COMP 250: Mid-term Exam Review

Lecture 19

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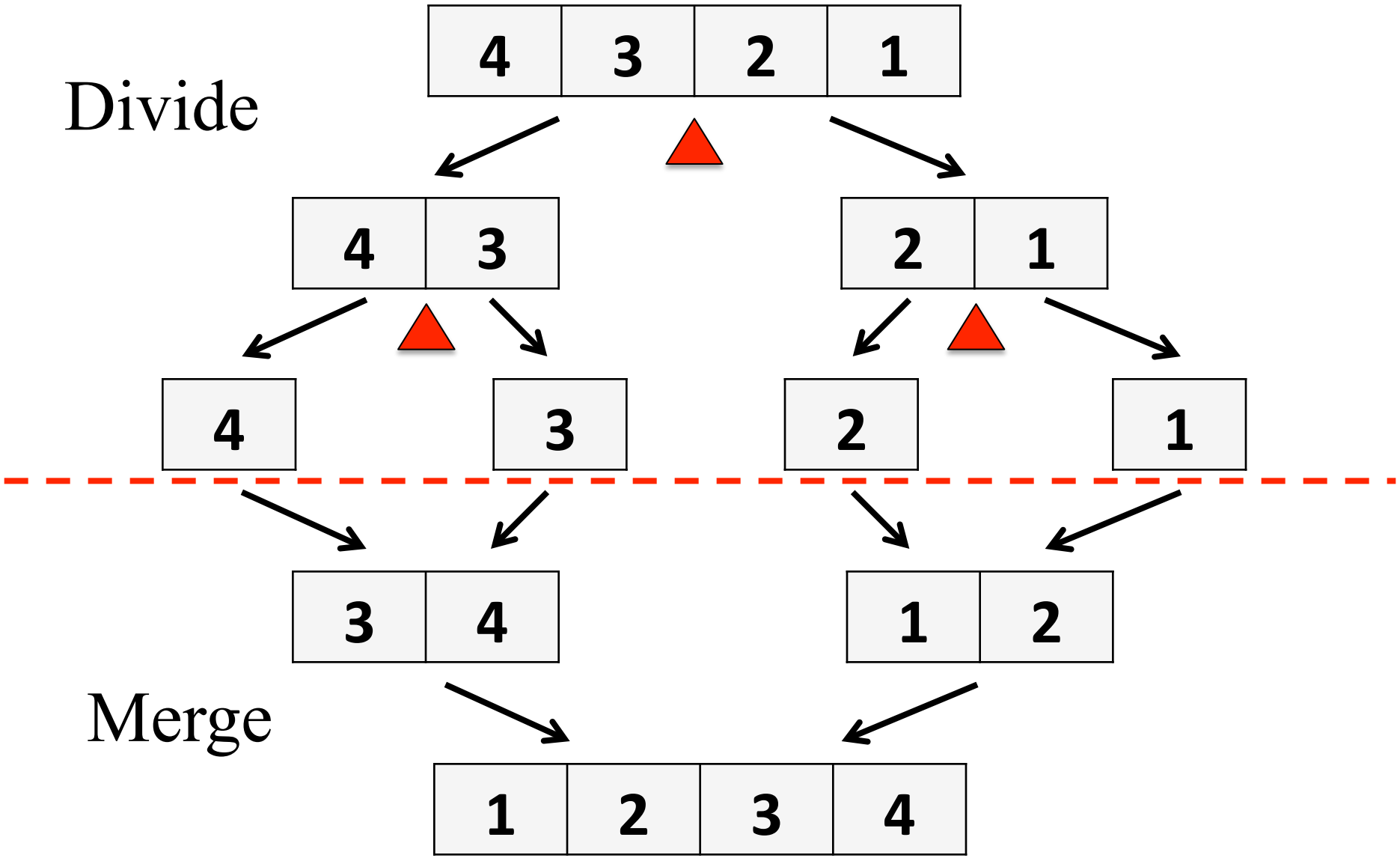
Review - Recursive algorithms

- To write a recursive algorithm:
  - Find how the problem can be broken up in one or more smaller problems of the same nature
  - Remember the base case!
- Usually, better running times are obtained when the size of the subproblems are approximately equal
  - power(a,n) = a * power(a,n-1) $\Rightarrow$ O(n)
  - power(a,n) = $(\text{power}(a,n/2))^2$ $\Rightarrow$ O(log n)
- Fibonnaci, BinarySearch, MergeSort…

Merge Sort



Recursive algorithms & inductions

- You can see an algorithm as a function.
- To prove the correctness of a recursive algorithm, we can use a similar technique to the one used to prove mathematical inductions.
- But we should also prove that our algorithm terminates!

