On the Face Lattice of the Metric Polytope

Antoine Deza[1], Komei Fukuda[2], Tomohiko Mizutani[1], and Cong Vo[1]

[1] Tokyo Institute of Technology, Math. and Comput. Sci. Dept., Tokyo, Japan
deza@is.titech.ac.jp
[2] McGill University, School of Computer Science, Montréal, Canada
fukuda@cs.mcgill.ca

Abstract. In this paper we study enumeration problems for polytopes arising from combinatorial optimization problems. While these polytopes turn out to be quickly intractable for enumeration algorithms designed for general polytopes, tailor-made algorithms using their rich combinatorial features can exhibit strong performances. The main engine of these combinatorial algorithms is the use of the large symmetry group of combinatorial polytopes. Specifically we consider a polytope with applications to the well-known max-cut and multicommodity flow problems: the metric polytope m_n on n nodes. We prove that for $n \geq 9$ the faces of codimension 3 of the metric polytope are partitioned into 15 orbits of its symmetry group. For $n \leq 8$, we describe additional upper layers of the face lattice of m_n. In particular, using the list of orbits of high dimensional faces of m_8, we prove that the description of m_8 given in [9] is complete with 1 550 825 000 vertices and that the Laurent-Poljak conjecture [14] holds for $n \leq 8$. Many vertices of m_9 are computed and additional results on the structure of the metric polytope are presented. Computational issues for the orbitwise face and vertex enumeration algorithms are also discussed.

1 Introduction

A full d-dimensional convex (bounded) polytope P can be defined either by the linear inequalities associated to the set $\mathcal{F}(P)$ of its facets or as the convex hull of its vertex set $\mathcal{V}(P)$. More generally, any proper face f of P can be defined either by the subset $\mathcal{F}(f)$ of facets containing f or as the convex hull of the vertices $\mathcal{V}(f)$ belonging to f. Given the facet set $\mathcal{F}(P)$, the vertex enumeration problem consists in enumerating all the vertices $\mathcal{V}(P)$ and the face enumeration problem consists in enumerating all the faces f of P in terms of facet sets $\mathcal{F}(f)$. These computationally difficult problems have been well studied; see [2, 3, 13] and references there. In this paper, we consider combinatorial polytope, i.e. polytopes arising from combinatorial optimization problems. These polytopes are often trivial for the very first cases and then the so-called combinatorial explosion occurs even for small instances. On one hand, combinatorial polytopes are quickly intractable for enumeration algorithm designed for solving general polytope, but on the other hand, tailor-made algorithms using their rich combinatorial features can exhibit strong performance. For example, large instances of the traveling salesman polytope, the linear ordering polytope, the cut polytope and the metric polytope were computed in [4, 9]. In this paper, pursuing the same approach, we propose an orbitwise face enumeration algorithm for combinatorial polytope. Focusing on the face lattice of the metric polytope m_n on n nodes, we compute the first instances for $n \leq 6$ and its upper layers for $n \leq 9$. These results allow us to prove that that the description of m_8 given in [9] is complete with 1 550 825 000 vertices and that the *dominating set* and *no cut-set* conjectures, see [9, 14], hold for m_8. A description of the faces of codimension 3 for any n is given as well as some preliminary results on the vertices of m_9.

2 Face Enumeration for Combinatorial Polytopes

2.1 Combinatorial polytopes

We present some polytopes associated to optimization problems arising from the complete directed graph D_n or the undirected graph K_n on n nodes: the traveling salesman polytope tsp_n which is the convex hull of all the incidence vectors of all Hamiltonian cycles of K_n, the linear ordering polytope lo_n which is the convex hull of the incidence vectors of all acyclic tournaments of D_n and the cut polytope c_n which is the convex hull of the incidence vectors of all the cuts of K_n. Another example is the metric polytope which can be defined as a relaxation of c_n by the triangular inequalities, see Section 3.1. One important feature of most combinatorial polytopes is their very large symmetry group. We recall that the symmetry group $Is(P)$ of a polytope P is the group of isometries preserving P. The isometries preserving ts_n are induced by the $n!$ permutations on $V_n = \{1, \ldots, n\}$, that is, $Is(tsp_n) \simeq Sym(n)$. We have $Is(m_n) = Is(c_n)$ for $n \geq 5$ and both are induced by permutations on V_n and additional isometries, see Section 3.2. For $n \geq 5$, we have $|Is(m_n)| = 2^{n-1}n!$, see [10]. As these symmetries preserve the adjacency relations and the linear independency, all faces are partitioned into orbits of faces equivalent under permutations and switchings. An orbitwise vertex enumeration algorithm was proposed in [9] and, in a similar vein, we propose an orbitwise face enumeration algorithm.

2.2 Orbitwise face enumeration algorithm

The input is a full d-dimensional polytope P defined by its (non-redundant) facet set $\mathcal{F}(P) = \{f_1^{d-1}, \ldots, f_m^{d-1}\}$. The symmetry group $Is(P)$ is assumed to be large. The main two subroutines are the computation of the canonical representative $\tilde{f}$ of the orbit O_f generated by a face f and the computation of the dimension $dim(f)$. The algorithm first computes the list $\mathcal{L}^{d-1} = \{\tilde{f}_1^{d-1}, \ldots, \tilde{f}_{I^{d-1}}^{d-1}\}$ of all the canonical representatives of the orbits of facets. Then the algorithm generates the set $L^{d-2} = \{\tilde{f}_s^{d-1} \cap f_r^{d-1} : s = 1, \ldots, I^{d-1}, r = 1, \ldots, m\}$. After computing the dimension of each subface $\tilde{f}_s^{d-1} \cap f_r^{d-1}$ and keeping only the $(d-2)$-faces, the algorithm reduces L^{d-2} to the list of canonical representatives of orbits of $(d-2)$-faces $\mathcal{L}^{d-2} = \{\tilde{f}_1^{d-2}, \ldots, \tilde{f}_{I^{d-2}}^{d-2}\}$. In general, after generating the list

$\mathcal{L}^{d-t+1}$, the algorithm generates the set L^{d-t} by intersecting each canonical representative $\tilde{f}_s^{d-t+1}$ with each facet F_r for $s = 1, \ldots, I^{d-t+1}$ and $r = 1, \ldots, m$ and then computes $\mathcal{L}^{d-t}$. The algorithm terminates after the list $\mathcal{L}^0$ of canonical representatives of the orbits of vertices is computed.

