On the Cocircuit-Graph of an Oriented Matroid

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Abstract

In this paper we consider the cocircuit graph $G_{\mathcal{M}}$ of an oriented matroid $\mathcal{M}$, the 1-skeleton of the cell complex $\mathcal{W}$ formed by the span of the cocircuits of $\mathcal{M}$.

In general, $\mathcal{W}$ is not determined by $G_{\mathcal{M}}$. However, we show that if the vertex set [resp. edge set] of $G_{\mathcal{M}}$ is properly labeled by the hyperplanes [resp. colines] of $\mathcal{M}$, $G_{\mathcal{M}}$ determines $\mathcal{W}$. Also we prove that, when $\mathcal{M}$ is uniform, the cocicuit graph together with all antipodal pairs of vertices being marked determines $\mathcal{W}$. These results can be considered as variations of Blind-Mani's theorem that says the one skeleton of a simple convex polytope determines its face lattice.

1 Introduction and notation

Let E be a finite set of vectors of $\mathbb{R}^{d+1}$ and, for each vector $e \in E$, consider the intersection (a $(d-1)$-dimensional sphere) of the hyperplane perpendicular to e through the origin with the unit sphere S^d. The cell decomposition of the sphere thus obtained does not change when vectors are replaced by non-zero scalar multiples. Note that in this cell complex the normal vectors of the hyperplanes passing through a 0-cell span a linear hyperplane in $\mathbb{R}^{d+1}$, and those associated with a 1-cell span a coline (i.e. $(d-1)$-space).

More generally, given the Folkman-Lawrence representation of an oriented matroid $\mathcal{M}$ on a finite set E as *an arrangement of pseudospheres indexed by E in S^d*, the cell complex $\mathcal{W}$ is combinatorially represented by the span of the cocircuits of $\mathcal{M}$. In general (see Example 2.1), the 1-skeleton of $\mathcal{W}$ (the cocircuit graph $G_{\mathcal{M}}$ of $\mathcal{M}$) does not determine $\mathcal{W}$. However, we prove that it does determine $\mathcal{W}$ if the vertices [resp. edges] of the graph are labeled by the corresponding subset of E.

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This result can be improved in the rank 3 and in the uniform case under the condition that the opposite pairs of vertices are marked. Then, as an application of the previous statement, Theorem 2.3, we prove that the graph is by itself sufficient to determine the complex.

The results presented in here are also closely related to the work of Roudneff in [13]. In particular Theorem 2.3 is a generalization by duality of Theorem 3.1 in [13], and the two theorems coincide in the uniform case. Applied to the rank 3 case, our main result implies as well a theorem of Goodman and Pollack, namely [8, Theorem 2.9]. It should be mentioned that our results can be considered as variations of Blind-Mani's theorem [4] that says the one skeleton of a simple convex polytope determines its face lattice (see also [11]). When an oriented matroid is representable, the complex $\mathcal{W}$ coincides with the face lattice of the dual of a $(d+1)$-zonotope. Thus our results show how theorems of Blind-Mani's type can be phrased for a certain class of non-simple polytopes.

For a long time our results remain unpublished, although they have been used more than once, mainly for algorithmic purposes. Recently, they were also the base for the method of proof of the main result of [5].

We suppose the reader is familiar with the basic terminology most commonly used in matroid theory and in oriented matroid theory, like those presented [2, 14, 15].