Induction proofs

- To prove that a proposition P(n) holds for all n ≥ a:
  - **Base case:**

    Prove that P(a) holds
  - **Induction step on n:**

    Induction Hypothesis: Assume P(n) holds

    Prove that I.H. implies that P(n+1) holds

Generalized induction proofs

To prove that a proposition P(n) holds for all $n \geq a$:

- **Base case:**

  Prove that P(a), P(a+1)... (as many as needed) hold

- **Induction step on n:**

  Induction Hypothesis: Assume P(k) holds for all $k \leq n$

  Prove that induction hypothesis implies that P(n+1) holds

Induction proofs: Example

Claim: for all positive integers k, n, we have $(1 + k)^n \geq 1 + kn$.

Proof:

- Base case: n=1, $(1 + k)^1 \geq 1 + k$
- Induction hypothesis: for all k and a given n, we have we have $(1 + k)^n \geq 1 + kn$
- Induction step (show that then it is true for n+1):

$$
\begin{aligned}
(1+k)^{n+1} &= (1+k)(1+k)^n \\
&\geq (1+k)(1+kn) \\
&= 1 + kn + k + k^2n \\
&= 1 + k(n+1) + (1+k) \\
&\geq 1 + k(n+1)
\end{aligned}
$$

Proving algorithms

```
AF(int x) {
  f = 2;
  while ( x > 1 ) {
    if ( x % f == 0 ) {
      System.out.println(f);
      x = x / f;
    } else {
      f++;
    }
  }
}
```

Iterative algorithm
$\Rightarrow$ Loop invariant

```
FM(int[] A, int l, int r) {
  if (l<r) {
    m=⌊(l+r)/2⌋
    return max(
        FM(A,l,m),
        FM(A,m+1,r));
  } else {
    return A[l];
}
```

Recursive algorithm
$\Rightarrow$ Induction proof

Proving algorithms

```
FM(int[] A, int l, int r) {
  if (l<r) {
    m=⌊(l+r)/2⌋
    return max(FM(A,l,m),FM(A,m+1,r));
  } else { return A[l]; }
```

- Pre-condition: $0 \leq l \leq r \leq A.length$
- Post-condition: return max value stored in A[l:r]
- Base case: the max of an array of size 1 is the value stored.
- Induction Hypothesis: for all l and r such that $|r-l| \leq n$, FM(A,l,r) returns the max in A[l:r]
- Induction step: for all $|r-l|=n+1$, then

$$FM(A,l,r) = \max(FM(A,l,m),FM(A,m+1,r))$$

Proving algorithms

```
FM(int[] A, int l, int r) {
  if (l<r) {
    m=⌊(l+r)/2⌋
    return max(FM(A,l,m),FM(A,m+1,r));
  } else { return A[l]; }
```

- Induction Hypothesis: for all l and r such that $|r-l| \leq n$, FM(A,l,r) returns the max in A[l:r]
- Induction step: Assume $|r-l|=n+1$.
  FM(A,l,r) = max(FM(A,l,m),FM(A,m+1,r))
  (I.H.) FM(A,l,m) & FM(A,m+1,r) return max of A[l,m] & A[m+1,r]
  Since max of A[l,r] is in A[l,m] or A[m+1,r], then FM(A,l,r) returns max of A[l:r]
- Termination: At every recursive call, $|r-l|$ decreases strictly until $|r-l|=0$, which is the base case.

