On the Face Lattice of the Metric Polytope

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Abstract. In this paper we study enumeration problems for polytopes arising from combinatorial optimization problems. While these polytopes turn out to be quickly intractable for enumeration algorithms designed for general polytopes, algorithms using their large symmetry groups can exhibit strong performances. Specifically we consider the metric polytope m_n on n nodes and prove that for $n \geq 9$ the faces of codimension 3 of m_n are partitioned into 15 orbits of its symmetry group. For $n \leq 8$, we describe additional upper layers of the face lattice of m_n. In particular, using the list of orbits of high dimensional faces, we prove that the description of m_8 given in [9] is complete with 1 550 825 000 vertices and that the Laurent-Poljak conjecture [15] holds for $n \leq 8$. Computational issues for the orbitwise face and vertex enumeration algorithms are also discussed.

1 Introduction

A full d-dimensional convex (bounded) polytope P can be defined either by the linear inequalities associated to the set $\mathcal{F}(P)$ of its facets or as the convex hull of its vertex set $\mathcal{V}(P)$. The computation of $\mathcal{F}(P)$ from $\mathcal{V}(P)$ is the facet enumeration problem and the computation of $\mathcal{V}(P)$ from $\mathcal{F}(P)$ is the vertex enumeration problem. These two problems are equivalent by the vertex/facet duality. More generally, any proper face f of P can be defined either by the subset $\mathcal{F}(f)$ of facets containing f or as the convex hull of the vertices $\mathcal{V}(f)$ belonging to f. Given the facet set $\mathcal{F}(P)$, the face enumeration problem consists in enumerating all the faces f of P in terms of facet sets $\mathcal{F}(f)$. These computationally difficult problems have been well studied; see [2, 3, 13] and references there. In this paper, we consider combinatorial polytope, i.e. polytopes arising from combinatorial optimization problems. These polytopes are often trivial for the very first cases and then the so-called combinatorial explosion occurs even for small instances. On one hand, combinatorial polytopes are quickly intractable for enumeration algorithm designed for solving general polytope, but on the other hand, algorithms using their large symmetry groups allow enumerations which were not possible otherwise. For example, large instances of the traveling salesman polytope, the linear ordering polytope, the cut polytope and the metric polytope were computed in [4, 9] using the same algorithm called *adjacency decomposition method* in [4] and *orbitwise vertex enumeration algorithm* in [9] which, given a vertex, find the adjacent ones, see Section 5.3 for more details. In this paper, pursuing the same approach, we propose an orbitwise face enumeration algorithm for combinatorial polytope. Focusing on the face lattice of the metric polytope m_n, we compute its upper layers for $n \leq 9$. These results allow us to prove that the description of m_8 given in [9] is complete with 1 550 825 000 vertices and that the *dominating set* and *no cut-set* conjectures, see [9, 15], hold for m_8. A description of the faces of codimension 3 for any n is given as well as some preliminary results on the vertices of m_9.

2 Face Enumeration for Combinatorial Polytopes

2.1 Combinatorial polytopes

Many combinatorial polyhedra are associated to optimization problems arising from the complete directed graph D_n or the undirected graph K_n on n nodes. Well studied combinatorial polyhedra include the cut polytope c_n and the metric polytope m_n. While c_n is the convex hull of the incidence vectors of all the cuts of K_n, m_n can be defined as a relaxation of c_n by the triangular inequalities, see Section 3.1 and see [11] for more details. One important feature of most combinatorial polytopes is their very large symmetry group. We recall that the symmetry group $Is(P)$ of a polytope P is the group of isometries preserving P. For $n \geq 5$, $Is(m_n) = Is(c_n)$ is induced by the $n!$ permutations on V_n and the 2^{n-1} *switching reflections*, see Section 3.2, and $|Is(m_n)| = 2^{n-1}n!$, see [10]. For $n = 4$, $m_4 = c_4$ and $|Is(m_4)| = 2(4!)^2$. Clearly all faces are partitioned into orbits of faces equivalent under permutations and switchings. An orbitwise vertex enumeration algorithm was proposed in [4, 9] and, in a similar vein, we propose an orbitwise face enumeration algorithm.

2.2 Orbitwise face enumeration algorithm

The input is a full d-dimensional polytope P defined by its (non-redundant) facet set $\mathcal{F}(P) = \{f_1^{d-1}, \ldots, f_m^{d-1}\}$. The algorithm first computes the list $\mathcal{L}^{d-1} = \{\tilde{f}_1^{d-1}, \ldots, \tilde{f}_{I^{d-1}}^{d-1}\}$ of all the canonical representatives of the orbits of facets. Then the algorithm generates the set $L^{d-2} = \{\tilde{f}_s^{d-1} \cap f_r^{d-1} : s = 1, \ldots, I^{d-1}, r = 1, \ldots, m\}$. After computing the dimension of each subface $\tilde{f}_s^{d-1} \cap f_r^{d-1}$ and keeping only the $(d-2)$-faces, the algorithm reduces L^{d-2} to the list of canonical representatives of orbits of $(d-2)$-faces $\mathcal{L}^{d-2} = \{\tilde{f}_1^{d-2}, \ldots, \tilde{f}_{I^{d-2}}^{d-2}\}$. In general, after generating the list $\mathcal{L}^{d-t+1}$, the algorithm computes $\mathcal{L}^{d-t}$ by:

