1. (a) Take $c = 2$ and $n_0 = 0$. We have $\sqrt{n} + n\sqrt{n} = n^{0.5} + n^{1.5} \leq n^2 + n^2 = 2n^2$. So $\sqrt{n} + n\sqrt{n} \leq cn^2$ for $n > 0$.

(b) We need to show $n^5 = O((n + \log_2 n)^5)$ and $(n + \log_2 n)^5 = O(n^5)$. For the first equality, note that $n \leq n + \log_2 n$ and therefore $n^5 \leq (n + \log_2 n)^5$. So $c = n_0 = 1$ satisfies the definition of big-O. To show $(n + \log_2 n)^5 = O(n^5)$, note that $(n + \log_2 n)^5 \leq (n + n)^5 = (2n)^5 = 32n^5$. So in this case $c = 32$ and $n_0 = 1$ works.

(c) Observe that $\frac{n!}{n^n} = \prod_{i=1}^{n} \frac{i}{n} \leq \frac{1}{n}$ since each term in the product is at most 1. So

$$\lim_{n \to \infty} \frac{n!}{n^n} \leq \lim_{n \to \infty} \frac{1}{n} = 0,$$

and we are done by the definition of small-o.

(d) We can use L'Hôpital's rule. So

$$\lim_{n \to \infty} \frac{\log_2 n}{n^{1/100}} = \lim_{n \to \infty} \frac{100n^{99/100}}{n \ln 2} = \lim_{n \to \infty} \frac{100}{n^{1/100} \ln 2} = 0.$$

(e) From part (d) we know that $\log_2 n = o(n^{1/100})$. In fact the same proof shows that $\log_2 n = o(n^{1/2})$. This implies that there is some $n_0$ such that $\log n < \frac{1}{2} \sqrt{n}$ for all $n > n_0$. Using this, we have

$$\frac{n\sqrt{n}}{2^n} = 2^{\sqrt{n} \log_2 n} \leq 2^{\sqrt{n} \sqrt{n}/2} = 2^{n/2} = \frac{1}{2^{n/2}},$$

for $n > n_0$. Therefore $\lim_{n \to \infty} \frac{n\sqrt{n}}{2^n} = 0$ and hence $n\sqrt{n} = o(2^n)$.

2. (a) False. It can be shown that $2^{2^n} = o(2^{2^{n+1}})$ since

$$\frac{2^{2^n}}{2^{2^{n+1}}} = \frac{2^{2^n}}{2^{2^n} \cdot 4^{2^n}} = \frac{2^{2^n}}{4^{2^n}} = \frac{1}{2^{2^n}},$$

and hence

$$\lim_{n \to \infty} \frac{2^{2^n}}{2^{2^{n+1}}} = 0.$$

Because $2^{2^n} = o(2^{2^{n+1}})$, it cannot be the case that $2^{2^n+1} = O(2^{2^n})$. 


(b) False. It can be shown that $\log_2 n^5 = o((\log n)^5)$ since

$$\frac{\log_2 n^5}{(\log n)^5} = \frac{5 \log_2 n}{(\log n)^5} \leq \frac{5}{(\log n)^4},$$

and hence

$$\lim_{n \to \infty} \frac{\log_2 n^5}{(\log n)^5} = 0.$$

(c) True. To show $n^{1/n} = O(1)$, we need to show there are $c$ and $n_0$ such that $n^{1/n} \leq c$ for $n > n_0$. Pick $c = 2$ and $n_0 = 1$. Observe that $n \leq 2^n$ for $n \geq 1$ and this implies $n^{1/n} \leq 2$. To show $n^{1/n} = \Omega(1)$, we need to show there are $c$ and $n_0$ such that $1 \leq cn^{1/n}$ for $n > n_0$. Let $c = n_0 = 1$ and observe that $1^n \leq n$ implies $1 \leq n^{1/n}$.

3. Two out of many possible solutions are given below.

(1) $s\text{c}d\text{b}t + 3, s\text{c}a\text{b}t + 2, s\text{c}d\text{t} + 2, s\text{c}a\text{d}t + 2$, maximum flow is 9.

\[ \begin{array}{c}
\text{s} \\
2/5 \\
7/7
\end{array} \quad \begin{array}{c}
\text{a} \\
2/2 \\
7/6
\end{array} \quad \begin{array}{c}
b \\
2/6
\end{array} \quad \begin{array}{c}
t \\
2/2
\end{array} \quad \begin{array}{c}
\text{c} \\
3/3 \\
4/5
\end{array} \quad \begin{array}{c}
d \\
3/3 \\
7/9
\end{array} \quad \begin{array}{c}
\text{d} \\
5/6 \\
4/9
\end{array} \quad \begin{array}{c}
\text{t} \\
0/3
\end{array} \]

(2) $s\text{a}b\text{t} + 2, s\text{c}b\text{t} + 3, s\text{c}d\text{t} + 4$, maximum flow is 9.

\[ \begin{array}{c}
\text{s} \\
2/5 \\
7/7
\end{array} \quad \begin{array}{c}
\text{a} \\
2/2 \\
0/6
\end{array} \quad \begin{array}{c}
b \\
5/6
\end{array} \quad \begin{array}{c}
t \\
0/3
\end{array} \quad \begin{array}{c}
\text{c} \\
3/3 \\
4/5
\end{array} \quad \begin{array}{c}
d \\
0/3 \\
4/9
\end{array} \quad \begin{array}{c}
\text{d} \\
7/7 \\
4/3
\end{array} \]