Loop Invariants

```
AF(int x) {
  f = 2;
  while ( x > 1 ) {
    if ( x % f == 0 ) {
      System.out.println(f);
      x = x / f;
    } else { f++; }
  }
}
```

Invariant:

x = the number that remains to be factor

and we have removed all factors < f

Loop invariant

- **Initialization:** x contains all factors
- **Maintenance:** At the beginning of the loop we removed all factors < f. After one iteration of the loop, if f is a factor of x, then we remove it and do not increase f. Otherwise, we can safely increment f and x is such that we removed all factors ≤ f.
- **Termination:** The number that remains to be factored = 1 and we have removed all factors including the largest factor (I.e.: we have found all the factors)

Discussion

- Difficulty is to identify a good loop invariant. The proof is usually simpler (initialization, maintenance, termination).
- A loop invariant can be any predicate as long as it helps to prove that the loop helps to solve the problem.
- With recursive algorithm, identifying the property we want to prove is easier because it is the post-condition.

Invariant vs. Induction

Iterative algorithm		Recursive algorithm
Loop invariant	$\Leftrightarrow$	Induction hypothesis
Initialization	$\Leftrightarrow$	Base case
Maintenance	$\Leftrightarrow$	Induction step
Termination	$\Leftrightarrow$	Termination

Running time

- Primitive operations
  - Running time is constant, indep. of problem size
    - Assigning a value to a variable
    - Calling a method; returning from a method
    - Arithmetic operations, comparisons
    - Indexing into an array
    - Following object reference
    - Conditionals
- Running time ≡ Number of primitive operations
- Loops: Sum the running time of each iteration
- findMin, insertionSort

Recurrences

- For recursive algorithms, we express the running time $T(n)$ for an input of size n as a function of $T(a)$ for some $a<n$
- Example:
  - Binary search: $T(n) = T(n/2) + a$
  - MergeSort: $T(n) = 2\,T(n/2) + c\,n$

Solving recurrences

- Solving recurrence ≡ give explicit formula for T(n)
- Substitution method:
  - Replace occurrences of T() by their value
  - Repeat until pattern emerges
- Prove by induction that guess is correct

Example

Solution of $T(n) = 2T(n/2) + cn$, $T(1)=0$ for $n \geq 0$ a power of 2 ?

$$
\begin{aligned}
T(n) &= 2\,(2T(n/4) + cn/2) + cn \\
&= 2^2\, T(n/4) + 2\, cn \\
&= 2^2\, (2T(n/8) + cn/4) + 2\, cn \\
&= 2^3\, T(n/8) + 3\, cn \\
&\ldots \\
&= 2^k\, T(n/2^k) + k\, cn
\end{aligned}
$$

We have $n=2^k$, thus $k=\log_2(n)$ and we replace k in T(n).

$T(n) = 2^{\log(n)}\, T(1) + \log_2(n)\ \ cn = cn \log(n)$

Then prove by induction...

Proof by induction

Proposition. If $T(n)$ satisfies the following recurrence, then $T(n) = n \log_2 n$.

assuming n is a power of 2

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ 2\,T(n/2) + n & \text{otherwise} \end{cases}$$

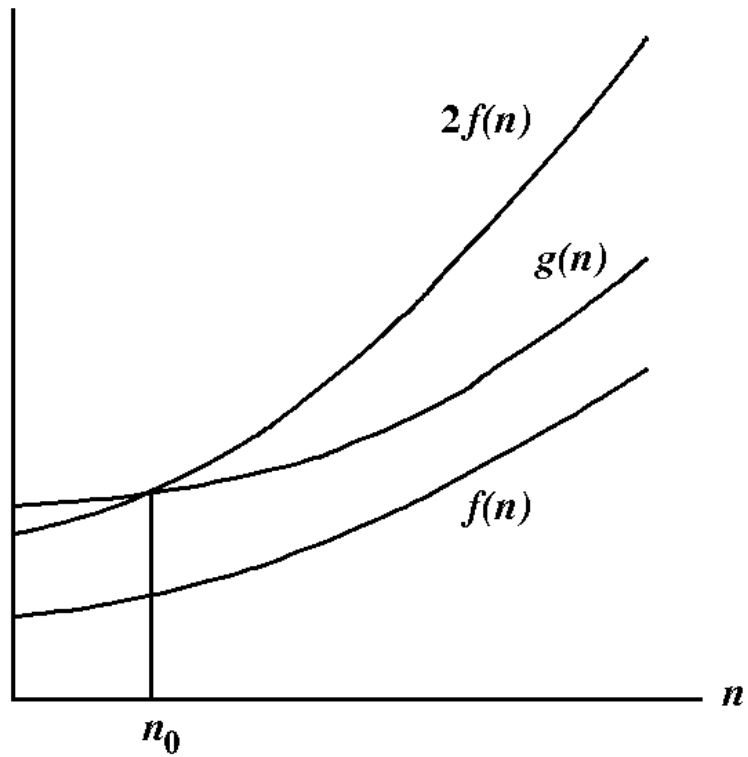
Pf 2. [by induction on n]