(i) generating the set L^{d-t} by intersecting each canonical representative $\tilde{f}_s^{d-t+1}$ with each facet F_r for $s = 1, \ldots, I^{d-t+1}$ and $r = 1, \ldots, m$,
(ii) computing the set $\mathcal{F}(\tilde{f}^{d-t+1} \cap f^{d-1})$ of all facets containing $\tilde{f}^{d-t+1} \cap f^{d-1}$ and then its dimension $dim(\tilde{f}^{d-t+1} \cap f^{d-1})$
(iii) for $dim(\tilde{f}^{d-t+1} \cap f^{d-1}) = d-t$, computing the canonical representative $\tilde{f}^{d-t}$ of $\tilde{f}^{d-t+1} \cap f^{d-1}$

The algorithm terminates after the list $\mathcal{L}^0$ of canonical representatives of the orbits of vertices is computed. Clearly the algorithm works faster when the symmetry group $Is(P)$ is larger. The main two subroutines are the computation of the canonical representative $\tilde{f}$ of the orbit O_f generated by a face f and the computation of the dimension $dim(f)$. The determination of $\mathcal{F}(\tilde{f}^{d-t+1} \cap f^{d-1})$ amounts to a redundancy check for the remaining facets of $\mathcal{F}(P) \backslash \{\mathcal{F}(\tilde{f}^{d-t+1} \cap f^{d-1})\}$. This operation can be done using *ccclib* (*redcheck*), see [12], and is polynomially equivalent to linear programming; see [3]. The rank of $\mathcal{F}(\tilde{f}^{d-t+1} \cap f^{d-1})$ directly gives $dim(\tilde{f}^{d-t+1} \cap f^{d-1})$. The computation of the canonical representative $\tilde{f}^{d-t}$ is done using a brute-force approach; that is, by generating all the elements belonging to the orbit $O_{\tilde{f}^{d-t+1} \cap f^{d-1}}$.

Remark 1.

1. With I^{d-t} the number of orbits of $(d-t)$-faces and m the number of facets, the dimension (resp. canonical representative) computation subroutine is called exactly (resp. at most) $m(1 + \sum_{t=1,\ldots,d-1} I^{d-t})$ times.
2. The output; that is, for $t = 1, \ldots, d$ the list $\mathcal{L}^{d-t}$ of canonical representatives $\tilde{f}_s^{d-t} : s = 1, \ldots, I^{d-t}$, is extremely compact. The full list of $(d-t)$-faces can be generated by the action of the symmetry group on each representative face $\tilde{f}_s^{d-t}$. With $|O_{\tilde{f}_s^{d-t}}|$ the size of the orbit generated by $\tilde{f}_s^{d-t}$, the total number of faces is $\sum_{t=1,\ldots,d} \sum_{s=1,\ldots,I^{d-t}} |O_{\tilde{f}_s^{d-t}}|$.

Item 1 of Remark 1 indicates that the algorithm runs smoothly as long as the number I^{d-t} of orbits of $(d-t)$-faces is relatively small. The number of $(d-t)$-faces usually grows extremely large with t getting close to $\lfloor \frac{d}{2} \rfloor$; that is: "Face lattices are very fat". Therefore the computation of the full face lattice of a polytope is generally extremely hard. Besides small dimensional polytopes and specific cases such as the d-cube, we can expect a similar pattern for the values of I^{d-t}; that is: "Orbitwise face lattices are also fat". On the other hand, one can expect the combinatorial explosion to occur at a deeper layer for the orbitwise face lattice than for the ordinary one. Actually, this algorithm is particularly suitable for the computation of the upper τ layers of the orbitwise face lattice for a small given τ. In that case the algorithm stops when $\mathcal{L}^{d-\tau}$ is computed. The computation of the orbitwise upper face lattice can be efficiently combined with classical vertex enumeration. See Section 5.1 for an application to the complete description of the vertices of m_8.

3 Faces of the Metric Polytope

3.1 Cut and metric polytopes

The $\binom{n}{2}$-dimensional cut polytope c_n is usually introduced as the convex hull of the incidence vectors of all the cuts of K_n. More precisely, given a subset S of $V_n = \{1, \ldots, n\}$, the *cut* determined by S consists of the pairs (i, j) of elements of V_n such that exactly one of i, j is in S. By $\delta(S)$ we denote both the cut and its incidence vector in $\mathbb{R}^{\binom{n}{2}}$; that is, $\delta(S)_{ij} = 1$ if exactly one of i, j is in S and 0 otherwise for $1 \leq i < j \leq n$. By abuse of notation, we use the term cut for both the cut itself and its incidence vector, so $\delta(S)_{ij}$ are considered as coordinates of a point in $\mathbb{R}^{\binom{n}{2}}$. The cut polytope c_n is the convex hull of all 2^{n-1} cuts, and the *cut cone* C_n is the conic hull of all $2^{n-1} - 1$ nonzero cuts. The cut polytope and one of its relaxation - the metric polytope - can also be defined in terms of a finite metric space in the following way. For all 3-sets $\{i, j, k\} \subset \{1, \ldots, n\}$, we consider the following inequalities:

