Zonotopes

A zonotope P in $\mathbb{R}^d$ is a convex polytope that is representable as the Minkowski sum of finite line segments:

$$P \quad = \quad L_1 + L_2 + \cdots + L_m,$$

where L_i is the line segment $[a_i, b_i]$ with $a_i, b_i \in \mathbb{R}^d$ for $i = 1, \ldots, m$. The line segments $[a_i, b_i]$ $(i = 1, \ldots, m)$ are called the generators of the zonotope P.

A simple example of zonotope is the d-dimensional unit hypercube

$$C_d = [0, e^1] + [0, e^2] + \cdots + [0, e^d],$$

where e^i denotes the ith unit vector in $\mathbb{R}^d$. The unit hypercube is the convex hull of all (2^d) 0/1 vectors in $\mathbb{R}^d$:

$$C_d = conv(\{0,1\}^d)$$

Also the unit hypercube can be represented by its $2d$ facet inequalities:

$$C_d = \{x \in \mathbb{R}^d \mid 0 \leq x_j \leq 1, \text{ for all } j = 1, \ldots, d\}.$$

While the unit hypercube is a very special zonotope, every zonotope P generated by m line segments is an image (projection) of the m-dimensional hypercube C^m by an affine map.

How many vertices and facets can a d-dimensional zonotope P given by m generators? Let us denote by $f_k(P)$ the number of k-dimensional faces of P. In particular, $f_0(P)$ and $f_{d-1}(P)$ denote the number of vertices (= extreme points) and facets of P. It follows easily from the definition of zonotope that $f_0(P) \leq 2^m$. But the trivial bound does not depend on d and is not very useful.

Here is a classical theorem stating the exact upper bounds, which was given as the maximum number of cells in an arrangement of m hyperplanes in $\mathbb{R}^d$ [3, 6]:

(a) Let P be a d-dimensional zonotope given by m generators $(m \geq d)$. Then, $f_0(P) \leq 2\ \sum_{i=0}^{d-1} \binom{m-1}{i}$ and $f_{d-1} \leq 2\binom{m}{d-1}$. Furthermore, the equalities can be attained by certain zonotopes and therefore the bounds are best possible.

It is good to understand the order of these upper bounds for any **fixed** d. Clearly, the upper bound of f_{d-1} is $O(m^{d-1})$. Now, the bound for the number of vertices is the expression $\sum_{i=0}^{d-1} \binom{m-1}{i}$ and the dominating one is the last term $\binom{m-1}{d-1}$ which is of order $(m-1)^{d-1}$

What are the zonotopes attaining the upper bounds? Any reasonable randomness measure of generators will result in such an zonotope attaining the maximum in probability one. On the other hand, zonotopes attaining the upper bounds are not necessarily random. For example, the hypercubes attain these upper bounds (why? Please check yourself.)

Let's see how random zonopes look like. Figure 1 depicts three dimensional zonotopes given by generators of form $[\mathbf{0}, b_i]$ for random vector $b_i \in \mathbb{R}^3$.

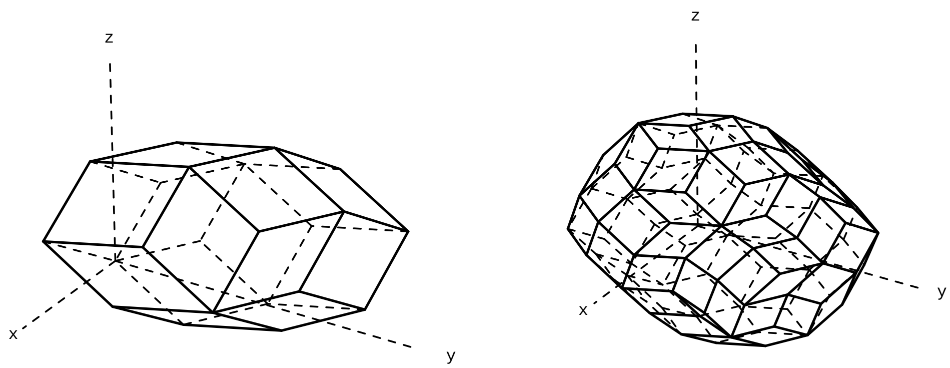

Figure 1: 3 dimensional zonotopes with 5 and 10 generators.

One characteristic of random zonotopes is the number of vertices in every facet is exactly $2(d-1)$, i.e. all facets are centrally symmetric quadrangle (i.e. parallelogram with four corners) in $\mathbb{R}^3$. Also, the combinatorial diameter of such a zonotope is the number of generators. Here the combinatorial diameter of P is the maximum of the shortest graph distances between two vertices of P. Essentially these two properties characterizes the zonotopes attaining the upper bounds in the theorem (a), as shown by Zaslavsky [6].

A truncated 3 dimensional cube below is a simple example of zonotopes which do not attain the upper bounds. It has three types of facets, 4-gon, 6-gon and 8-gon.

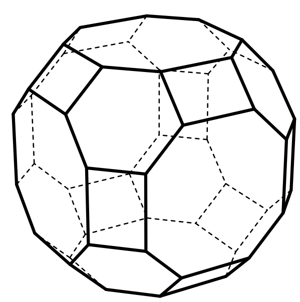

Figure 2: A truncated cube: 3 dimensional zonotope with 6 non-random generators.

Computational Aspects of Zonotopes

Suppose you are given m generators of a zonotope P in $\mathbb{R}^d$. How can one compute a minimal V-representation and a minimal H-representation?

First of all, we may assume that the generators are of form $L_i = [\mathbf{0}, b_i]$ for all $i = 1, \ldots, m]$, since we can easily shift the polytope by any amount later.

There is a simple (but inefficient) way to find a highly redundant V-representation. Just select $\mathbf{0}$ or b_i for each i, and compute the sum. Each vertex of P is such a sum. There are 2^m sums that constitute a V-representation. Then one can remove all redundant ones using linear programming methods. Both cddlib and lrslib come with redundancy removal programs and libraries.

There is a much more efficient algorithm [2] which is polynomial (in the size of both input and output) and very practical. In fact, there is an implementation [5] which was designed to solve the binary quadratic optimization using the zonotope vertex computation as suggested in [1, 4].

Finally to compute an H-representation efficiently is still an open question, while Seymour's algorithm for hyperplane generation works at least theoretically. Finding an algorithm that is polynomial and compact (i.e. memory requirement is polynomial in the input size only) is a challenging question.

References

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