Let us fix some notations we will use. We use E to denote the set $\{1, 2, \ldots, n\}$ of n elements, which is also denoted by $[n]$. For an element C of $\{-, 0, +\}^E$, we consider as usual the *positive support* $C^+ := \{e \in E : C_e = +\} \subset E$, and similarly the *zero support* C^0 and the *negative support* C^-. The *support* of C is $\underline{C} := C^+ \cup C^-$. For any subset A of E, by reversing the signs of all the coordinates of C indexed by A, we obtain ${}_{\overline{A}}C$. Given an oriented matroid $\mathcal{M}$ (over E), with set of (signed) cocircuits $\mathcal{C}$ and a subset A of E, the set of form ${}_{\overline{A}}\mathcal{C} := \{{}_{\overline{A}}C \mid C \in \mathcal{C}\}$ is the set of cocircuits of a *reorientation* of $\mathcal{M}$. We represent the underlying non-oriented matroid by $\underline{\mathcal{M}}$, whose (non-oriented) cocircuits are merely the supports of cocircuits of $\mathcal{M}$. We denote the collection of all possible reorientations of $\mathcal{M}$ by $\mathrm{OC}(\mathcal{M})$, called the *reorientation class* of $\mathcal{M}$. Two oriented matroids $\mathcal{M}$ on E and $\mathcal{M}'$ on E' are said to be *isomorphic* if there exists a bijection f from E to E' such that $OC(f(\mathcal{M})) = OC(\mathcal{M}')$, where $f(\mathcal{M})$ denotes the oriented matroid on E' obtained from $\mathcal{M}$ by the map f. The *isomorphism class* of all oriented matroids isomorphic to an oriented matroid $\mathcal{M}$ is denoted by $\mathrm{IC}(\mathcal{M})$.

As usual, we consider the set $\mathcal{W}$ of *covectors* or the *span of cocircuits* of $\mathcal{M}$, as the *compositions* of the signed cocircuits (cf. [2], p. 141), defined by $(X \circ Y)^{\pm} = X^{\pm} \cup (Y^{\pm} \setminus X^{\mp})$. The partial order defined by the relation

$$X \leq Y \iff \text{for every } e \in E,\ Y_e = X_e \text{ if } X_e \neq 0$$

captures the natural facial incidence relation among cells of the associated arrangement of pseudo-spheres.

By Folkman-Lawrence representation theorem [2], we may think of $\mathcal{W}$ as a cell complex of cells of form

$$[X] = \{Y \in \mathcal{W} \mid Y \leq X\}.$$

The topological dimension of $[X]$ is one less than the corank of X^0, a flat of $\underline{\mathcal{M}}$. $\mathcal{W}$ is a piecewise linear sphere of dimension rank $(\mathcal{M}) - 1$ and the *topes*, i.e. the cells of maximal dimension, are in one-to-one correspondence with the acyclic reorientations of $\mathcal{M}$.

The abstract cell complex $\mathcal{W}$ determines the isomorphism class of $\mathcal{M}$. The span $\mathcal{W}'$ of cocircuits of $\mathcal{M}'$ is isomorphic to $\mathcal{W}$ if and only if $\mathcal{M}'$ is isomorphic to $\mathcal{M}$. Indeed, it is possible to reconstruct the isomorphism class of $\mathcal{M}$ from the top two levels of the complex, the *tope graph*, whose vertices are the topes and whose edges are the cells of dimension one less (cf. [1, 6]). For a polynomial algorithm for the reconstruction, see [7]. The present paper studies how much information the bottom two levels of the complex contain.

Definition 1.1. We call the 1-skeleton of the cell complex $\mathcal{W}$ associated with $\mathcal{M}$ the *cocircuit graph* $\mathrm{G}_{\mathcal{M}}$ of $\mathcal{M}$.

Hence, $\mathrm{G}_{\mathcal{M}}$ consists of:

(i) the set of vertices $\mathrm{VG}_{\mathcal{M}}$ formed by the signed cocircuits of $\mathcal{M}$;

(ii) the set of edges $\mathrm{EG}_{\mathcal{M}}$ of pairs of form $\{V, W\} \subset \mathrm{VG}_{\mathcal{M}}$ for which there exists $X \in \mathcal{W}$, $X \geq V, W$, such that X covers V (and hence W, by the *diamond property* [3]): $X = V \circ W = W \circ V$; the set complement of its support, X^0, is a coline of $\underline{\mathcal{M}}$.

Let $\mathcal{M}$ be an oriented matroid of rank 2. It is easy to see that its cocircuit graph $\mathrm{G}_{\mathcal{M}}$ can be seen as an even-length cycle $(C_1, C_2, \ldots, C_s, -C_1, -C_2, \ldots, -C_s)$ of cocircuits such that any two consecutive ones are adjacent. Its reorientation class $\mathrm{OC}(\mathcal{M})$ is thus representable as the cyclic reordering of the sequence $(H_1, H_2, \ldots, H_s)$ where $H_j := C_j^0$, the hyperplane associated with C_j.