$$x_{ij} - x_{ik} - x_{jk} \leq 0, \tag{1}$$

$$x_{ij} + x_{ik} + x_{jk} \leq 2. \tag{2}$$

(1) induce the $3\binom{n}{3}$ facets which define the *metric cone* M_n. Then, bounding the latter by the $\binom{n}{3}$ facets induced by (2) we obtain the metric polytope m_n. The facets defined by (1) (resp. by (2)) can be seen as triangle (resp. perimeter) inequalities for distance x_{ij} on $\{1, \ldots, n\}$ and are denoted by $\Delta_{i,j,\bar{k}}$ (resp. by $\Delta_{i,j,k}$). While the cut cone is the conic hull of all, up to a constant multiple, $\{0, 1\}$-valued extreme rays of the metric cone, the cut polytope c_n is the convex hull of all $\{0, 1\}$-valued vertices of the metric polytope. For a detailed study of those polytopes and their applications in combinatorial optimization we refer to Deza and Laurent [11] and Poljak and Tuza [16].

3.2 Combinatorial and Geometric Properties

The polytope c_n is a $\binom{n}{2}$-dimensional $\{0, 1\}$-polyhedron with 2^{n-1} vertices and m_n is a polytope of the same dimension with $4\binom{n}{3}$ facets inscribed in the cube $[0, 1]^{\binom{n}{2}}$. We have $c_n \subseteq m_n$ with equality only for $n \leq 4$. Any facet of the metric polytope contains a face of the cut polytope and the vertices of the cut polytope are vertices of the metric polytope. In fact, the cuts are precisely the integral vertices of the metric polytope. The metric polytope m_n wraps the cut polytope c_n very tightly. Indeed, in addition to the vertices, all edges and 2-faces of c_n are also faces of m_n, for 3-faces it is false for $n \geq 4$, see [7]. Any two cuts are adjacent both on c_n and on m_n; in other words m_n is *quasi-integral*; that is, the skeleton of the convex hull of its integral vertices, i.e. the skeleton of c_n, is an induced subgraph of the skeleton of the metric polytope itself. We recall that the skeleton of a polytope is the graph formed by its vertices and edges. While the diameters of the cut polytope and the dual metric polytope satisfy $\delta(c_n) = 1$ and $\delta(m_n^*) = 2$, the diameters of their dual are conjectured to be $\delta(c_n^*) = 4$ and $\delta(m_n) = 3$, see [6, 15]. One important feature of the metric and cut polytopes is their very large symmetry group. More precisely, for $n \geq 5$, $Is(m_n) = Is(c_n)$ is induced by the permutations on $V_n = \{1, \ldots, n\}$ and the switching reflections by a cut and $|Is(m_n)| = 2^{n-1}n!$, see [10]. Given a cut $\delta(S)$, the switching reflection $r_{\delta(S)}$ is defined by $y = r_{\delta(S)}(x)$ where $y_{ij} = 1 - x_{ij}$ if $(i, j) \in \delta(S)$ and $y_{ij} = x_{ij}$ otherwise. For $n = 4$, $c_4 = m_4$ and there are some additional symmetries: $|Is(m_4)| = 2(4!)^2$. Note that the symmetries preserve the adjacency relations and the linear independency.

3.3 Faces of the Metric Polytope

We recall some results and conjectures on the faces of the metric polytope. The cuts are the only integral vertices of m_n. Consider the following map $\phi_0 : \mathbb{R}^{\binom{n-1}{2}} \longrightarrow \mathbb{R}^{\binom{n}{2}}$, defined by: $\phi_0(v)_{ij} = v_{ij}$ for $1 \leq i < j \leq n-1$, $\phi_0(v)_{i,n} = v_{1,i}$ for $2 \leq i \leq n-1$ and $\phi_0(v)_{1,n} = 0$. Both $\phi_0(v)$ and its switching by $\delta(\{n\})$ are called *trivial extensions* of v. Note that a trivial extension of a vertex of m_{n-1} is a vertex of m_n. Besides the cuts, all vertices with are not fully fractional are trivial extensions; that is, the *new vertices* of m_n are the fully fractional ones. The $(\frac{1}{3}, \frac{2}{3})$-valued fully fractional vertices are well studied, see [11, 14, 15], and include the anticut orbit formed by the 2^{n-1} *anticuts* $\bar{\delta}(S) = \frac{2}{3}(1, \ldots, 1)$-$\frac{1}{3}\delta(S)$. If $G = (V_n, E)$ is a connected graph, we denote by d_G its path metric, where $d_G(i,j)$ is the length of a shortest path from i to j in G for $i \neq j \in V_n$. Then $\tau(d_G) = max(d_G(i,j) + d_G(i,k) + d_G(j,k) : i,j,k \in G)$ is called the *triameter of* G and we set $x_G = \frac{2}{\tau(d_G)} d_G$. Any vertex of m_n of the form x_G for some graph is called a *graphic* vertex, see [11, 14, 15] and Fig. 1 for the graphs of 2 graphic $(\frac{1}{3}, \frac{2}{3})$-valued vertices of m_8. Note that for any connected graph $G = (V_n, E)$, we have $\tau(d_G) \leq 2(n-1)$ and that any $(\frac{1}{3}, \frac{2}{3})$-valued vertex v of m_n is (up to switching) graphic; that is, there exist a graph G and a cut $\delta(S)$ such that $v = r_{\delta(S)}(x_G)$. Since $m_3 = c_3$ and $m_4 = c_4$, the vertices of m_3 and m_4 are made of 4 and 8 cuts forming 1 orbit. The 32 vertices of m_5 are 16 cuts and 16 anticuts, i.e., form 2 orbits. The metric polytope m_6 has 544 vertices, see [15], partitioned into 3 orbits: cuts, anticuts and 1 orbit of trivial extensions; and m_7 has 275 840 vertices, see [8], partitioned into 13 orbits: cuts, anticuts, 3 orbits of trivial extensions, 3 $(\frac{1}{3}, \frac{2}{3})$-valued orbits and 5 other fully fractional orbits. For m_8, 1 550 825 600 vertices partitioned into 533 orbits (cuts, anticuts, 28 trivial extensions, 37 $(\frac{1}{3}, \frac{2}{3})$-valued and 466 other fully fractional) were found assuming Conjecture 2, see [9]. The description was conjectured to be complete.