Orbitwise Face Enumeration Algorithm
begin
  for $t = 1, \ldots, d$ initialize $\mathcal{L}^{d-t} := \emptyset$; **endfor**;
  for each facet $f^{d-1} \in \{f_1^{d-1}, \ldots, f_m^{d-1}\}$
    compute the canonical representative $\tilde{f}^{d-1}$ of the orbit generated by f^{d-1};
    if $\tilde{f}^{d-1} \notin \mathcal{L}^{d-1}$ then $\mathcal{L}^{d-1} := \mathcal{L}^{d-1} \cup \{\tilde{f}^{d-1}\}$; **endif**;
  endfor;   /* $\mathcal{L}^{d-1}$: list of representatives of orbits of facets */
  output $\mathcal{L}^{d-1} = \{\tilde{f}_1^{d-1}, \ldots, \tilde{f}_{I^{d-1}}^{d-1}\}$;
  for $t = 2, \ldots, d$
    for each $(d-t+1)$-face $\tilde{f}^{d-t+1} \in \mathcal{L}^{d-t+1} = \{\tilde{f}_1^{d-t+1}, \ldots, \tilde{f}_{I^{d-t+1}}^{d-t+1}\}$
      for each facet $f^{d-1} \in \{f_1^{d-1}, \ldots, f_m^{d-1}\}$
        if $dim(\tilde{f}^{d-t+1} \cap f^{d-1}) = d - t$ then compute the canonical
        representative $\tilde{f}^{d-t}$ of the orbit generated by $\tilde{f}^{d-t+1} \cap f^{d-1}$;
          if $\tilde{f}^{d-t} \notin \mathcal{L}^{d-t}$ then $\mathcal{L}^{d-t} := \mathcal{L}^{d-t} \cup \{\tilde{f}^{d-t}\}$; **endif**;
        endif;
      endfor;
    endfor;   /* $\mathcal{L}^{d-t}$: list of representatives of orbits of $(d-t)$-faces */
    output $\mathcal{L}^{d-t} = \{\tilde{f}_1^{d-t}, \ldots, \tilde{f}_{I^{d-t}}^{d-t}\}$;
    $t := t + 1$;
  endfor;
end.

In order to compute the canonical representative $\tilde{f}^{d-t}$ of $\tilde{f}^{d-t+1} \cap f^{d-1}$ and its dimension $dim(\tilde{f}^{d-t+1} \cap f^{d-1})$, we have first to determine $\mathcal{F}(\tilde{f}^{d-t+1} \cap f^{d-1})$ the set of facets containing $\tilde{f}^{d-t+1} \cap f^{d-1}$. The determination of $\mathcal{F}(\tilde{f}^{d-t+1} \cap f^{d-1})$ amounts to a redundancy check for the remaining facets of $\mathcal{F}(P) \backslash \{\mathcal{F}(\tilde{f}^{d-t+1} \cap f^{d-1})\}$. This operation can be done using *ccclib* (*redcheck*), see [12], and is polynomially equivalent to linear programming; see [3]. The rank of $\mathcal{F}(\tilde{f}^{d-t+1} \cap f^{d-1})$ directly gives $dim(\tilde{f}^{d-t+1} \cap f^{d-1})$.

Remark 1.

1. With I^{d-t} the number of orbits of $(d-t)$-faces and m the number of facets, the dimension (resp. canonical representative) computation subroutine is called exactly (resp. at most) $m(1 + \sum_{t=1,\ldots,d-1} I^{d-t})$ times.
2. The output; that is, for $t = 1, \ldots, d$ the list $\mathcal{L}^{d-t}$ of canonical representatives $\tilde{f}_s^{d-t} : s = 1, \ldots, I^{d-t}$, is extremely compact. The full list of $(d-t)$-faces can be generated by the action of the symmetry group on each representative face $\tilde{f}_s^{d-t}$. With $|O_{\tilde{f}_s^{d-t}}|$ the size of the orbit generated by $\tilde{f}_s^{d-t}$, the total number of faces is $\sum_{t=1,\ldots,d} \sum_{s=1,\ldots,I^{d-t}} |O_{\tilde{f}_s^{d-t}}|$.

3. Additional combinatorial properties could allow to considerate only a fraction of the subfaces $\tilde{f}^{d-t+1} \cap f^{d-1}$. In that case, both subroutines are called for only a fraction of the numbers given in Item 1.

