Assignment #2 (Jan 28, 2002, Due: Wed, Feb. 6)

1. Write a complete proof of Farkas-Type Lemma (Gale’s Theorem) using Fourier’s elimination. (In particular, show how one finds $\lambda$ in (b) when the first statement is false.)

For a matrix $A \in \mathbb{R}^{m \times d}$ and a vector $b \in \mathbb{R}^m$, exactly one of the following two statements holds:

(a) $\exists x \in \mathbb{R}^d$ such that $Ax \leq b$;
(b) $\exists \lambda \in \mathbb{R}^m$ such that $A^T \lambda = 0$, $b^T \lambda < 0$ and $\lambda \geq 0$.

2. Derive the following alternative theorems. (“Exactly one of (a) and (b) holds” is omitted below. Also $g$ is any fixed index in $\{1, \ldots, d\}$.)

**Farkas’ Lemma.**

(a) $\exists x \in \mathbb{R}^d$ such that $Ax = b$ and $x \geq 0$;
(b) $\exists \lambda \in \mathbb{R}^m$ such that $A^T \lambda \geq 0$, $b^T \lambda < 0$.

**Gordan’s Theorem.**

(a) $\exists x \in \mathbb{R}^d$ such that $Ax = 0$, $x \geq 0$ and $x \neq 0$;
(b) $\exists \lambda \in \mathbb{R}^m$ such that $A^T \lambda > 0$.

**No Name Theorem.**

(a) $\exists x \in \mathbb{R}^d$ such that $Ax = 0$, $x \geq 0$ and $x_g > 0$;
(b) $\exists \lambda \in \mathbb{R}^m$ such that $A^T \lambda \geq 0$ and $(A^T \lambda)_g > 0$.

3. (Optional) Let $A \in \mathbb{R}^{m \times d}$ and $c \in \mathbb{R}^d$. Write a Farkas-type alternative statement (b) to

(a) $\exists x \in \mathbb{R}^d$ such that $Ax \leq 1$ and $c^T x > 1$.

4. Using a polyhedral representation conversion program (e.g. cddlib, lrslib) to compute minimal H-representation of the fractional matching polytope $P_{FMA}$ of the complete graph $K_4$ and $K_6$ whose edges are labeled as follows. Are they 0/1 polytopes? If not, how many blossom inequalities do we have to add to determine the matching polytope $P_{MA}$?