Conjecture 1. [15] Any vertex of the metric polytope m_n is adjacent to a cut.

Conjecture 2. [9] For $n \geq 6$, the restriction of the skeleton of the metric polytope m_n to the non-cut vertices is connected.

Conjecture 2 can be seen as complementary to the Conjecture 1 both graphically and computationally: For any pair of vertices, while Conjecture 1 implies that there is a path made of cuts joining them, Conjecture 2 means that there is a path made of non-cuts vertices joining them. In other words, the cut vertices would form a *dominating set* but not a *cut-set* in the skeleton of m_n. On the other hand, while Conjecture 1 means that the enumeration of the extreme rays of the metric cone M_n is enough to obtain the vertices of the metric polytope m_n; Conjecture 2 means that we can obtain the vertices of m_n without enumerating the extreme rays of M_n. Note that for arbitrary graphs these are clearly independent. Conjecture 1 underlines the extreme connectivity of the cuts. Recall that the cuts form a clique in both the cut and metric polytopes. Therefore, if Conjecture 1 holds, the cuts would be a dominant clique in the skeleton of m_n implying that its diameter would satisfy $\delta(m_n) \leq 3$. The orbitwise description of the facets and ridges (faces of codimension 2) of m_n for any n was given in [6] as well as the face $\Delta_{1,2,3} \cap \Delta_{1,2,\bar{3}}$ of codimension $n-1$ and the face $\Delta_{1,2,3} \cap \Delta_{1,\bar{3},4}$ of codimension 3. We have $\mathcal{L}^{d-1}(m_n) = \{\Delta_{1,2,3}\}$ and $\mathcal{L}^{d-2}(m_{n\geq 6}) = \{\Delta_{1,2,3} \cap \Delta_{1,2,4},\ \Delta_{1,2,3} \cap \Delta_{1,4,5},\ \Delta_{1,2,3} \cap \Delta_{4,5,6}\}$, $\mathcal{L}^{d-2}(m_5) = \{\Delta_{1,2,3} \cap \Delta_{1,2,4},\ \Delta_{1,2,3} \cap \Delta_{1,4,5}\}$ and $\mathcal{L}^{d-2}(m_4) = \{\Delta_{1,2,3} \cap \Delta_{1,2,4}\}$. The full orbitwise face lattices of m_4 and m_5 were given in [7]. In Section 4.1 we compute additional orbits of faces of small metric polytopes and in Section 4.2 we characterize $\mathcal{L}^{d-3}(m_n)$ for any n.

4 Generating Faces of the Metric Polytope

4.1 Faces of small metric polytopes

As stated earlier, generating the full face lattice is usually extremely hard. We restricted the computation to the enumeration of the upper τ layers of the orbitwise face lattice of m_n. We choose to set $\tau = 4$ for the partial orbitwise enumeration of m_6 (resp. m_7 and m_8). The first 4 entries of the f-vectors of m_6, m_7 and m_8 are: $f(m_6) = \{1, 3, 10, 34, \ldots\}$, $f(m_7) = \{1, 3, 13, 61, \ldots\}$ and $f(m_8) = \{1, 3, 14, 79, \ldots\}$, Due to space limitation, we refer to [5] for a detailed presentation. The set $\mathcal{L}^{d-3}(m_n)$ is easy to check for reasonable values of n as $I^{d-3}(m_n) \leq 15$, see Theorem 1. Additional properties of m_n can be used to increase the efficiency of the algorithm. In particular, the set L^{d-t} can be generated by considering for each s only the facets which are not equivalent under isometries preserving $\tilde{f}_s^{d-t+1}$. The *support* of $\Delta_{i,j,k}$ (or $\Delta_{i,j,\bar{k}}$) is $\sigma(\Delta_{i,j,k}) = \sigma(\Delta_{i,j,\bar{k}}) = \{i, j, k\}$. Let assume that, as in Section 5.2, we are interested only in the upper $n-1$ layers of the face lattice of m_n. In that case, when generating $\tilde{f}_s^{d-t+1} \cap \Delta_r$ with $t < n$, we can disregard Δ_r if $\sigma(\Delta_r) = \sigma(\Delta)$ for any $\Delta \in \mathcal{F}(\tilde{f}_s^{d-t+1})$ as for such Δ_r we have $codim(\tilde{f}_s^{d-t+1} \cap \Delta_r) \geq n-1$.