Item 1 of Remark 1 indicates that the algorithm runs smoothly as long as the number I^{d-t} of orbits of $(d-t)$-faces is relatively small. The number of $(d-t)$-faces usually grows extremely large with t getting close to $\lfloor \frac{d}{2} \rfloor$ and can be already very large for $\lfloor \frac{d}{4} \rfloor \leq t \leq \lfloor \frac{3d}{4} \rfloor$; that is: "Face lattices are very fat". Therefore the computation of the full face lattice of a polytope is generally extremely hard. Besides small dimensional polytopes and specific cases such as the d-cube, we can expect a similar pattern for the values of I^{d-t}; that is: "Orbitwise face lattices are also fat". On the other hand, one can expect the combinatorial explosion to occur at a deeper layer for the orbitwise face lattice than for the ordinary one. Actually, this algorithm is particularly suitable for the computation of the upper τ layers of the orbitwise face lattice for a small given $\tau \leq \lfloor \frac{d}{4} \rfloor$. In that case the algorithm stops when $\mathcal{L}^{d-\tau}$ is computed. The computation of the orbitwise upper face lattice can be efficiently combined with classical vertex enumeration. See Section 5.1 for an application to the complete description of the vertices of m_8.

3 Faces of the Metric Polytope

3.1 Cut and metric polytopes

The $\binom{n}{2}$-dimensional cut polytope c_n is usually introduced as the convex hull of the incidence vectors of all the cuts of K_n. More precisely, given a subset S of $V_n = \{1, \ldots, n\}$, the *cut* determined by S consists of the pairs (i, j) of elements of V_n such that exactly one of i, j is in S. By $\delta(S)$ we denote both the cut and its incidence vector in $\mathbb{R}^{\binom{n}{2}}$; that is, $\delta(S)_{ij} = 1$ if exactly one of i, j is in S and 0 otherwise for $1 \leq i < j \leq n$. By abuse of notation, we use the term cut for both the cut itself and its incidence vector, so $\delta(S)_{ij}$ are considered as coordinates of a point in $\mathbb{R}^{\binom{n}{2}}$. The cut polytope c_n is the convex hull of all 2^{n-1} cuts, and the *cut cone* C_n is the conic hull of all $2^{n-1} - 1$ nonzero cuts. The cut polytope and one of its relaxation - the metric polytope - can also be defined in terms of a finite metric space in the following way. For all 3-sets $\{i, j, k\} \subset \{1, \ldots, n\}$, we consider the following inequalities:

$$x_{ij} - x_{ik} - x_{jk} \leq 0, \tag{1}$$

$$x_{ij} + x_{ik} + x_{jk} \leq 2. \tag{2}$$

(1) induce the $3\binom{n}{3}$ facets which define the *metric cone* M_n. Then, bounding the latter by the $\binom{n}{3}$ facets induced by (2) we obtain the metric polytope m_n. The facets defined by (1) (resp. by (2)) can be seen as triangle (resp. perimeter) inequalities for distance x_{ij} on $\{1, \ldots, n\}$ and are denoted by $\Delta_{i,j,\bar{k}}$ (resp. by $\Delta_{i,j,k}$). While the cut cone is the conic hull of all, up to a constant multiple,

$\{0,1\}$-valued extreme rays of the metric cone, the cut polytope c_n is the convex hull of all $\{0,1\}$-valued vertices of the metric polytope.

One of the motivations for the study of these polyhedra comes from their applications in combinatorial optimization, the most important being the max-cut and multicommodity flow problems. For a detailed study of those polytopes and their applications in combinatorial optimization we refer to Deza and Laurent [11] and Poljak and Tuza [15].

3.2 Combinatorial and Geometric Properties

The polytope c_n is a $\binom{n}{2}$-dimensional $\{0,1\}$-polyhedron with 2^{n-1} vertices and m_n is a polytope of the same dimension with $4\binom{n}{3}$ facets inscribed in the cube $[0,1]^{\binom{n}{2}}$. We have $c_n \subseteq m_n$ with equality only for $n \leq 4$. Any facet of the metric polytope contains a face of the cut polytope and the vertices of the cut polytope are vertices of the metric polytope. In fact, the cuts are precisely the integral vertices of the metric polytope. The metric polytope m_n wraps the cut polytope c_n very tightly. Indeed, in addition to the vertices, all edges and 2-faces of c_n are also faces of m_n, for 3-faces it is false for $n \geq 4$, see [7]. Any two cuts are adjacent both on c_n and on m_n; in other words m_n is *quasi-integral*; that is, the skeleton of the convex hull of its integral vertices, i.e. the skeleton of c_n, is an induced subgraph of the skeleton of the metric polytope itself. We recall that the skeleton of a polytope is the graph formed by its vertices and edges. While the diameters of the cut polytope and the dual metric polytope satisfy $\delta(c_n) = 1$ and $\delta(m_n^*) = 2$, the diameters of their dual are conjectured to be $\delta(c_n^*) = 4$ and $\delta(m_n) = 3$, see [6, 14].

One important feature of the metric and cut polytopes is their very large symmetry group. More precisely, for $n \geq 5$, $Is(m_n) = Is(c_n)$ and both are induced by permutations on $V_n = \{1, \dots, n\}$ and *switching reflections by a cut* and, for $n \geq 5$, we have $|Is(m_n)| = 2^{n-1}n!$, see [10]. Given a cut $\delta(S)$, the switching reflection $r_{\delta(S)}$ is defined by $y = r_{\delta(S)}(x)$ where $y_{ij} = 1 - x_{ij}$ if $(i,j) \in \delta(S)$ and $y_{ij} = x_{ij}$ otherwise. As these symmetries preserve the adjacency relations and the linear independency, all faces of m_n are partitioned into orbits of faces equivalent under permutations and switchings.