- Base case: when $n = 1$, $T(1) = 0$.
- Inductive hypothesis: assume $T(n) = n \log_2 n$.
- Goal: show that $T(2n) = 2n \log_2 (2n)$.

$$\begin{aligned} T(2n) &= 2\,T(n) + 2n \\ &= 2\,n \log_2 n + 2n \\ &= 2\,n\,(\log_2 (2n) - 1) + 2n \\ &= 2\,n \log_2 (2n). \quad \blacksquare \end{aligned}$$

Big-Oh notation

g(n) is O(f(n)) iff there exist constants c and n_0 such that $g(n) \leq c\, f(n)$ for all $n \geq n_0$



- f(n) is Ω(g(n)) iff g(n) is O(f(n))
- f(n) is Θ(g(n)) iff f(n) is O(g(n)) and f(n) is Ω(g(n))

Example

Show that Fib(n) is $O(2^n)$

Proof: We will show by induction that for all $n \geq 0$, $Fib(n) < 2^n$

- Base case: $n=0$, $Fib(0)=0<2^0=1$; $n=1$, $Fib(1)=1<2^1=2$;
- Induction hypothesis: for all $k \leq n$ $Fib(k)<2^k$. Fib(n+1) ?
- Induction step:

$$
\begin{aligned}
Fib(n+1) &= Fib(n) + Fib(n-1) \\
&< 2^n + 2^{n-1} \\
&< 2^n + 2^n = 2^{n+1}
\end{aligned}
$$

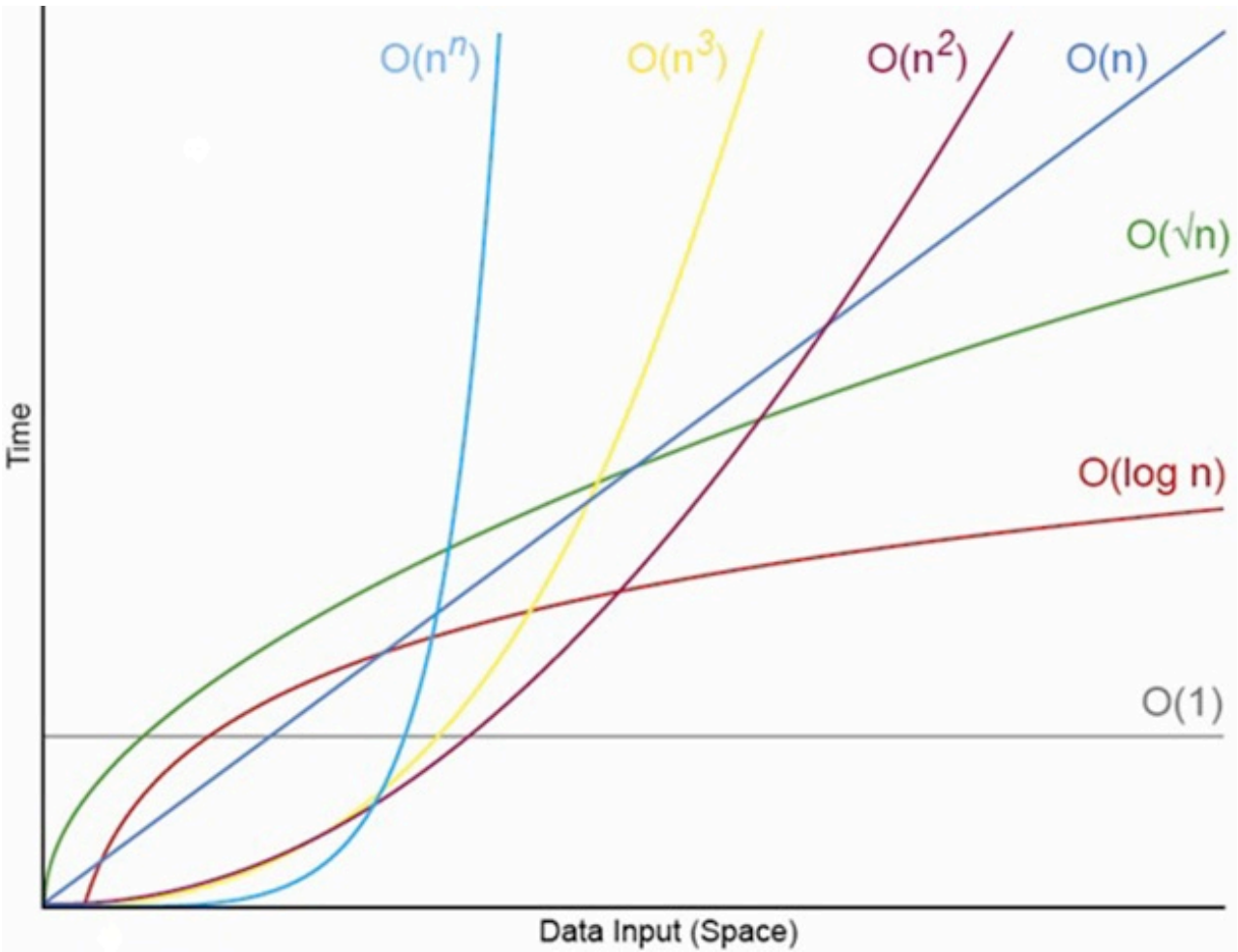
Using the previous definition, we set $c=1$ and $n_0=0$, then Fib(n) is $O(2^n)$.