It should be noted that a cocircuit graph is merely a graph, represented as abstract sets of vertices and edges which do not carry themselves any explicit information about the original sign vectors.

2 Main theorem

We note first that the isomorphism class of an oriented matroid $\mathcal{M}$ is not always determined by the cocircuit graph $\mathrm{G}_{\mathcal{M}}$:

Example 2.1. *Let $\mathcal{M}$ and $\mathcal{M}'$ be the affine oriented matroids of the point configurations in Figure 2.1, where we have the vertices of a tetrahedron together with three extra points interior to three different edges, forming a triangle parallel to a base, in the first case, and the same without the vertex opposite to the base. It is clear that two configurations generate the same set of linear hyperplanes (i.e. planes). There are exactly two cocircuits associated with each hyperplane, representing the partition of the points not on the hyperplane. One can easily verify that the extra point 7 of the left has no effect on the adjacency of cocicuits, and thus $\mathrm{G}_{\mathcal{M}}$ and $\mathrm{G}_{\mathcal{M}'}$ are isomorphic, although clearly $\mathcal{M}'$ is not isomorphic to $\mathcal{M}$.*

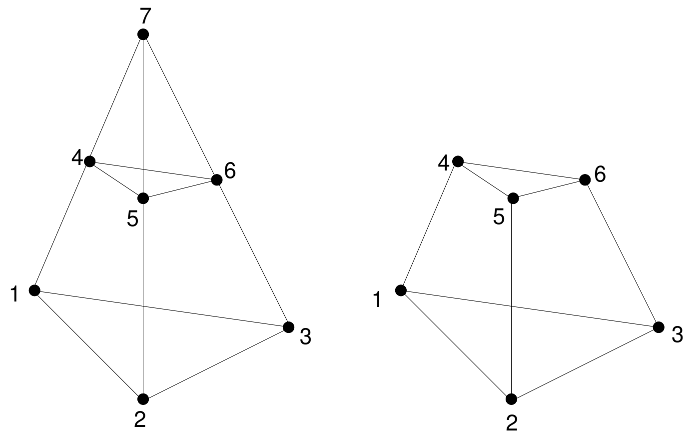

Figure 2.1: Two configurations whose cocircuit graphs are identical

We will show that this is not the case when the graph $\mathrm{G}_{\mathcal{M}}$ is properly labeled. More precisely, we consider:

Definition 2.2. We say the cocircuit graph $\mathrm{G}_{\mathcal{M}}$ is *vertex-labeled* if every vertex $C \in \mathrm{VG}_{\mathcal{M}}$ is labeled by its zero support, i.e. the hyperplane $C^0 = E \setminus \underline{C}$. $\mathrm{G}_{\mathcal{M}}$ is *edge-labeled* if every edge $\{C, D\} \in \mathrm{EG}_{\mathcal{M}}$ is labeled by its zero support, i.e. the coline $W^0 = E \setminus \underline{W}$, for $W = C \circ D$.

Then, we can state the first main theorem:

Theorem 2.3. *Let $\mathcal{M}$ be a simple oriented matroid, of rank greater or equal to 2. Then, any one of the following structures determines all of them:*

(a) $\mathrm{OC}(\mathcal{M})$*;*

(b) *the vertex-labeled graph* $\mathrm{G}_{\mathcal{M}}$*;*

(c) *the edge-labeled graph* $\mathrm{G}_{\mathcal{M}}$*;*

(d) *the (non-oriented) matroid $\underline{\mathcal{M}}$, together with the classes* $\mathrm{OC}(\mathcal{M}/L)$ *for all colines L of $\underline{\mathcal{M}}$.*