4.2 Faces of codimension 3 of the metric polytope

As recalled earlier the first 2 upper layers of m_n are known for any n. We have $I^{d-1}(m_n) = 1$, $I^{d-2}(m_{n\geq 6}) = 3$ and, by Theorem 1, we get $I^{d-3}(m_{n\geq 9}) = 15$.

Theorem 1. *For $n \geq 9$, the faces of codimension 3 of the metric polytope m_n are partitioned into 15 orbits equivalent under permutations and switchings. The first 15 (resp. 14, 13, 10 and 6) representatives given in Table 1 generate the 15 (resp. 14, 13, 10 and 6) orbits of faces of codimension 3 of $m_{n\geq 9}$ (resp. m_8, m_7, m_6 and m_5). The first 2 representatives and the last one generate the 3 orbits of faces of codimension 3 of m_4.*

Table 1. The orbits of faces of codimension 3 of m_n for $n \geq 4$

Orbit $O_{f_i^3}$	Representative f_i^3	m_n for which f_i^3 is a $(d-3)$-face	$\lvert O_{f_i^3} \rvert$
$O_{f_1^3}$	$\Delta_{1,2,3} \cap \Delta_{1,2,4} \cap \Delta_{1,3,4}$	$m_{n\geq 4}$	$32\binom{n}{4}$
$O_{f_2^3}$	$\Delta_{1,2,3} \cap \Delta_{1,2,4} \cap \Delta_{1,\bar{3},4}$	$m_{n\geq 4}$	$24\binom{n}{4}$
$O_{f_3^3}$	$\Delta_{1,2,3} \cap \Delta_{1,2,4} \cap \Delta_{1,2,5}$	$m_{n\geq 5}$	$160\binom{n}{5}$
$O_{f_4^3}$	$\Delta_{1,2,3} \cap \Delta_{1,2,4} \cap \Delta_{1,3,5}$	$m_{n\geq 5}$	$960\binom{n}{5}$
$O_{f_5^3}$	$\Delta_{1,2,3} \cap \Delta_{1,2,4} \cap \Delta_{3,4,5}$	$m_{n\geq 5}$	$480\binom{n}{5}$
$O_{f_6^3}$	$\Delta_{1,2,3} \cap \Delta_{1,2,4} \cap \Delta_{\bar{3},4,5}$	$m_{n\geq 5}$	$480\binom{n}{5}$
$O_{f_7^3}$	$\Delta_{1,2,3} \cap \Delta_{1,2,4} \cap \Delta_{1,5,6}$	$m_{n\geq 6}$	$5\ 760\binom{n}{6}$
$O_{f_8^3}$	$\Delta_{1,2,3} \cap \Delta_{1,2,4} \cap \Delta_{3,5,6}$	$m_{n\geq 6}$	$5\ 760\binom{n}{6}$
$O_{f_9^3}$	$\Delta_{1,2,3} \cap \Delta_{1,4,5} \cap \Delta_{2,4,6}$	$m_{n\geq 6}$	$3\ 840\binom{n}{6}$
$O_{f_{10}^3}$	$\Delta_{1,2,3} \cap \Delta_{1,4,5} \cap \Delta_{\bar{2},4,6}$	$m_{n\geq 6}$	$3\ 840\binom{n}{6}$
$O_{f_{11}^3}$	$\Delta_{1,2,3} \cap \Delta_{1,2,4} \cap \Delta_{5,6,7}$	$m_{n\geq 7}$	$6\ 720\binom{n}{7}$
$O_{f_{12}^3}$	$\Delta_{1,2,3} \cap \Delta_{1,4,5} \cap \Delta_{1,6,7}$	$m_{n\geq 7}$	$6\ 720\binom{n}{7}$
$O_{f_{13}^3}$	$\Delta_{1,2,3} \cap \Delta_{1,4,5} \cap \Delta_{2,6,7}$	$m_{n\geq 7}$	$40\ 320\binom{n}{7}$
$O_{f_{14}^3}$	$\Delta_{1,2,3} \cap \Delta_{1,4,5} \cap \Delta_{6,7,8}$	$m_{n\geq 8}$	$53\ 760\binom{n}{8}$
$O_{f_{15}^3}$	$\Delta_{1,2,3} \cap \Delta_{4,5,6} \cap \Delta_{7,8,9}$	$m_{n\geq 9}$	$17\ 920\binom{n}{9}$
$O_{f_{16}^3}$	$\Delta_{1,2,3} \cap \Delta_{1,2,4} \cap \Delta_{1,3,4}$	m_4	$2\binom{n}{2}$