Example 2

- Previously, we used the recursive formula $T(n)=2T(n/2)+n$, with $T(1)=0$, to characterize the number of operations of MergeSort.
- We showed that $T(n) = n \log_2(n)$
- Then, MergeSort runs in $O(n \log_2(n))$

Big-Oh notation

Hierarchy of big-Oh classes



Big-Oh notation

Simplification rules

- If $f_1(n) \in O(g(n))$ and $f_2(n) \in O(g(n))$ then

  $f_1(n) + f_2(n) \in O(g(n))$

- If $f_1(n) \in O(g(n))$ then $k.f_1(n) \in O(g(n))$

- If $f_1(n) \in O(g(n))$ and $f_2(n) \in O(h(n))$ then

  $f_1(n).f_2(n) \in O(g(n).h(n))$

…

Log identities

- $\log(a\ b) = \log(a) + \log(b)$
- $\log(a^n) = n \log(a)$
- $\log_a(n) = \log_b(n) / \log_b(a)$
- $a^{\log_b(n)} = n^{\log_b(a)}$

Running time of a For loop

```
for (i=1; i<N; i=i*2) { … }
```

Something running in O(1)

What is the running time of this loop?

Value of i after k iterations: 2^k

We have $i < N \Rightarrow 2^k < N \Rightarrow k < \log_2(N)$.

There is less than $\log_2(N)$ iterations.

The running time of this loop is O(log(n)).

#operations: O(log(n)) O(1) = O(log(n))

Abstract Data Types

- Implementation: Arrays or linked-lists
- Basic operations
  - getFirst(), get(n), getLast()
  - removeFirst(), removeLast(), remove(o)
  - addFirst(o), addLast(o), add(o)
  - empty(), size()
- Advantages and disadvantages over arrays
- Stack, Queues, deques, rotating arrays

Operations on deques with Array

- Queue() {
```
    L = new Array[N];
    head = tail = -1;
}
```
- isEmpty() {
```
    if ((head == -1)&&(tail==-1)) return true;
    else return false;
}
```
- isFull() {
```
    if ((head - tail % N) == 1) return true;
    else return false;
}
```

Operations on deques with Array

- Enqueue(o) throw Exception {

```
    if ( isFull() ) { throw new Exception("Full stack") }
    if ( isEmpty() ) { head = tail = 0; }
    else {  tail = ( ( tail + 1 ) % N ); }
    L[tail] = o;
}
```

- Dequeue() {

```
    if ( isEmpty() ) { throw new Exception("Empty stack") }
    Object o = L[head];
    if (((head – tail) %  N) = 1) { head = tail = -1; }
    else { head = ( ( head + 1 ) % N ); }
    return o;
}
```