3.3 Faces of the Metric Polytope

We recall some results and conjectures on the faces of the metric polytope. The cuts are the only integral vertices of m_n. All other vertices with are not fully fractional are so-called *trivial extensions* of a vertex of m_{n-1}. Consider the following two mappings:

$$
\begin{array}{lll}
\mathbb{R}^{\binom{n-1}{2}} \longrightarrow \mathbb{R}^{\binom{n}{2}} & \mathbb{R}^{\binom{n-1}{2}} \longrightarrow \mathbb{R}^{\binom{n}{2}} & \\
v \longrightarrow \phi_0(v) & v \longrightarrow \phi_1(v) & \\
 & & \\
\phi_0(v)_{ij} = v_{ij} & \phi_1(v)_{ij} = v_{ij} & \text{for } 1 \leq i < j \leq n-1 \\
\phi_0(v)_{i,n} = v_{1,i} & \phi_1(v)_{i,n} = 1 - v_{1,i} & \text{for } 2 \leq i \leq n-1 \\
\phi_0(v)_{1,n} = 0 & \phi_0(v)_{1,n} = 1. &
\end{array}
$$

The vertices $\phi_0(v)$ and $\phi_1(v)$ are called trivial extensions of v. Note that $\phi_1(v) = r_{\delta(\{n\})}(\phi_0(v))$. In other words, the *new vertices* of m_n - i.e. not trivial extensions of vertices of m_{n-1} - are the fully fractional ones. The $(\frac{1}{3}, \frac{2}{3})$-valued fully fractional vertices are well studied and include the anticut orbit formed by the 2^{n-1} *anticuts* $\bar{\delta}(S) = \frac{2}{3}(1, \ldots, 1)$-$\frac{1}{3}\delta(S)$. If $G = (V_n, E)$ is a connected graph, we denote by d_G its path metric, where $d_G(i, j)$ is the length of a shortest path from i to j in G for $i \neq j \in V_n$. Then $\tau(d_G) = max(d_G(i, j) + d_G(i, k) + d_G(j, k) : i, j, k \in G)$ is called the *triameter of* G and we set $x_G = \frac{2}{\tau(d_G)} d_G$. Any vertex of m_n of the form x_G for some graph is called a *graphic* vertex, see Fig. 1 for the graphs of 2 graphic $(\frac{1}{3}, \frac{2}{3})$-valued vertices of m_8. Note that for any connected graph $G = (V_n, E)$, we have $\tau(d_G) \leq 2(n - 1)$ and that any $(\frac{1}{3}, \frac{2}{3})$-valued vertex of m_n is (up to switching) graphic.

Since $m_3 = c_3$ and $m_4 = c_4$, the vertices of m_3 and m_4 are made of 4 and 8 cuts forming 1 orbit. The 32 vertices of m_5 are 16 cuts and 16 anticuts, i.e., form 2 orbits. The metric polytope m_6 has 544 vertices, see [14], partitioned into 3 orbits: cuts, anticuts and 1 orbit of trivial extensions; and m_7 has 275 840 vertices, see [8], partitioned into 13 orbits: cuts, anticuts, 3 orbits of trivial extensions, 3 $(\frac{1}{3}, \frac{2}{3})$-valued orbits and 5 other fully fractional orbits. For m_8, 1 550 825 600 vertices partitioned into 533 orbits (cuts, anticuts, 28 trivial extensions, 37 $(\frac{1}{3}, \frac{2}{3})$-valued and 466 other fully fractional) were found using an heuristic, see [9]. The description was conjectured to be complete.

Conjecture 1. [14] Any vertex of the metric polytope is adjacent to a cut.

Conjecture 2. [9] For $n \geq 6$, the restriction of the skeleton of the metric polytope m_n to the non-cut vertices is connected.

Conjecture 2 can be seen as complementary to the Conjecture 1 both graphically and computationally: For any pair of vertices, while Conjecture 1 implies that there is a path made of cuts joining them, Conjecture 2 means that there is a path made of non-cuts vertices joining them. In other words, the cut vertices would form a *dominating set* but not a *cut-set* in the skeleton of m_n. On the other hand, while Conjecture 1 means that the enumeration of the metric cone M_n is enough to obtain the metric polytope m_n; Conjecture 2 means that we can obtain m_n without enumerating M_n. Note that for arbitrary graphs these are clearly independent. Conjecture 1 underlines the extreme connectivity of the cuts. Recall that the cuts form a clique in both the cut and metric polytopes. Therefore, if Conjecture 1 holds, the cuts would be a dominant clique in the skeleton of m_n implying that its diameter would satisfy $\delta(m_n) \leq 3$.

The orbitwise description of the facets and ridges (faces of codimension 2) of m_n for any n was given in [6] as well as the face $\Delta_{1,2,3} \cap \Delta_{1,2,\bar{3}}$ of codimension $n - 1$ and the face $\Delta_{1,2,3} \cap \Delta_{1,\bar{3},4}$ of codimension 3 (this face is given in second position in Theorem 1). We have $\mathcal{L}^{d-1}(m_n) = \{\Delta_{1,2,3}\}$ and $\mathcal{L}^{d-2}(m_{n\geq 6}) = \{\Delta_{1,2,3} \cap \Delta_{1,2,4},\ \Delta_{1,2,3} \cap \Delta_{1,4,5},\ \Delta_{1,2,3} \cap \Delta_{4,5,6}\}$. The full orbitwise face lattices of m_4 and m_5 were given in [7]. In Section 4.1 we compute additional faces of small metric polytopes and in Section 4.2 we characterize $\mathcal{L}^{d-3}(m_n)$ for any n.

