COMP 330 - Fall 2016 - Assignment 5

Due 7:00 pm Dec 2, 2016

**General rules:** In solving these questions you may consult books but you may not consult with each other. You should either drop your solutions in the assignment drop-off box located in the Trottier Building on the 2nd floor, or submit it through mycourses. Late homeworks can be submitted until 48 hours after the deadline. Late submissions must be submitted through mycourses as the drop-off box will not be checked after the deadline. There will be a penalty of -25% for one-day delays, and -40% for two-day delays on late homeworks.


2. A polynomial is called non-negative on integers, if it is non-negative for every assignment of integers to its variables. Answering Hilberts 10th problem, Matiyasevich showed that the following language is undecidable: $P$ is the set of all $\langle p \rangle$ such that $p$ is a multivariate polynomial with integer coefficients that evaluates to zero for some assignment of positive integers to its variables. For example $\langle x_1^2 + x_2^2 - 5 \rangle \in P$ as it evaluates to 0 if we set $x_1 = 1$ and $x_2 = 2$. On the other hand $\langle x_1^2 - 5 \rangle$ is not in $P$.

   (a) (5 Points) Use Matiyasevich’s result to prove that the following language is undecidable: $L$ is the set of all $\langle p \rangle$ such that $p$ is a multivariate polynomial with integer coefficients that can evaluate to a prime number for some assignment of positive integers to its variables.

   (b) (10 Points) Use Matiyasevich’s result to prove that the following language is undecidable: $L$ is the set of all pairs of multivariate polynomial $\langle p, q \rangle$, both with integer coefficients, such that for some assignment of integer values to the variables $p(\cdot) > q(\cdot)$.

3. (10 Points) Show that the following language is undecidable relative to $A_{TM}$.

   $$B_{TM} = \{ \langle M^{A_{TM}}, w \rangle \mid M^{A_{TM}} \text{ is an oracle TM that accepts } w \}.$$ 

4. (10 Points) Prove that a language $L$ is recursively enumerable if and only if it can be expressed as

   $$L = \{ x \mid \exists y \text{ such that } \langle x, y \rangle \in R \}$$

   where $R$ is a decidable language. You need to prove that every language of this form is recursively enumerable, and that every recursively enumerable language can be described as above for some decidable language $R$.

5. (15 Points) You are allowed to use Rice’s theorem (See Problem 5.28 of the textbook) to answer this question. For each one of the following three languages, either prove that they are decidable, or prove that they are undecidable.

   $$L_r = \{ \langle M \rangle \mid L(M) \text{ is a regular language} \}.$$ 

   $$L_{re} = \{ \langle M \rangle \mid L(M) \text{ is a recursively enumerable language} \}.$$ 

   $$L_d = \{ \langle M \rangle \mid L(M) \text{ is a decidable language} \}.$$
6. (5 Points) Prove that every recursively enumerable language \( L \) satisfies \( L \leq_m A_{TM} \).

7. (5 Points) Prove that for any two languages \( A \) and \( B \), there exists a language \( J \) such that \( A \leq_T J \) and \( B \leq_T J \).

8. (10 Points) Prove that for any languages \( A \), there exists a language \( J \) such that \( A \leq_T J \) and \( J \not\leq_T A \).

9. (5 Points) Show that the Post Correspondence Problem is decidable relative to \( A_{TM} \).

10. (15 Points) Determine whether the following language is decidable or undecidable

\[
X = \{\langle M \rangle \mid \text{given any input } w, M \text{ eventually leaves the start state}\}.
\]

11. (0 Points) Problems 6.16 and 6.17 from the textbook (3rd Edition). First problem asks you to prove the existence of languages \( A, B \) such that \( A \not\leq_T B \) and \( B \not\leq_T A \). The second problem asks you to describe two disjoint recursively enumerable languages \( A \) and \( B \) such that there is no decidable \( C \) with \( A \subseteq C \) and \( B \cap C = \emptyset \).