Proof. Note that when rank $(\mathcal{M}) = 2$, clearly the structures in (a), (b) and (d) provide the information for each other. Furthermore, the structures in (b) and (c) are equivalent, in that vertex- and edge-labeling clearly determine each other. On the other hand, the knowledge of (a) implies the knowledge of (b), according to the definitions. Hence, we start by proving that the knowledge of (b)($\Longleftrightarrow$ (c)) implies that of (d):

Suppose that we know the vertex-labeled graph $\mathrm{G}_{\mathcal{M}}$. Then $\underline{\mathcal{M}}$ is determined by the labels. For a coline L of this matroid, consider the subgraph of $\mathrm{G}_{\mathcal{M}}$ of edges labeled by L. This new graph is, by definition, the cocircuit graph $\mathrm{G}_{\mathcal{M}/L}$ of $\mathcal{M}/L$. Since this new oriented matroid has rank 2, its graph is a simple cycle, and thus the reorientation class is represented by the subgraph of the graph $\mathrm{G}_{\mathcal{M}}$ whose edge labels contain $H \setminus L$.

Now, we prove that, given, as in (d), the underlying matroid $\underline{\mathcal{M}}$ of $\mathcal{M}$ and the reorientation classes $\mathrm{OC}(\mathcal{M}/L)$ for all colines L of $\underline{\mathcal{M}}$, the reorientation class of $\mathcal{M}$ is uniquely determined. More precisely, we prove by induction on n the statement

(*) if an oriented matroid $\mathcal{M}'$ satisfies $\underline{\mathcal{M}'} = \underline{\mathcal{M}}$ and $\mathcal{M}'/L \in \mathrm{OC}(\mathcal{M}/L)$ for every coline L of $\underline{\mathcal{M}}$, then $\mathrm{OC}(\mathcal{M}') = \mathrm{OC}(\mathcal{M})$.

Clearly (*) is true if rank $(\mathcal{M}) = 2$. In addition, (*) is true if rank $(\mathcal{M}) = n$, because then every element $e \in E$ is a coloop and clearly $\underline{\mathcal{M}}$ has just one orientation.

Let $\mathcal{M}$ be an oriented matroid on $E = [n]$ with rank $r > 2$ and $n > r$. By induction hypothesis, we assume that the statement (*) is true for any oriented matroid of the same rank and a smaller number of elements. Now suppose that an oriented matroid $\mathcal{M}'$ satisfies $\underline{\mathcal{M}'} = \underline{\mathcal{M}}$ and $\mathcal{M}'/L \in \mathrm{OC}(\mathcal{M}/L)$ for every coline L of $\underline{\mathcal{M}}$. We will show $\mathcal{M}' \in \mathrm{OC}(\mathcal{M})$. For this, we may reorient $\mathcal{M}'$ as we wish for the proof below.

Since $n > r$, $\underline{\mathcal{M}}$ contains an element which is not a coloop. By renumbering if necessary, we may suppose n is not a coloop. Thus $\mathcal{M}'([n-1])$ has rank r as well. Since the knowledge of (d) for $\mathcal{M}$ clearly induces the knowledge of (d) for $\mathcal{M} \setminus n$, and by the induction hypothesis, $\mathrm{OC}(\mathcal{M}'([n-1])) = \mathrm{OC}(\mathcal{M}([n-1]))$. By reorientation of $\mathcal{M}'$ if necessary, we may assume $\mathcal{M}'([n-1]) = \mathcal{M}([n-1])$. Then the following statement is valid:

(d') Given a coline L of $\underline{\mathcal{M}}$ such that $n \notin L$, either $\mathcal{M}/L = \mathcal{M}'/L$ or $\mathcal{M}/L = {}_{\overline{\{n\}}}\mathcal{M}'/L$.

We need the following property to complete the proof:

(**) Let M be a matroid on $[n]$ of rank ≥ 2. For any two hyperplanes H_0, H with $n \notin H_0 \cup H$, there is a sequence of hyperplanes $H_0, H_1, \ldots, H_k = H$ such that $H_i \cap H_{i+1}$ is a coline, and $n \notin H_i$, $i = 0, \ldots, k$.