Proof. *For $n \leq 9$ Theorem 1 can be directly checked using the orbitwise face enumeration algorithm with $\tau = 3$; that is, the algorithm is set to compute only the upper 3 layers of the face lattice of m_n. Let assume $n \geq 9$, the faces of codimension 2 of m_n are partitioned into 3 orbits generated by $\Delta_{1,2,3} \cap \Delta_{1,2,4}$, $\Delta_{1,2,3} \cap \Delta_{1,4,5}$ and $\Delta_{1,2,3} \cap \Delta_{4,5,6}$. Any faces of codimension 3 of m_n can therefore be written as the intersection of a facet Δ of m_n with one of these 3 faces $\Delta' \cap \Delta''$ of codimension 2. If the support $\sigma(\Delta) \not\subset \{1, \ldots, 9\}$, by elementary permutations preserving Δ' and Δ'' we can generate $\tilde{\Delta} \in O_\Delta$ with $O_{\Delta' \cap \Delta'' \cap \tilde{\Delta}} = O_{\Delta' \cap \Delta'' \cap \Delta}$ and $\sigma(\tilde{\Delta}) \subset \{1, \ldots, 9\}$. In other words, to generate orbitwise all the subfaces of the canonical faces of codimension 2 it is enough to consider the case $n = 9$. This way one can easily obtain 28 faces f_i of codimension at least 3. Then, as for the orbitwise face enumeration algorithm, we have to compute for $i = 1, \ldots, 28$ and for any n the dimension $dim(f_i)$ and - if $codim(f_i) = 3$ - to compute the canonical representative $\tilde{f_i}$. Therefore we have to first determine the set $\mathcal{F}_n(f_i)$ of facets of m_n containing f_i. Clearly, if an inequality (i) defining a facet of m_n is forced to be satisfied with equality by the inequalities defining Δ', Δ'' and $\tilde{\Delta}$*

being satisfied with equality, then the same inequality (i) *- now seen as defining a facet of* m_{n+1} *- will also be forced to be satisfied with equality. In other words the set* $\mathcal{F}_n(f_i)$ *can only increase with* n *and* $dim(f_i)$ *can only decrease with* n. *Therefore, among the* 28 *faces* f_i, *only the* 15 *first faces of codimension* 3 *for* m_9 *given in Table 1 are candidates for being faces of codimension* 3 *for* $m_{n\geq 9}$. *A case by case study of the* 15 *faces, gives* $\mathcal{F}_n(f_i)$ *and proves that indeed these* 15 *faces generate* 15 *orbits of faces of codimension* 3 *for* $n \geq 9$. *The idea is simply to notice that the pattern of* $\mathcal{F}_n(f_i)$ *is essentially given by the value of* $\mathcal{F}_{12}(f_i)$. *Since all the cases are similar, we only present the computation of* $\mathcal{F}_n(f_{15})$ *where* $f_{15} = \Delta_{1,2,3} \cap \Delta_{4,5,6} \cap \Delta_{7,8,9}$. *Using the orbitwise face enumeration algorithm with* $\tau = 3$, *one can easily check that* $\mathcal{F}_{12}(f_{15}) = \{\Delta_{1,2,3},\ \Delta_{4,5,6},\ \Delta_{7,8,9}\}$. *Let* $n \geq 12$ *and* Δ *be a facet of* m_n *with* $\sigma(\Delta) \not\subset \{1,\ldots,12\}$. *By elementary permutations preserving* $\mathcal{F}_{12}(f_{15})$ *we can generate* $\tilde{\Delta} \in O_\Delta$ *with* $\sigma(\tilde{\Delta}) \subset \{1,\ldots,12\}$. *Let now consider* $\tilde{\Delta}$ *as a facet of* m_{12}. *Since* $\tilde{\Delta} \notin \mathcal{F}_{12}(f_{15})$ *at least one vertex* v *of* m_{12} *satisfies* $v \in f_{15}$ *and* $v \notin \tilde{\Delta}$. *Then, the* $(n-12)$*-times* 0*-extension* v_{ext} *of* v *is a vertex of* m_n *satisfying* $v_{ext} \in f_{15}$ *but* $v_{ext} \notin \tilde{\Delta}$ *where* $\tilde{\Delta}$ *is now considered as a facet of* m_n. *Thus,* $\tilde{\Delta} \notin \mathcal{F}_n(f_{15})$ *and, by the same elementary permutations,* $\Delta \notin \mathcal{F}_n(f_{15})$*; that is,* $\mathcal{F}_n(f_{15}) = \{\Delta_{1,2,3},\ \Delta_{4,5,6},\ \Delta_{7,8,9}\}$ *and* $codim(f_{15}) = 3$ *for any* $n \geq 9$. *In the same way, for* $\mathcal{F}_n(f)$ *increasing with* n, *the pattern of* $\mathcal{F}_n(f)$ *is essentially given by small values of* n. *Consider for example* $\mathcal{F}_n(\Delta_{1,2,3} \cap \Delta_{1,2,\bar{3}})$*: We have* $\mathcal{F}_n(\Delta_{1,2,3} \cap \Delta_{1,2,\bar{3}}) = \{\Delta_{1,2,i},\ \Delta_{1,2,\bar{i}} : i = 3,\ldots,n\}$ *and therefore* $|\mathcal{F}_n(\Delta_{1,2,3} \cap \Delta_{1,2,\bar{3}})| = 2(n-2)$ *and* $codim(\Delta_{1,2,3} \cap \Delta_{1,2,\bar{3}}) = n-1$. *As for* $\mathcal{F}_n(f_{15})$, *one can compute* $\mathcal{F}_4(\Delta_{1,2,3} \cap \Delta_{1,2,\bar{3}})$ *and notice that* $\Delta \in \mathcal{F}_{n\geq 5}(\Delta_{1,2,3} \cap \Delta_{1,2,\bar{3}}) \iff \tilde{\Delta} \in \mathcal{F}_4(\Delta_{1,2,3} \cap \Delta_{1,2,\bar{3}})$*; that is,* $\Delta = \Delta_{1,2,i}$ *or* $\Delta_{1,2,\bar{i}} : i = 4,\ldots,n$. □

