Note: There are in total 110 points, but your grade will be considered out of 100. Don’t forget to write your name.

1. (20 points) Construct a DFA for the following language over the alphabet $\Sigma = \{0, 1\}$:

   $L = \{w \mid w \text{ contains } 010 \text{ as a substring}\}$.

   **Answer:** Have four states $a, b, c, d$ with only $d$ being an accept state. The 010 goes from $a$ to $b$ to $c$ to $d$. The other arrows are added as: 0 takes us from $b$ to $b$, and $d$ to $d$, and a 1 takes us from $a$ to $a$, $c$ to $a$, and $d$ to $d$.

2. (20 points) Construct a context-free grammar for the following language over the alphabet $\Sigma = \{a, b, c\}$:

   $L = \{a^i b^j c^k \mid k = i + j \text{ and } i, j, k \geq 0\}$.

   **Answer:** $S \rightarrow aSc | B$ and $B \rightarrow bBc | \epsilon$.

3. (25 points) Prove that the following language over the alphabet $\Sigma = \{a, b, c\}$ is not context-free.

   $L = \{a^i b^j c^k \mid k = i \times j \text{ and } i, j, k \geq 0\}$.

   **Answer:** Suppose that it is. Let $p$ be the pumping constant, and consider $w = a^p b^p c^{p^2}$. The pumping lemma in particular tells us that there is a decomposition $w = uvxyz$ with $0 < |vy| \leq p$ such that $uv^2xz \in L$.

   Note $|w^2xyz| = |w| + |vy| \leq |w| + p = p^2 + 3p$. If $uv^2xyz = a^{i'} b^{j'} c^{k'}$, then $i' \geq p$ and $j' \geq p$, and moreover at least one of these two inequalities must be strict (Since we are not removing any letters from $w$ and we are adding some extra letters). This shows $|uv^2xyz| = |a^{i'} b^{j'} c^{k'}| \geq p + (p + 1) + p(p + 1) = p^2 + 3p + 1$ which is a contradiction.

4. (25 points) Is it true that for every non-empty regular language $L$, there exists a DFA that recognizes $L$ and has exactly one accept state? If your answer is positive, provide a proof. If your answer is negative, provide a counter-example and justify why it is a counter-example.

   **Answer:** False! Consider $L = \{\epsilon, 1\}$. If a DFA with one accept state recognizes this, then the start state must be an accept state as $\epsilon \in L$, and moreover there must be a 1-arrow from this state to itself as 1 must be accepted. Then 11 will be accepted by this DFA. A contradiction!

5. (20 points) Is it true that for every non-empty regular language $L$, there exists an NFA that recognizes $L$ and has exactly one accept state? If your answer is positive, provide a proof. If your answer is negative, provide a counter-example and justify why it is a counter-example.

   **Answer:** True! Consider a DFA for $L$, add a new accept state, and connect all the old accept states with $\epsilon$-arrows to this new state. Turn the old accept states into non-accept states. This NFA has one accept state and recognizes $L$. 

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