General rules: In solving these questions you may consult books but you may not consult with each other. There are in total 105 points, but your grade will be considered out of 100. You should either drop your solutions in the assignment drop-off box located in the Trottier Building on the 2nd floor, or submit it through mycourses. Late homeworks can be submitted until 48 hours after the deadline. Late submissions must be submitted through mycourses as the drop-off box will not be checked after the deadline. There will be a penalty of -25% for one-day delays, and -40% for two-day delays on late homeworks.

1. (10 Points) Let $M_1$ and $M_2$ be two Turing machines. Consider the following Turing machine:

   On input $w$:
   
   - Step 1: Run $M_1$ on $w$. If $M_1$ accepts $w$, then accept.
   - Step 2: Run $M_2$ on $w$. If $M_2$ accepts $w$, then accept.

What is the language of this Turing Machine? Explain.

Answer: Let $H(M_1)$ be the set of the strings on which $M_1$ halts. The language of the above TM is $L(M_1) \cup (L(M_2) \cap H(M_1))$. If $w \in L(M_1)$ then it will be accepted, and if it is in $L(M_2) \cap H(M_1)$ it will be accepted on the second line. Note that if $w \notin H(M_1)$ then it will loop in the first line, and thus will not be accepted.

2. (15 Points) Is the following language decidable?

   $$L = \{\langle M \rangle \mid M = (\{1, 2, \ldots, 100\}, \{0, 1\}, \{0, 1, \sqcup\}, \delta, 1, 2, 3) \text{ is a decider}\}.$$ 

Answer: Yes, this is a finite language, and every finite language is decidable. Note that there are at most $100 \times 3 \times 3 \times 100 \times 2$ choices for $\delta$.

3. A polynomial is called non-negative, if it is non-negative for every assignment of real numbers to its variables. For example $x_1^2 + x_2^2 - 2x_1x_2$ is non-negative because we can express it as $(x_1 - x_2)^2$.

   (a) (15 Points) Prove that if a multi-variate polynomial $p$ is not non-negative, then there is an assignment of rational numbers to its variables that makes it negative.

   Answer: Suppose that $p$ is negative for assigning the real values $a_1, \ldots, a_k$ to its variables. We pick a sequence of rational numbers $r_1(n), \ldots, r_k(n)$ such that $\lim_{n \to \infty} r_i(n) = a_i$. Then $\lim_{n \to \infty} p(r_1(n), \ldots, r_k(n)) = p(a_1, \ldots, a_k) < 0$ and thus for some value of $n$ the assignment $r_1(n), \ldots, r_k(n)$ will lead to a negative evaluation.

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1Recall that a TM is formally defined as $(Q, \Sigma, \Gamma, \delta, q_{\text{start}}, q_{\text{accept}}, q_{\text{reject}})$. A Turing Machine is called a decider if it halts on every input.
(b) (15 Points) Use the previous part to show that the following language is recursively enumer-able (Turing recognizable):

\[ L = \{ \langle p \rangle \mid p \text{ is a multi-variate polynomial with integer coefficients, and } p \text{ is not non-negative.} \} \]

**Answer:** Let \( p \) be on \( k \) variables. The set of rational numbers \( \mathbb{Q} \) is countable, and thus the set of \( k \)-tuples \((r_1, \ldots, r_k)\) where \( r_1, \ldots, r_k \) are rational is countable as well. To see this note that for every \( m \), the set of all rational numbers \( (\frac{a_1}{m}, \ldots, \frac{a_k}{m}) \) with \( |a_1|, \ldots, |a_k|, |b_1|, \ldots, |b_k| \leq m \) is finite and thus can be listed. For \( m = 1, 2, \ldots \) we can create these lists, and for every tuple \((r_1, \ldots, r_k)\) in the list check to see whether \( p(r_1, \ldots, r_k) \) is negative. If \( p \) is not non-negative, by the previous part we will find an assignment that establishes this.

(c) (25) Consider a multi-variate polynomial \( p \) with integer coefficients (e.g. \( x_1^5x_2 + x_3x_4 - x_1^2 \)). It is known that if \( p \) is non-negative, then there exists an integer \( k \) and polynomials \( q_1, \ldots, q_k \) and \( r \) with integer coefficients such that \( p = (\frac{q_1}{r})^2 + \ldots + (\frac{q_k}{r})^2 \). Use this fact (and the previous part) to show that the following language is decidable:

\[ \{ \langle p \rangle \mid p \text{ is a non-negative multi-variate polynomial with integer coefficients} \} \]

**Answer:** We show that it is possible to enumerate the elements of the following set

\[ A = \{ (q_1, \ldots, q_k, r) \mid q_1, \ldots, q_k, r \text{ are polynomials with integer coefficients} \} \]

Indeed there is an algorithm that for \( m = 1, 2, 3, \ldots \), lists the finite set of elements \((q_1, \ldots, q_k, r) \in A, \) with \( k \leq m, \) \( \deg(q_i) \leq m \) for every \( 1 \leq i \leq k, \) and all the coefficients of \( q_1, \ldots, q_k \) and \( r \) are bounded in magnitude by \( m. \) The we can use this algorithm to enumerate the elements of \( A \) as \( s_1, s_2, \ldots. \) Now the following TM, accepts the complement of \( L: \)

- \( M = \) “on input \( p \) which is a polynomial on \( k \) variables:
  - For \( i = 1, 2, \ldots \)
  - Check whether \( p = (\frac{q_1}{r})^2 + \ldots + (\frac{q_k}{r})^2 \) where \( (q_1, \ldots, q_k, r) = s_i. \) If it is accept.

This and Part (b) show that both \( L \) and its complement are recursively enumerable. It follows that \( L \) and \( L^c \) are decidable.

4. (10 Points) Prove that the following language is not decidable

\[ L = \{ \langle M \rangle \mid L(M) = L(1^*01010^*) \}, \]

where \( L(1^*01010^*) \) denotes the language of the regular expression \( 1^*01010^*. \)

**Answer:** We give a reduction from \( A_{TM}. \) Let \( \langle M, w \rangle \) be an input for \( A_{TM}. \) Construct the following TM.

- \( N = \) “on input \( x: \)
  - Simulate \( M \) on \( w, \) if it rejects, reject and halt.
  - Otherwise check to see if \( x \in L(1^*01010^*). \) If yes, accept.”

Note that if \( \langle M, w \rangle \in A_{TM}, \) then \( L(N) = L(1^*01010^*) \) and if \( \langle M, w \rangle \notin A_{TM}, \) then \( L(N) = \emptyset. \) Thus we showed \( A_{TM} \leq_m L \) as

\[ \langle N \rangle \in L \Leftrightarrow \langle M, w \rangle \in A_{TM}. \]
5. (15 Points) Prove that the following language is not decidable

\[ L = \{ \langle M \rangle \mid L(M) \text{ is finite} \}. \]

**Answer:** We give a reduction from \( A_{\text{TM}}^c \). Let \( \langle M, w \rangle \) be an input for \( A_{\text{TM}}^c \). Construct the following TM.

- \( N = \) “on input \( x \):
- Simulate \( M \) on \( w \), if it accepts, accept. If it rejects, reject.”

Note that if \( \langle M, w \rangle \in A_{\text{TM}}^c \), then \( L(N) = \emptyset \) and if \( \langle M, w \rangle \notin A_{\text{TM}}^c \), then \( L(N) = \{0, 1\}^* \). Thus we showed \( A_{\text{TM}}^c \leq_m L \) as

\[ \langle N \rangle \in L \iff \langle M, w \rangle \in A_{\text{TM}}^c. \]