4 Generating Faces of the Metric Polytope

4.1 Faces of small metric polytopes

As stated earlier, generating the full face lattice is usually extremely hard. While the face lattice of m_7 (defined by 140 facets in dimension 21) is currently intractable, we computed the full orbitwise face lattice of m_6 (80 facets in dimension 15) and some upper layers of the lattice of m_7 and m_8. As mentioned earlier, I^{d-t} can be quite large for $t = \lfloor \frac{d}{4} \rfloor$ and therefore we set $t = 5$ (resp. 7) for the partial orbitwise enumeration of the m_7 (resp. m_8). Due to space limitation, we refer to [5] for a detailed presentation. The set $\mathcal{L}^{d-3}(m_n)$ is easy to check for reasonable values of n as $I^{d-3}(m_n) \leq 15$, see Theorem 1. As noticed in Item 3 of Remark 1, additional properties of m_n can be used to significantly increase the efficiency of the algorithm. In particular, the set L^{d-t} can be generated by considering for each s only the facets of m_n which are not equivalent under isometries preserving $\tilde{f}_s^{d-t+1}$. Two invariants of $O_{f_s^{d-t}}$ are the size $|\mathcal{F}(f_s^{d-t})|$ and the joint-support $\cup_{\Delta \in \mathcal{F}(f_s^{d-t})} \sigma(\Delta)$. The *support* of $\Delta_{i,j,k}$ (or $\Delta_{i,j,\bar{k}}$) is $\sigma(\Delta_{i,j,k}) = \sigma(\Delta_{i,j,\bar{k}}) = \{i, j, k\}$. These two invariants can be easily used to test non-membership to an orbit. Similarly, for small dimensional faces, the computation of $\mathcal{V}(f_s^{d-t})$ can be efficiently used for checking non-membership as the number of vertices, cuts, and anticuts of f_s^{d-t} are invariants of $O_{f_s^{d-t}}$. Let assume that, as in Section 5.2, we are interested only in the upper $n-1$ layers of the face lattice of m_n. In that case, when generating $\tilde{f}_s^{d-t+1} \cap \Delta_r$ with $t < n$, we can disregard Δ_r if $\sigma(\Delta_r) = \sigma(\Delta)$ for any $\Delta \in \mathcal{F}(\tilde{f}_s^{d-t+1})$ as for such Δ_r we have $codim(\tilde{f}_s^{d-t+1} \cap \Delta_r) \geq n-1$.

4.2 Faces of codimension 3 of the metric polytope

As recalled earlier the first 2 upper layers of m_n are known for any n. We have $I^{d-1}(m_n) = 1$, $I^{d-2}(m_{n\geq 6}) = 3$ and, by Theorem 1, we get $I^{d-3}(m_{n\geq 9}) = 15$.

Theorem 1. *For $n \geq 9$, the faces of codimension 3 of the metric polytope m_n are partitioned into 15 orbits equivalent under permutations and switchings. The 15 orbits can be represented by the following triplets of facets:* $\Delta_{1,2,3} \cap \Delta_{1,2,4} \cap \Delta_{1,3,4}$, $\Delta_{1,2,3} \cap \Delta_{1,2,4} \cap \Delta_{1,\bar{3},4}$, $\Delta_{1,2,3} \cap \Delta_{1,2,4} \cap \Delta_{1,2,5}$, $\Delta_{1,2,3} \cap \Delta_{1,2,4} \cap \Delta_{1,3,5}$, $\Delta_{1,2,3} \cap \Delta_{1,2,4} \cap \Delta_{3,4,5}$, $\Delta_{1,2,3} \cap \Delta_{1,2,4} \cap \Delta_{\bar{3},4,5}$, $\Delta_{1,2,3} \cap \Delta_{1,2,4} \cap \Delta_{1,5,6}$, $\Delta_{1,2,3} \cap \Delta_{1,2,4} \cap \Delta_{3,5,6}$, $\Delta_{1,2,3} \cap \Delta_{1,4,5} \cap \Delta_{2,4,6}$, $\Delta_{1,2,3} \cap \Delta_{1,4,5} \cap \Delta_{\bar{2},4,6}$, $\Delta_{1,2,3} \cap \Delta_{1,2,4} \cap \Delta_{5,6,7}$, $\Delta_{1,2,3} \cap \Delta_{1,4,5} \cap \Delta_{1,6,7}$, $\Delta_{1,2,3} \cap \Delta_{1,4,5} \cap \Delta_{2,6,7}$, $\Delta_{1,2,3} \cap \Delta_{1,4,5} \cap \Delta_{6,7,8}$ *and* $\Delta_{1,2,3} \cap \Delta_{4,5,6} \cap \Delta_{7,8,9}$. *The first 14 (resp. 13, 10 and 6) representatives generate the 14 (resp. 13, 10 and 6) orbits of faces of codimension 3 of m_8 (resp. m_7, m_6 and m_5). The first 2 representatives and $\Delta_{1,2,3} \cap \Delta_{1,2,\bar{3}}$ generate the 3 orbits of faces of codimension 3 of m_4.*

Proof. *For $n \leq 9$ Theorem 1 can be directly checked using the orbitwise face enumeration algorithm with $\tau = 3$; that is, the algorithm is set to compute only the upper 3 layers of the face lattice of m_n. Let assume $n \geq 9$, the faces of codimension 2 of m_n are partitioned into 3 orbits generated by* $\Delta_{1,2,3} \cap \Delta_{1,2,4}$,

