A polynomial case of unconstrained zero-one quadratic optimization

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Abstract

Unconstrained zero-one quadratic maximization problems can be solved in polynomial time when the symmetric matrix describing the objective function is positive semidefinite of fixed rank with known spectral decomposition.

1 Introduction

The unconstrained quadratic maximization in zero-one variables

$$\begin{array}{ll} \max & f(x) = x^T Q x \\ \text{subject to} & x \in \{0,1\}^n, \end{array}$$

where Q is an $n \times n$ rational symmetric matrix, is a classical NP-hard combinatorial optimization problem. It is well known [4] that the weighted max-cut problem can be considered as a special case. It remains NP-hard even when Q is positive definite or when it is indefinite of rank two [6]. If a linear term is added to the objective function, the problem remains NP-hard even when Q is negative definite. Well known polynomial cases are, for instance, when

(a) the matrix Q is of rank one, in which case the solution can be found by inspection,

(b) Q has nonpositive off-diagonal elements [5], and

(c) the graph underlying the associated max-cut problem is series parallel [3].

We assume that Q is positive semidefinite and $d = rank(Q)$ is fixed. Hence there is a $d \times n$ matrix V representing a decomposition of Q into a sum of rank one matrices, i.e. $Q = V^T V$. The rows of V are suitably scaled eigenvectors of Q. We do not dwell here on the complexity of obtaining this representation of matrix Q to a sufficient degree of accuracy but rather suppose that this can be done. One way to dodge the issue is of course simply to assume that V rather than Q be given. Eliminating this assumption is a sufficiently interesting problem on its own.

We will give a construction of polynomial complexity in n to find an optimal solution of the problem, when Q is given by its matrix V of fixed rank d.

2 Polynomiality of instances with positive semidefinite Q of fixed rank

In the sequel, positive semidefinite QP problems of the following form will be considered.

$$\begin{aligned} &\max f(x) = x^T Q x = x^T V^T V x = \sum_{i=1}^{d} (v_i, x)^2 \\ &\text{subject to} \quad x \in \{0,1\}^n \end{aligned} \tag{1}$$

where (a, b) is the inner product of two vectors a and b, d is a fixed positive integer, $v_i \in \mathbb{R}^n$ for $i = 1, \ldots, d$ and V is the $d \times n$ matrix which has v_i^T as its ith row. Consider the linear map $\mathbb{R}^n \to \mathbb{R}^d : z = Vx$. The image of the hypercube $[0,1]^n$ under this map is a convex polytope Z, known as a *zonotope*. Notice that for every extreme point $\tilde{z}$ of Z there exists an extreme point $\tilde{x}$ of $[0,1]^n$, i.e. a point $\tilde{x} \in \{0,1\}^n$, such that $\tilde{z} = V\tilde{x}$. For the optimal value f^* of (1) one has

$$\begin{aligned} f^* &= \max\left\{ \sum_{i=1}^{d} (v_i, x)^2 \quad : x \in \{0,1\}^n \right\} \\ &= \max\left\{ \sum_{i=1}^{d} (v_i, x)^2 \quad : x \in [0,1]^n \right\} \\ &= \max\left\{ \sum_{i=1}^{d} z_i^2 \quad : z \in Z \right\}, \end{aligned}$$

where the equality in the second line is a direct consequence of the convexity of the function $\sum_{i=1}^{d} (v_i, x)^2$. The last expression is a maximization of a convex function of z over the convex set Z. Therefore the maximum is attained at some extreme point $\tilde{z}$ of Z. Let us denote by S the set of extreme points of the zonotope Z. Continuing the previous manipulations:

$$f^* \;=\; \max\left\{ \sum_{i=1}^{d} z_i^2 \quad : z \in Z \right\} = \max_{z \in S}\left\{ \sum_{i=1}^{d} z_i^2 \right\}. \tag{2}$$

The zero-one QP problem (1) is thus reduced to the enumeration of extreme points of the zonotope Z in $\mathbb{R}^d$. Denoting by v^j the jth column of V, Z is the Minkowski sum of the n closed line segments $[\mathbf{0}, v^j]$ for $j = 1, \ldots, n$. ¿From the theory of discrete geometry it is known that

1) $|S| \;=\; O(n^{d-1})$, and

2) The points of S together with a corresponding $x \in \{0,1\}^n$ for each $z \in S$ can be calculated in $O(n^{d-1})$ time for $d \geq 3$, and $O(n^d)$ time for $d \leq 2$.

