total flow out of source.} \)

Value of flow \( f = |f| = 3 \).
Flow properties

- Flow in == Flow out
- Source $s$ has outgoing flow
- Sink $t$ has ingoing flow
- Flow out of source $s$ == Flow in the sink $t$
  - Source $s$ has only outgoing flow
Maximum-flow problem

Given $G$, $s$, $t$, and $c$, find a flow whose value is maximum.
Applications

https://ais.web.cern.ch/ais/

(http://driverlayer.com)
Naïve algorithm

Initialize $f = 0$
While true {
    if ($\exists$ path $P$ from $s$ to $t$ such that all edges have a flow less than capacity)
        then
            increase flow on $P$ up to max capacity
    else
        break
}
Naïve algorithm

Initialize $f = 0$
While true {
   if (exists a path $P$ from $s$ to $t$ s.t. all edges $e \in P$ $f(e) < c(e)$)
      then {
         $\beta = \min\{ c(e) - f(e) \mid e \in P \}$
         for all $e \in P$ { $f(e) += \beta$ }
      } else { break }
}
Example where algorithm works
Example where algorithm works

|f| = 2
Example where algorithm works

|f| = 4
Example where algorithm works

$|f| = 5$
Example where algorithm fail!
Example where algorithm fail!

$|f| = 3$  And terminates...
Challenges

How to choose paths such that:

• We do not get stuck
• We guarantee to find the maximum flow
• The algorithm is efficient!
A better algorithm

Motivation: If we could subtract flow, then we could find it.

Algo 1 terminates here...

Negative value on edge that does not satisfy the definition
Residual graphs

Given a flow network $G=(V,E)$ with edge capacities $c$ and a given flow $f$, define the residual graph $G_f$ as:

- $G_f$ has the same vertices as $G$
- The edges $E_f$ have capacities $c_f$ (called residual capacities) that allow us to change the flow $f$, either by:
  1. Adding flow to an edge $e \in E$
  2. Subtracting flow from an edge $e \in E$
Residual graphs

for each edge $e = (u, v) \in E$
  if $f(e) < c(e)$
    then {
      put a **forward edge** $(u,v)$ in $E_f$ 
      with residual capacity $c_f(e) = c(e) - f(e)$
    }
  if $f(e)>0$
    then {
      put a **backward edge** $(v,u)$ in $E_f$
      with residual capacity $c_f(e) = f(e)$
    }
}
Flow network

Flow

Residual graph

forward

backward
Example 2/3

Flow network

Flow

Residual graph

forward

backward

1 = 3-2

3-2=1

1/1
Example 3/3
Example 3/3

Flow

Residual graph
Augmenting path

An augmenting path is a path from the source $s$ to the sink $t$ in the residual graph $G_f$ that allows us to increase the flow.

Q: By how much can we increase the flow using this path?
Example

Flow in $G$

Residual graph $G_f$
Example

Residual graph $G_f$

Flow in $G_f$
Example

$|f| = 3$

$G_f$

$|f| = 5$

$\beta = 2$
Methodology

• Compute the residual graph $G_f$
• Find a path $P$
• Augment the flow $f$ along the path $P$
  1. Let $\beta$ be the bottleneck (smallest residual capacity $c_f(e)$ of edges on $P$)
  2. Add $\beta$ to the flow $f(e)$ on each edge of $P$.

Q: How do we add $\beta$ into $G$?
Augmenting a path

```java
f.augment(P) {
    \beta = \min \{ c(e) - f(e) \mid e \in P \}
    for each edge e = (u,v) \in P {
        if e is a forward edge {
            f(e) += \beta
        } else { // e is a backward edge
            f(e) -= \beta
        }
    }
}
```
Ford-Fulkerson algorithm

\[
\begin{align*}
f & \leftarrow 0 \\
G_f & \leftarrow G \\
\text{while} \ (\text{there is a s-t path in } G_f) \{ \\
\quad & f.\text{augment} \ (P) \\
\quad & \text{update } G_f \text{ based on new } f \\
\}
\end{align*}
\]
Correctness (termination)

**Claim:** The Ford-Fulkerson algorithm terminates.

**Proof:**

- The capacities and flows are strictly positive integers.
- The sum of capacities leaving $s$ is finite.
- Bottleneck values $\beta$ are strictly positive integers.
- The flow increase by $\beta$ after each iteration of the loop.
- The flow is an increasing sequence of integers that is bounded.
Complexity (Running time)

- Let \( C = \sum_{e \in E} c(e) \)
- Finding an augmenting path from \( s \) to \( t \) takes \( O(|E|) \) (e.g. BFS or DFS).
- The flow increases by at least 1 at each iteration of the main while loop.
- The algorithm runs in \( O(C \cdot |E|) \)