$\Delta_{1,2,3} \cap \Delta_{1,4,5}$ *and* $\Delta_{1,2,3} \cap \Delta_{4,5,6}$. *Any faces of codimension* 3 *of* m_n *can therefore be written as the intersection of a facet* Δ *of* m_n *with one of these* 3 *faces* $\Delta' \cap \Delta"$ *of codimension* 2. *If the support* $\sigma(\Delta) \not\subset \{1,\ldots,9\}$, *by elementary permutations preserving* Δ' *and* $\Delta"$ *we can generate* $\tilde{\Delta} \in O_\Delta$ *with* $O_{\Delta' \cap \Delta" \cap \tilde{\Delta}} = O_{\Delta' \cap \Delta" \cap \Delta}$ *and* $\sigma(\tilde{\Delta}) \subset \{1,\ldots,9\}$. *In other words, to generate orbitwise all the subfaces of the canonical faces of codimension* 2 *it is enough to consider the case* $n = 9$. *This way one can easily obtain* 28 *faces* f_i *of codimension at least* 3. *Then, as for the orbitwise face enumeration algorithm, we have to compute for* $i = 1,\ldots,28$ *and for any* n *the dimension* $dim(f_i)$ *and - if* $codim(f_i) = 3$ *- to compute the canonical representative* $\tilde{f}_i$. *Therefore we have to first determine the set* $\mathcal{F}_n(f_i)$ *of facets of* m_n *containing* f_i. *Clearly, if an inequality* (i) *defining a facet of* m_n *is forced to be satisfied with equality by the inequalities defining* Δ', $\Delta"$ *and* $\tilde{\Delta}$ *being satisfied with equality, then the same inequality* (i) *- now seen as defining a facet of* m_{n+1} *- will also be forced to be satisfied with equality. In other words the set* $\mathcal{F}_n(f_i)$ *can only increase with* n *and* $dim(f_i)$ *can only decrease with* n. *Therefore, among the* 28 *faces* f_i, *only the* 15 *faces of codimension* 3 *for* m_9 *given in Theorem 1 are candidates for being faces of codimension* 3 *for* $m_{n\geq 9}$. *A case by case study of the* 15 *faces, gives* $\mathcal{F}_n(f_i)$ *and proves that indeed these* 15 *faces generate* 15 *orbits of faces of codimension* 3 *for* $n \geq 9$. *The idea is simply to notice that the pattern of* $\mathcal{F}_n(f_i)$ *is essentially given by the value of* $\mathcal{F}_{12}(f_i)$. *Since all the cases are similar, we only present the computation of* $\mathcal{F}_n(f_{15})$ *where* $f_{15} = \Delta_{1,2,3} \cap \Delta_{4,5,6} \cap \Delta_{7,8,9}$. *Using the orbitwise face enumeration algorithm with* $\tau = 3$, *one can easily check that* $\mathcal{F}_{12}(f_{15}) = \{\Delta_{1,2,3},\ \Delta_{4,5,6},\ \Delta_{7,8,9}\}$. *Let* $n \geq 12$ *and* Δ *be a facet of* m_n *with* $\sigma(\Delta) \not\subset \{1,\ldots,12\}$. *By elementary permutations preserving* $\mathcal{F}_{12}(f_{15})$ *we can generate* $\tilde{\Delta} \in O_\Delta$ *with* $\sigma(\tilde{\Delta}) \subset \{1,\ldots,12\}$. *Let now consider* $\tilde{\Delta}$ *as a facet of* m_{12}. *Since* $\tilde{\Delta} \notin \mathcal{F}_{12}(f_{15})$ *at least one vertex* v *of* m_{12} *satisfies* $v \in f_{15}$ *and* $v \notin \tilde{\Delta}$. *Then, the* $(n-12)$*-times* 0*-extension* v_{ext} *of* v *is a vertex of* m_n *satisfying* $v_{ext} \in f_{15}$ *but* $v_{ext} \notin \tilde{\Delta}$ *where* $\tilde{\Delta}$ *is now considered as a facet of* m_n. *Thus,* $\tilde{\Delta} \notin \mathcal{F}_n(f_{15})$ *and, by the same elementary permutations,* $\Delta \notin \mathcal{F}_n(f_{15})$*; that is,* $\mathcal{F}_n(f_{15}) = \{\Delta_{1,2,3},\ \Delta_{4,5,6},\ \Delta_{7,8,9}\}$ *and* $codim(f_{15}) = 3$ *for any* $n \geq 9$. *In the same way, for* $\mathcal{F}_n(f)$ *increasing with* n, *the pattern of* $\mathcal{F}_n(f)$ *is essentially given by small values of* n. *Consider for example* $\mathcal{F}_n(\Delta_{1,2,3} \cap \Delta_{1,2,\bar{3}})$*: We have* $\mathcal{F}_n(\Delta_{1,2,3} \cap \Delta_{1,2,\bar{3}}) = \{\Delta_{1,2,i},\ \Delta_{1,2,\bar{i}}\ :\ i = 3,\ldots,n\}$ *and therefore* $|\mathcal{F}_n(\Delta_{1,2,3} \cap \Delta_{1,2,\bar{3}})| = 2(n-2)$ *and* $codim(\Delta_{1,2,3} \cap \Delta_{1,2,\bar{3}}) = n-1$. *As for* $\mathcal{F}_n(f_{15})$, *one can compute* $\mathcal{F}_4(\Delta_{1,2,3} \cap \Delta_{1,2,\bar{3}})$ *and notice that* $\Delta \in \mathcal{F}_{n\geq 5}(\Delta_{1,2,3} \cap \Delta_{1,2,\bar{3}}) \iff \tilde{\Delta} \in \mathcal{F}_4(\Delta_{1,2,3} \cap \Delta_{1,2,\bar{3}})$*; that is,* $\Delta = \Delta_{1,2,i}$ *or* $\Delta_{1,2,\bar{i}} : i = 4,\ldots,n$. □