The statement (**) is clear if rank of M is equal to 2. Suppose rank $(M) > 2$. We use induction on the rank. If rank $(H_0 \cap H) \geq 1$, there exists a non-loop element $e \in H_0 \cap H$ and the result is true in the contraction M/e by the induction hypothesis and then in M. Suppose that rank $(H_0 \cap H) = 0$, and consider a coline $L \subset H_0$. There is at least an element $a \in H \setminus L$ such that n is not in the hyperplane $H' := \overline{L \cup \{a\}}$. Indeed, otherwise, $\bigcap_{a \in H \setminus L} \overline{L \cup \{a\}}$ must contain the hyperplane $\overline{L \cup \{n\}}$ and this is impossible because the former set is the intersection of at least two hyperplanes by the assumptions rank $(M) > 2$ and rank $(H_0 \cap H) = 0$. Thus the property (**) follows from the first case for the pair H' and H.

Let us consider now a hyperplane H_0 of $\underline{\mathcal{M}}$ such that $n \notin H_0$. Now, consider another hyperplane $H, n \notin H$. From (**) we know that there is a sequence of hyperplanes $H_0, H_1, \ldots, H_m = H$ such that $H_i \cap H_{i+1}$ is a coline, and $n \notin H_i$, $i = 0, \ldots, m$. Hence, by the property (d'), all the cocircuits with n in $\mathcal{M}$ and $\mathcal{M}'$, are either equal, or all the cocircuits differ only in the sign of n. By Las Vergnas' single element extension Theorem [12] (see also [2, Proposition 7.1.4]), $\mathcal{M} = \mathcal{M}'$ or $\mathcal{M} = {}_{\overline{\{n\}}}\mathcal{M}'$, and clearly $\mathcal{M}' \in \mathrm{OC}(\mathcal{M})$. □

The equivalence (a) $\iff$ (d) of Theorem 2.3 is a generalization of Theorem 3.1 in [13] by duality. Also, in the rank 3 case, by identifying opposite points in the Folkman-Lawrence representation in the sphere S^2, we obtain an *arrangement of pseudolines* in the real projective plane (see [9] for definitions); then the equivalence (a) $\iff$ (b) reads as a theorem of J.E. Goodman and R. Pollack ([8, Theorem 2.9]).

An immediate consequence of the equivalence (a) $\iff$ (d) of Theorem 2.3 is also:

Corollary 2.4 ([5], Theorem 3.1). *Given an oriented matroid of rank r, for each s, $2 \leq s < r$, the labeled rank s contractions determine the reorientation class of the oriented matroid.*

We remark that Theorem 2.3 can be used to represent the reorientation class $\mathcal{M}$ in a simple graphic way: Indeed, the vertex-labeled graph $\mathrm{G}_{\mathcal{M}}$ can be recovered from its decomposition in the circuits corresponding to the same labels of edges. Thus, in a way, this representation "generalizes" the notion of the *Gale diagram* [10, 16].

As a corollary, in the rank 3 case $\mathrm{G}_{\mathcal{M}}$ determines by itself the isomorphism class of $\mathcal{M}$. More precisely, we have:

Corollary 2.5. *Let $\mathcal{M}$ be a simple rank 3 oriented matroid. Then any of the following structures determine all the remaining:*

(a) $\mathrm{IC}(\mathcal{M})$;

(b) *the* (unlabeled) *cocircuit graph* $\mathrm{G}_{\mathcal{M}}$ *of* $\mathcal{M}$;

(c) *the (non-oriented) matroid* $\underline{\mathcal{M}'}$ *for some* $\mathcal{M}' \in \mathrm{IC}(\mathcal{M})$, *together with the class* $\mathrm{OC}(\mathcal{M}'/e)$ *for every element* $e \in E$ *that is not a loop.*

Proof. We prove that we may recover the structure in (c) of Theorem 2.3 from the one in (b) of the corollary, i.e. that we may read from $\mathrm{G}_{\mathcal{M}}$ a proper edge-labeling, which is uniquely determined up to isomorphism.

