1. (10 points) Prove that the following problem belongs to \( P \): Given a graph \( G \), we want to know whether \( G \) has an independent set of size 100.

2. (10 Points) Prove that the following problem belongs to \( \text{PSPACE} \): Given a graph \( G \) and an integer \( k \), we want to know whether the number of independent sets in \( G \) is equal to \( k \).

3. (10 Points) Show that the following problem is \( \text{NP}\)-complete:
   - Input: An undirected graph \( G \) and an edge \( e \).
   - Question: Does \( G \) have a Hamiltonian cycle that passes through the edge \( e \).

4. (10 points) In the MAX-CUT problem, given an undirected graph \( G \) we want to partition the vertices of \( G \) into two parts \((A, B)\) such that the number of edges between \( A \) and \( B \) is maximized. Prove that the following is a \( \frac{1}{2} \)-factor approximation algorithm for this problem:
   - Let \( v_1, \ldots, v_n \) be all the vertices of \( G \).
   - Initially set \( A = B = \emptyset \).
   - For \( i = 1, \ldots, n \) do
     - IF \( v_i \) has more neighbours in \( A \) than in \( B \) THEN
       - add \( v_i \) to \( B \)
     - Else
       - add \( v_i \) to \( A \)
   - EndFor

5. (10 points) Prove that the following algorithm is a 2-factor approximation algorithm for the minimum vertex cover problem:
   - While there is still an edge \( e \) left in \( G \):
     - Delete all the two endpoints of \( e \) from \( G \)
   - EndWhile
   - Output the set of the deleted vertices

6. (10 points) Prove that the following algorithm is a \( \frac{1}{2} \)-factor approximation algorithm for the MAX-SAT problem: Given a CNF \( \phi \) on \( n \) variables \( x_1, \ldots, x_n \):
   - For \( i = 1, \ldots, n \) do
     - IF \( x_i \) appears in more clauses than \( \overline{x_i} \) THEN
       - Set \( x_i = T \)
7. (10 points) A kite is a graph on an even number of vertices, say $2k$, in which $k$ of the vertices form a clique and the remaining $k$ vertices are connected in a tail that consists of a path joined to one of the vertices of the clique. Prove that KITE problem defined as in the following is NP-complete.

- Input: An undirected graph $G$, and a positive integer $k$.
- Question: Does $G$ contain a kite on $2k$ vertices as a subgraph?

8. Consider a graph $G = (V, E)$. The chromatic number of $G$ is the minimum number of colors required to color the vertices of $G$ properly. Let $\mathcal{I}$ be the set of all independent sets in $G$ (Note that every element in $\mathcal{I}$ is a set).

(a) (10 Points) Prove that the solution to the following linear program provides a lower-bound for the chromatic number of $G$.

$$\begin{align*}
\min & \sum_{I \in \mathcal{I}} x_I \\
\text{s.t.} & \sum_{I : v \in I} x_I \geq 1 \quad \forall v \in V \\
& x_I \geq 0 \quad \forall I \in \mathcal{I}
\end{align*}$$

(b) (10 Points) Write the dual of the above linear program.

(c) (10 Points) Prove that every clique in $G$ provides a solution to the dual linear program.