5 Generating Vertices of the Metric Polytope

5.1 Combining orbitwise face enumeration with classical vertex enumeration

As emphasized earlier, the face lattice is usually much larger that the number of vertices. Therefore, computing the full face lattice in order to obtain the vertices is extremely costly. On the other hand, the upper layers of the orbitwise face lattice might be relatively small. In that case the orbitwise face enumeration can be efficiently combined with a classical vertex enumeration methods in the following way. First, for an appropriate small $\tau \leq \lfloor \frac{d}{4} \rfloor$, compute the upper orbitwise face lattice till the list $\mathcal{L}^{d-\tau}$ of canonical $(d-\tau)$-faces is obtained. Then for $s = 1, \ldots, I^{d-\tau}$, compute by a classical vertex enumeration method the set $\mathcal{V}(\tilde{f}_s^{d-\tau})$ of vertices belonging to $\tilde{f}_s^{d-\tau}$. Finally, compute the canonical representative $\tilde{v}$ for each vertex $v \in \mathcal{V}(\tilde{f}_s^{d-\tau})$. The set of all such vertices $\tilde{v}$ is exactly $\mathcal{L}^0$ as each canonical vertex of $\mathcal{L}^0$ belongs, up to an isometry of $Is(P)$, to at least one of the $(d-\tau)$-faces $\tilde{f}_s^{d-\tau}$. Clearly, the choice of τ is critical. Typically, for the first values of t, by going down one layer from $\mathcal{L}^{d-t+1}$ to $\mathcal{L}^{d-t}$ the number of orbits increases ($I^{d-t+1} \leq I^{d-t}$) and the average sizes of faces decreases ($\mathcal{V}(\tilde{f}^{d-t+1}_{average}) \geq \mathcal{V}(\tilde{f}^{d-t}_{average})$). Therefore, a *good* τ should be such that $I^{d-\tau}$ and $\mathcal{V}(\tilde{f}^{d-\tau}_{average})$ are relatively small: In particular the largest $\tilde{f}_s^{d-\tau}$ should within problems currently solvable by vertex enumeration algorithms. In Section 5.2, assuming that the computation of m_{n-1} is just within current vertex enumeration abilities, we indicate that for m_n a good τ should satisfy $n-1 \leq \tau \leq \lfloor \frac{d}{4} \rfloor$ and that $\tau = 7$ is actually enough for the description of m_8. Note that $n-1 = 7 = \lfloor \frac{d}{4} \rfloor$ for m_8.

5.2 Vertices of the metric polytopes on 8 nodes

As mentioned earlier, the face $\tilde{f}_\mu^{d-n+1} = \Delta_{1,2,3} \cap \Delta_{1,2,\bar{3}}$ generates one orbit of faces of codimension $n-1$ of m_n which are combinatorially equivalent to m_{n-1}. In other words, the orbitwise face lattice of m_n contains a copy of m_{n-1} in $\mathcal{L}^{d-n+1}$. This implies that some canonical faces of $\mathcal{L}^{d-n+2}$ are quite larger than m_{n-1} and therefore intractable if we assume that m_{n-1} is just within current vertex enumeration methods abilities. For m_8, it means that we should compute at least $\mathcal{L}^{21}$ and it turns out to be enough as $\tilde{f}_\mu^{21}$ (which we do not need to enumerate since $\tilde{f}_\mu^{21} \simeq m_7$) and other elements of $\mathcal{L}^{21}$ are tractable. The whole computation is quite long as $\mathcal{L}^{21}$ is large as well as $\mathcal{V}(\tilde{f}^{21}_{average})$. For the same reasons, skipping $\tilde{f}_\mu^{21}$, the computation of the canonical vertices for each $\mathcal{V}(\tilde{f}_s^{21})$ is also long. Insertion algorithms usually handle high degeneracy better than pivoting algorithms, see [2] for a detailed presentation of the main vertex enumeration methods. The metric polytope m_n is quite degenerate as the cut incidence $Icd_{\delta(S)} = 3\binom{n}{3}$ is much larger than the dimension $d = \binom{n}{2}$. We recall that the incidence $Icd_v = |\mathcal{F}(v)|$. Thus we choose an insertion algorithm for the enumeration of each $\tilde{f}_s^{21}$: the *cddlib* implementation of the double description method [12]. The ordering of the facet is lexicographic with the rule $-1 \prec 1 \prec 0$. The result shows that $\mathcal{L}^0$ is made of the 533 canonical vertices found in [9]. Due to space limitation, we refer to [5] for a detailed presentation. The conjectured description of m_8 being complete, the following is straightforward.

Proposition 1.

1. *The metric polytope m_8 has exactly 1 550 825 600 vertices and its diameter is $\delta(m_8) = 3$. The metric cone M_8 has exactly 119 269 588 extreme rays.*

2. *The* LAURENT-POLJAK *dominating set Conjecture 1 and the no cut-set Conjecture 2 hold for m_8.*

While most of the vertices of m_8 are almost simple, the only simple vertices of m_8 belong to the orbits $O_{\tilde{v}_{532}}$ and $O_{\tilde{v}_{533}}$ of size $|O_{\tilde{v}_{532}}|$ =368 640 and $|O_{\tilde{v}_{533}}|$ =430 080; that is 0.05% of the total number of vertices of m_8. A vertex of a d-dimensional polytope is simple if $|\mathcal{F}(v)| = d$. Both canonical representative $\tilde{v}_{532}$ and $\tilde{v}_{533}$ are graphic $(\frac{1}{3}, \frac{2}{3})$-valued vertices, see Fig. 1. The largest denominator among vertices of m_8 is 15 and occurs for $v \in O_{\tilde{v}_{451}}$ with $|O_{\tilde{v}_{451}}|$ =2 580 480 and $\tilde{v}_{451} = \frac{1}{15}(2, 4, 4, 5, 5, 7, 8, 6, 6, 5, 5, 5, 10, 8, 5, 5, 5, 4, 9, 9, 3, 4, 10, 10, 5, 10, 5, 5)$.