Remark 2. The proof of Theorem 1 indicates that the number $I^{d-k}(m_n)$ of orbits of faces of codimension k of the metric polytope is probably constant for $n \geq 3k$. Another interesting issue is the determination of an upper bound for $I^{d-k}(m_n)$ for any k and n.

5 Generating Vertices of the Metric Polytope

5.1 Combining orbitwise face enumeration with classical vertex enumeration

As emphasized earlier, the face lattice is usually much larger than the number of vertices. Therefore, computing the full face lattice in order to obtain the vertices is extremely costly. On the other hand, the upper layers of the orbitwise face lattice might be relatively small. In that case the orbitwise face enumeration can be efficiently combined with a classical vertex enumeration methods in the following way. First, for an appropriate small τ, compute the upper orbitwise face lattice till the list $\mathcal{L}^{d-\tau}$ of canonical $(d-\tau)$-faces is obtained. Then for $s = 1,\ldots,I^{d-\tau}$, compute by a classical vertex enumeration method the set $\mathcal{V}(\tilde{f}_s^{d-\tau})$ of vertices belonging to $\tilde{f}_s^{d-\tau}$. Finally, compute the canonical representative $\tilde{v}$ for each vertex $v \in \mathcal{V}(\tilde{f}_s^{d-\tau})$. The set of all such vertices $\tilde{v}$ is exactly $\mathcal{L}^0$ as each canonical vertex of $\mathcal{L}^0$ belongs, up to an isometry of $Is(P)$, to at least one of the $(d-\tau)$-faces $\tilde{f}_s^{d-\tau}$. Clearly, the choice of τ is critical. Typically, for the first values of t, by going down one layer from $\mathcal{L}^{d-t+1}$ to $\mathcal{L}^{d-t}$ the number of orbits increases ($I^{d-t} \geq I^{d-t+1}$) and the average sizes of faces decreases ($\mathcal{V}(\tilde{f}^{d-t}_{average}) \leq \mathcal{V}(\tilde{f}^{d-t+1}_{average})$). Therefore, a *good* τ should be such that $I^{d-\tau}$ and $\mathcal{V}(\tilde{f}^{d-\tau}_{average})$ are relatively small: In particular the largest $\tilde{f}_s^{d-\tau}$ should within problems currently solvable by vertex enumeration algorithms. In Section 5.2, assuming that the computation of m_{n-1} is just within current vertex enumeration abilities, we indicate that for m_n a good τ should satisfy $n-1 \leq \tau$ and that $\tau = 7$ is actually enough for the description of m_8. Note that $n-1 = 7 = \lfloor \frac{d}{4} \rfloor$ for m_8.

5.2 Vertices of the metric polytope on 8 nodes

As mentioned earlier, the face $\tilde{f}_\mu^{d-n+1} = \Delta_{1,2,3} \cap \Delta_{1,2,\bar{3}}$ generates one orbit of faces of codimension $n-1$ of m_n which are combinatorially equivalent to m_{n-1}. In other words, the orbitwise face lattice of m_n contains a copy of m_{n-1} in $\mathcal{L}^{d-n+1}$. This implies that some canonical faces of $\mathcal{L}^{d-n+2}$ are quite larger than m_{n-1} and therefore intractable if we assume that m_{n-1} is just within current vertex enumeration methods abilities. For m_8, it means that we should compute at least $\mathcal{L}^{21}$ and it turns out to be enough as $\tilde{f}_\mu^{21}$ (which we do not need to enumerate since $\tilde{f}_\mu^{21} \simeq m_7$) and other elements of $\mathcal{L}^{21}$ are tractable. The whole computation is quite long as $\mathcal{L}^{21}$ is large as well as $\mathcal{V}(\tilde{f}^{21}_{average})$. For the same reasons, skipping $\tilde{f}_\mu^{21}$, the computation of the canonical vertices for each $\mathcal{V}(\tilde{f}_s^{21})$ is also long. Insertion algorithms usually handle high degeneracy better than pivoting algorithms, see [2] for a detailed presentation of the main vertex enumeration methods. The metric polytope m_n is quite degenerate as the cut incidence $Icd_{\delta(S)} = 3\binom{n}{3}$ is much larger than the dimension $d = \binom{n}{2}$. We recall that the incidence $Icd_v = |\mathcal{F}(v)|$. Thus we choose an insertion algorithm for the enumeration of each $\tilde{f}_s^{21}$: the *cddlib* implementation of the double description method [12]. The ordering of the facet is lexicographic with the rule $-1 \prec 1 \prec 0$. The result shows that $\mathcal{L}^0$ is made of the 533 canonical vertices found in [9]. Due to space limitation, we refer to [5] for a detailed presentation. The conjectured description of m_8 being complete, the following is straightforward.