For this purpose, consider $\mathcal{W}$, the span of cocircuits of $\mathcal{M}'$. By the Folkman-Lawrence representation theorem, there is an embedding of $\mathcal{W}$ into the sphere S^2. Since the cocircuit graph is 3-connected (in fact, it is 4-connected by Cordovil-Fukuda's theorem [6, Theorem 1.6]), the embedding is unique. All vertices of $\mathrm{G}_{\mathcal{M}'}$ are of even degree, and the edges incident with the same vertex are circularly ordered in S^2. Since two such edges correspond to the same point of $\underline{\mathcal{M}'}$ if and only if they are opposite in this ordering, the result follows now from Theorem 2.3. □

3 Uniform Case

In this section, we consider the main theorem in the previous section, Theorem 2.3, in the special case when an oriented matroid is uniform. It turns out that only a little extra information besides the cocircuit graph determines the isomorphism class of a uniform oriented matroid.

Theorem 3.1. *Let $\mathcal{M}$ be a uniform oriented matroid. Then the isomorphism class* $\mathrm{IC}(\mathcal{M})$ *of $\mathcal{M}$ is uniquely determined by the (unlabeled) cocircuit graph* $\mathrm{G}_{\mathcal{M}}$ *and the knowledge of the antipodal pairs of vertices in* $\mathrm{G}_{\mathcal{M}}$ *corresponding to the opposite pairs of cocircuits.*

To prove this theorem, it will be convenient to introduce a few definitions and a lemma. Let M be a matroid of rank r. We denote by $\mathcal{L} = \mathcal{L}(M)$ the set of all colines. Define the *distance* $\mathrm{dist}(H, L)$ in M from a hyperplane H to a coline L as the smallest number s such that there exists a sequence of colines: $L_0 \subseteq H, L_1, \ldots, L_s = L$ with rank $(L_{i-1} \cup L_i) = r - 1$ for $i = 1, \ldots, s$. We denote by $\mathcal{L}_k(H)$ the set of all colines whose distance from a hyperplane H is k, for $k = 0, 1, \ldots$.

Lemma 3.2. *Let M be a uniform matroid and let H be a hyperplane. Then*

$$\mathcal{L}_k(H) = \{L \subseteq E : |L| = r - 2 \textit{ and } |L \setminus H| = k\}.$$

Proof. This follows immediately from the uniformity. □

Proof. (of Theorem 3.1.) Again we prove that, besides permutation of the elements of E, we can read from $\mathrm{G}_{\mathcal{M}}$ its edge-labeling.

First, we prove that the edges of $\mathrm{G}_{\mathcal{M}}$ that correspond to the same coline of $\underline{\mathcal{M}}$ can be determined. By the uniformity, the rank r is easily detected, i.e. $\delta = 2(r-1)$ where δ is the degree of each vertex. The size n of the ground set is still unknown but we shall explain below how one can recover it.

Let $P = [V = V_0, a_0, \ldots, a_m, V_{m+1} = -V]$ be a path in $\mathrm{G}_{\mathcal{M}}$ connecting the opposite vertices V and $-V$. We assume for the moment that each vertex V_j and each edge a_j are both represented as sign vectors in the original matroid. It is easy to see that $m \geq n - (r - 1)$, since V and $-V$ have exactly $n - (r - 1)$ non-zero components, and the number of (non-zero) components which are opposite in V_{i-1} and V_{i+1} is at most one for any $i = 1, \ldots, m$. Hence the number of (non-zero) components which are opposite in V_0 and V_k is at most $k - 1$ for any $k = 2, \ldots, m + 1$. Furthermore, the path P is a part of the graph $\mathrm{G}_{\mathcal{M}/L}$ for some coline L if and only if $m = n - r + 1$. The "only if" part is obvious. To show the "if" part, assume that the path is not a part of the graph $\mathrm{G}_{\mathcal{M}/L}$ for any coline L. This means that there are two distinct elements $e, f \in V^0$ and two vertices, say V_i and V_j, such that $e \notin V_i^0$ and $f \notin V_j^0$. Since each of the e and f components of vertices must become nonzero and then zero independently in the path P, a simple count shows $m > n - r + 1$.