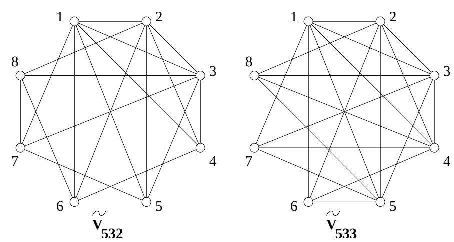

Fig. 1. Graphic canonical vertices of the only two orbits of simple vertices of m_8

5.3 Vertices of the metric polytopes on 9 nodes

The computation of the vertices of m_9 is most probably intractable as we expect this extremely degenerate 36-dimensional polytope to have around 10^{14} vertices partitioned into a couple of 100 000 orbits. Using an heuristic for the orbitwise vertex enumeration algorithm given in [9], we computed 180 021 orbits of m_n including 762 $(\frac{1}{3}, \frac{2}{3})$-valued orbits. The largest denominator found is 39 and most of the vertices are almost simple but the lowest incidence is 37, i.e. no simple vertex was found so far. By construction the restriction $m_{\simeq 9}$ of the skeleton of m_9 to the known 180 021 orbits of vertices satisfies Conjecture 2. All vertices of $m_{\simeq 9}$ are adjacent to at least 2 cuts; that is, satisfy Conjecture 1 and $\delta(m_{\simeq 9}) \leq 3$.

Remark 2. None of the currently known vertices of m_9 is simple. Since m_6 and m_7 have no simple vertex, the only known simple vertices of m_n for $n \geq 6$ belong to the orbits $O_{\tilde{v}_{532}}$ and $O_{\tilde{v}_{533}}$. Still, we believe that for n large enough m_n has many simple vertices, see [8], even if so far we have not yet found them.

6 Conclusions

We presented an orbitwise face enumeration algorithm for combinatorial polytopes with large symmetry group. In particular, considering the metric polytope on n nodes, we computed the full orbitwise face lattice for m_6 and its upper layers for m_7 and m_8. A description of the faces of codimension 3 of m_n for any n was also given. Using the list of the faces of codimension 7 of m_8, we proved that the conjectured description of the vertices of m_8 is complete and therefore that the Laurent-Poljak dominating set and the no cut-set conjectures hold for $n \leq 8$. Even if getting a certificate that the description m_9 is complete seams intractable, we could generate many of its vertices. Remark 2 indicates that either m_n has not as many simple vertices as believed or that m_8 is not large enough to reveal the general features of the metric polytope on n nodes.

References

1. Avis D.: lrs homepage, School of Computer Science, McGill University, Canada (2001) `http://www.cgm.cs.mcgill.ca/~ avis/C/lrs.html`
2. Avis D., Bremner D., Seidel R.: How good are convex hull algorithms? Computational Geometry: Theory and Applications 7 (1997) 265–301
3. Avis D., Fukuda K., Picozzi S.: On canonical representations of convex polyhedra. In: Cohen A. M., Gao X. S., Tokuyama N. (eds.): Mathematical Software, World Scientific (2002) 351–360
4. Christof T., Reinelt G.: Decomposition and parallelization techniques for enumerating the facets of 0/1-polytopes. Preprint, Heidelberg University (1998)
5. Deza A.: Metric polytope homepage, Math. and Comput. Sci. Dept., Tokyo Institute of Technology, Japan (2002) `http://www.is.titech.ac.jp/~ deza/metric.html`
6. Deza A., Deza M.: The ridge graph of the metric polytope and some relatives. In: Bisztriczky T., McMullen P., Schneider R., Weiss A. I. (eds.): Polytopes: Abstract, Convex and Computational (1994) 359–372
7. Deza A., Deza M.: The combinatorial structure of small cut and metric polytopes. In: Ku T. H. (eds.): Combinatorics and Graph Theory, World Scientific (1995) 70–88
8. Deza A, Deza M., Fukuda F.: On skeletons, diameters and volumes of metric polyhedra. In: Deza M., Euler R., Manoussakis Y. (eds.): Lecture Notes in Computer Science 1120 Springer-Verlag (1996) 112–128
9. Deza A., Fukuda K., Pasechnik D., Sato M.: On the skeleton of the metric polytope. In: Akiyama J., Kano M., Urabe M. (eds.): Lecture Notes in Computer Science 2098 Springer-Verlag (2001) 125–136
10. Deza M., Grishukhin V., Laurent M.: The symmetries of the cut polytope and of some relatives. In: Gritzmann P., Sturmfels B. (eds.): Applied Geometry and Discrete Mathematics, the "Victor Klee Festschrift" DIMACS Series in Discrete Mathematics and Theoretical Computer Science 4 (1991) 205–220
11. Deza M., Laurent M.: Geometry of cuts and metrics. Algorithms and Combinatorics 15 Springer-Verlag (1997)
12. Fukuda K.: cddlib reference manual, cddlib Version 0.92, ETHZ, Zürich, Switzerland (2001) `http://www.ifor.math.ethz.ch/~ fukuda/cdd_home/cdd.html`
13. Fukuda K., Rosta V.: Combinatorial face enumeration in convex polytopes. Computational Geometry: Theory and Applications 4 (1994) 191–198
14. Laurent M, Poljak S.: The metric polytope. In: Balas E., Cornuejils G., Kannan R. (eds.): Integer Programming and Combinatorial Optimization (1992) 247–286
15. Poljak S., Tuza Z.: Maximum cuts and large bipartite subgraphs. In: Cook W., Lovasz L., Seymour P. D. (eds.): DIMACS Series 20 (1995) 181–244