Proposition 1.

1. *The metric polytope m_8 has exactly 1 550 825 600 vertices and its diameter is $\delta(m_8) = 3$. The metric cone M_8 has exactly 119 269 588 extreme rays.*
2. *The* LAURENT-POLJAK *dominating set Conjecture 1 and the no cut-set Conjecture 2 hold for m_8.*

A vertex of a d-dimensional polytope is simple if $|\mathcal{F}(v)| = d$. While most of the vertices of m_8 are almost simple, the only simple vertices of m_8 belong to the orbits $O_{\tilde{v}_{532}}$ and $O_{\tilde{v}_{533}}$ of size $|O_{\tilde{v}_{532}}|$ =368 640 and $|O_{\tilde{v}_{533}}|$ =430 080; that is, only 0.05% of the total number of vertices of m_8 are simple. Both canonical representative $\tilde{v}_{532}$ and $\tilde{v}_{533}$ are graphic $(\frac{1}{3}, \frac{2}{3})$-valued vertices, see Fig. 1. The largest denominator among vertices of m_8 is 15 and occurs only for vertices of $O_{\tilde{v}_{451}}$ with $\tilde{v}_{451} = \frac{1}{15}(2,4,4,5,5,7,8,6,6,5,5,5,10,8,5,5,5,4,9,9,3,4,10,10,5,10,5,5)$ and $|O_{\tilde{v}_{451}}|$ =2 580 480. All vertices of m_8 are adjacent to at least 2 cuts and the vertices adjacent to exactly 2 cuts belong to $O_{\tilde{v}_{531}}$ with $|O_{\tilde{v}_{531}}|$ =1 290 240 and $\tilde{v}_{531} = \frac{1}{9}(2,2,3,3,4,4,5,4,3,3,6,6,3,5,5,2,6,3,6,3,3,6,3,3,6,4,5,3)$.

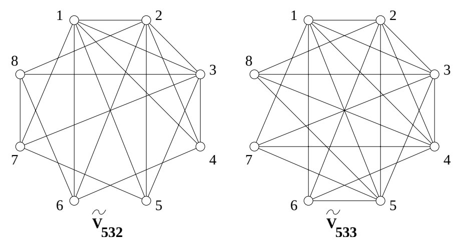

Fig. 1. Graphic canonical vertices of the only two orbits of simple vertices of m_8

5.3 Vertices of the metric polytope on 9 nodes

The computation of the vertices of m_9 is most probably intractable as we expect this extremely degenerate 36-dimensional polytope to have around 10^{14} vertices partitioned among several hundred thousand orbits. In this section, we present some computational results concerning the vertices of m_9. Given a vertex v, after computing the canonical representative $\tilde{v}$, the orbitwise vertex enumeration algorithm computes the set $N_{\tilde{v}}$ of vertices adjacent to $\tilde{v}$, then identifies all the orbits of vertices contained in $N_{\tilde{v}}$ and picks up the next representative whose neighborhood is not yet computed. The algorithm terminates when all the orbitwise neighborhoods $N_{\tilde{v}}$ are computed. To compute those neighborhoods, the algorithm performs one classic vertex enumerations for each orbit $O_{\tilde{v}}$ of vertices. The complexity of computing $N_{\tilde{v}}$ is closely related to the incidence of $\tilde{v}$. While the computation is easy for vertices having a small incidence, highly degenerated vertices can be intractable. For example, the algorithm failed to computed $N_{\delta(S)}$ the set of vertices adjacent to a cut $\delta(S)$ for m_8; we recall that $Icd_{\delta(S)} = 168$ and that $dim(m_8) = 28$. This remark leads to the following *skipping high degeneracy* heuristic: compute $N_{\tilde{v}}$ only for $Icd_{\tilde{v}} \leq Icd_{max}$ where Icd_{max} is an arbitrarily set in advance upper bound for the considered incidences, see [9]. In particular, assuming Conjecture 2 means that we can enumerate m_n by setting $Icd_{max} = Icd_{\delta(S)} - 1$. Setting $Icd_{max} = 44$, we computed 253 210 orbits of vertices of m_9. We wish to set $Icd_{max} = \frac{1}{2}\binom{n+1}{3} = 60$ (i.e. halfway from the dimension $\binom{n}{2}$ to the anticut incidence $Icd_{\bar{\delta}(S)} = \binom{n}{3}$, see [9]), but significantly raising the value of Icd_{max} is currently beyond our computational capacities. The largest denominator found is 39 and most of the vertices are almost simple but the lowest incidence is 37, i.e. no simple vertex was found so far. Conjecture 2 has been checked for the 253 210 orbits of vertices of m_9 computed so far.

Remark 3. None of the currently known vertices of m_9 is simple. Since m_6 and m_7 have no simple vertex, the only known simple vertices of m_n for $n \geq 6$ belong to the orbits $O_{\tilde{v}_{532}}$ and $O_{\tilde{v}_{533}}$. We believe that, while we can obtain nearly all the vertices of m_9 by setting $Icd_{max} = 44$, the algorithm can not reach all of them unless the value of Icd_{max} is raised significantly. We also believe that m_9 has simple vertices, see [8], which are among the currently unreachable vertices of m_9.

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