Therefore, a path of minimum length between a pair of opposite vertices always corresponds to a coline, and the number n can be determined from the length and r. In addition, two such paths correspond to the same coline if and only if they cross opposite vertices (in opposite ordering). Consequently we can partition the graph $\mathrm{G}_{\mathcal{M}}$ into an edge-disjoint union of cycles of length $2(n-r+2)$, each of which is the graph of a rank 2 contraction. We simply use "cycle" to mean these special cycles for the rest of the proof.

The only thing left to be shown is a construction of a proper label of each edge (or equivalently each cycle) by some $(r - 2)$ subset of $[n]$, since Theorem 2.3 takes care of the rest.

Below we shall explain how each edge and each vertex of $\mathrm{G}_{\mathcal{M}}$ can be labeled properly so that the resulting label is the labeling by an oriented matroid isomorphic to the original one.

First we select any vertex V_0 of $\mathrm{G}_{\mathcal{M}}$. By the freedom of isomorphism, one can label the vertex freely, and so we label it by $V_0 = [r - 1]$. Also there are exactly $(r - 1)$ cycles containing V_0. We can label these cycles by the $(r-1)$ subsets of cardinal $(r-2)$ of V_0. Now we must use the remaining elements from $[n]$, i.e. $r, r+1, \ldots, n$. For this, we select any labeled cycle, say the one labeled by $[r-2]$. A half of this cycle is of form $[V = V_0, a_0, \ldots, a_m, V_{m+1} = -V]$, where $m = n - r + 1$. We can then label the intermediate vertices and their antipodes by $V_i = [r-2] \cup \{r+i-1\}$ for $i = 1, \ldots, m$, since the remaining labels were never used before and must be used exactly once in this path.

Now the important claim is that all unlabeled vertices (and thus unlabelled edges) will get a unique label. This implies that the graph can be labelled without any ambiguity and thus properly. To prove this claim, we show how the labels can be assigned inductively. Let $H_0 = V_0^0$ and

$$\mathcal{L}_k(H_0) = \{L \subseteq [n] : |L| = r-2 \text{ and } |L \setminus H_0| = k\}.$$

By Lemma 3.2, $\mathcal{L}_k(H_0)$ is the set of labels for the cycles of distance k from V_0. The distance here is well-defined since we have $\mathrm{G}_{\mathcal{M}}$ and the adjacency of two cycles means they share a vertex. We denote by $\mathcal{C}_k$ the set of cycles in $\mathrm{G}_{\mathcal{M}}$ whose distance from V_0 is k.

All labels in $\mathcal{L}_0$ are already assigned to the cycles in $\mathcal{C}_0$, and the vertices of one cycle in $\mathcal{C}_0$ are assigned labels. The proof will be completed by showing the two statements:

(a) The labels of vertices (of cycles) in $\mathcal{C}_0$ are uniquely determined by the labels assigned already.

(b) The labels of vertices and cycles in $\mathcal{C}_k$ are uniquely determined by the labels of vertices and cycles in $\mathcal{C}_{k-1}$, for any $k \geq 1$.

To prove (a), consider the vertex V_i whose label is $[r-2] \cup \{r+i-1\}$ $(i = 1, \ldots, m)$. There are exactly $(r-2)$ cycles in $\mathcal{C}_1$ containing this vertex. Let C be any one of them. It is easy to see that the cycle meets exactly two cycles of $\mathcal{C}_0$, one labeled by $[r-2]$ and the other by $[r-1] \setminus \{j\}$ for some $j \in [r-2]$. The antipodal pair of vertices in the intersection of these two cycles must get the label $[r-1] \setminus \{j\} \cup \{r+i-1\}$, and the cycle C must be labeled $[r-2] \cup \{r+i-1\} \setminus \{j\}$. It is easy to see that every vertex on $\mathcal{C}_0$ will get a unique label this way. The proof of (b) is essentially the same, and we omit the detail here. □

We conclude the paper by posing the following problem:

Problem 3.3. *Does the (unlabeled) cocircuit graph* $\mathrm{G}_{\mathcal{M}}$ *of a* uniform *oriented matroid* $\mathcal{M}$ *determine its isomorphism class?*

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