These statements can be easily verified with geometric duality. The dual object of the zonotope generated by the columns of V is a central arrangement of n oriented hyperplanes in $\mathbb{R}^d$ each having a v^j as its normal vector. The extreme points of Z correspond one-to-one to the regions (d-dimensional faces) of the arrangement, and the facets of Z to the lines (1-dimensional faces) of the arrangement, see Figure 1 and see [2, Section 1.7]) for detail. With an incremental construction algorithm for arrangements (see, e.g. [2, Chapter 7]), one can generate the regions of the associated arrangement in $O(n^{d-1})$ time for $d \geq 3$ and $O(n^d)$ time for $d \leq 2$.

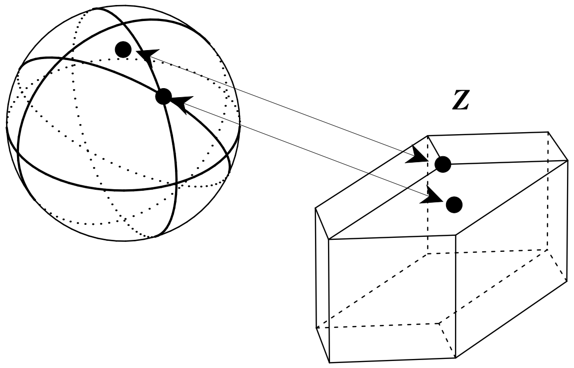

Figure 1: Duality of a zonotope and the associated arrangement (shown as a cut section with a sphere)

Note that the arrangement construction should be applied to an Euclidean arrangement in $\mathbb{R}^{d-1}$ which is the intersection of the central arrangement and a hyperplane in general position. This generates a half of the regions in $\mathbb{R}^d$ and the other half is obtained by negation. Each region is coded as a sign vector in $\{+1, -1\}^n$ whose jth component indicates the position of the region relative to the oriented jth hyperplane, and it is essentially the corresponding x vector except that the -1 components should be read as zero. Having found S, one finds f^* using (2). To calculate the objective value corresponding to each $z \in S$ takes $O(d) = O(1)$ work and hence the evaluation of the objective for all $z \in S$ is $O(n^{d-1})$. Consequently, a problem of the form (1) can be solved in $O(n^d)$ time, i.e. in time polynomial in the number of variables n.

It should be noted that our complexity analysis ignores the complexity of real number computation and assumed that each elementary arithmetic operation for real numbers can be done correctly in constant time. If input matrix V is rational, all computation can be done in time polynomial in the size of input with fixed d under the usual Turing machine model.

3 Concluding Remarks

We have shown that the zero-one QP problem (1) can be solved in $O(n^{d-1})$ time for $d \geq 3$ when the matrix Q is positive semidefinite and its spectral decomposition is explicitly given. The algorithm is based on the enumeration of full dimensional regions of an arrangement of n hyperplanes in R^{d-1}.

It is not known whether the incremental algorithm [2, Chap. 7] has practical importance in higher dimensions (even in dimension 4 or 5). One major obstacle is a requirement: it must store the whole cell complex of each incrementally constructed arrangement, that can be extremely complicated. There is a quite different algorithm, known as the reverse search algorithm [1], for generating the full dimensional regions. This might lead to a more practical solution to the positive semidefinite zero-one QP problem. While its worst case complexity is higher than that of the incremental algorithm, it runs in time polynomial in the input and the output sizes, it does not need to store regions generated in earlier steps, and it can be highly parallelized. Determining how practical the reverse search algorithm can be is a subject of future research.

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