General rules: In solving these questions you may consult books but you may not consult with each other. There are in total 105 points, but your grade will be considered out of 100. You should either drop your solutions in the assignment drop-off box located in the Trottier Building on the 2nd floor, or submit it through mycourses. Late homeworks can be submitted until 48 hours after the deadline. Late submissions must be submitted through mycourses as the drop-off box will not be checked after the deadline. There will be a penalty of -25% for one-day delays, and -40% for two-day delays on late homeworks.

1. (10 Points) Let $M_1$ and $M_2$ be two Turing machines. Consider the following Turing machine:

On input $w$:

- Step 1: Run $M_1$ on $w$. If $M_1$ accepts $w$, then accept.
- Step 2: Run $M_2$ on $w$. If $M_2$ accepts $w$, then accept.

What is the language of this Turing Machine? Explain.

2. (15 Points) Is the following language decidable$^1$?

$$L = \{ \langle M \rangle \mid M = (\{1, 2, \ldots, 100\}, \{0, 1\}, \{0, 1, \sqcup\}, \delta, 1, 2, 3) \text{ is a decider}\}.$$  

3. A polynomial is called non-negative, if it is non-negative for every assignment of real numbers to its variables. For example $x_1^2 + x_2^2 - 2x_1x_2$ is non-negative because we can express it as $(x_1 - x_2)^2$.

(a) (15 Points) Prove that if a multi-variate polynomial $p$ is not non-negative, then there is an assignment of rational numbers to its variables that makes it negative.

(b) (15 Points) Use the previous part to show that the following language is recursively enumerable (Turing recognizable):

$$L = \{ \langle p \rangle \mid p \text{ is a multi-variate polynomial with integer coefficients, and } p \text{ is not non-negative}\}.$$  

(c) (25) Consider a multi-variate polynomial $p$ with integer coefficients (e.g. $x_1^5x_2 + x_3x_4 - x_1^2$). It is known that if $p$ is non-negative, then there exists an integer $k$ and polynomials $q_1, \ldots, q_k$ and $r$ with integer coefficients such that $p = \left(\frac{q_1}{r}\right)^2 + \ldots + \left(\frac{q_k}{r}\right)^2$. Use this fact (and the previous part) to show that the following language is decidable:

$$\{ \langle p \rangle \mid p \text{ is a non-negative multi-variate polynomial with integer coefficients}\}.$$  

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$^1$Recall that a TM is formally defined as $(Q, \Sigma, \Gamma, \delta, q_{\text{start}}, q_{\text{accept}}, q_{\text{reject}})$. A Turing Machine is called a decider if it halts on every input.
4. (10 Points) Prove that the following language is not decidable

\[ L = \{ \langle M \rangle \mid L(M) = L(1^*01010^*) \} \],

where \( L(1^*01010^*) \) denotes the language of the regular expression \( 1^*01010^* \).

5. (15 Points) Prove that the following language is not decidable

\[ L = \{ \langle M \rangle \mid L(M) \text{ is finite} \}. \]