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Bridging Theories with Axioms: Boole, Stone, and Tarski

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1 Introduction

In discussions of mathematical practice the role axiomatics has often been confined to providing the starting points for formal proofs, with little or no effect on the discovery or creation of new mathematics. For example, quite recently Patras wrote that the axiomatic method "never allows for authentic creation" (Patras 2001, 159), and similar views have been popular with philosophers of science and mathematics throughout the 20th century.[1] Nevertheless, it is undeniable that axiomatic systems have played an *essential* role in a number of mathematical innovations, most famously in the discovery of non-Euclidean geometries. It was Euclid's axiomatization of geometry that motivated the investigations of Bolyai and Lobachevsky, and the later construction of models by Beltrami and Klein.[2] Moreover, it was not only through the investigation and modification of given systems of axioms that new mathematical notions were introduced, but also by using axiomatic characterizations to express analogies and to discover new

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[1]See (Schlimm 2006) for an overview of these discussions.

[2]See, e. g., (Bonola 1955) and (Gray 1998).

ones, as can be seen, for example, in the developments that led to the formation of algebraic structures, like groups and lattices.[3] In the present paper I would like to draw attention to a different use of axiomatics in mathematical practice, namely that of being a vehicle for bridging theories belonging to previously unrelated areas. How axioms have been instrumental in linking mathematical theories is illustrated by the investigations of Boole, Stone, and Tarski, all of which revolve around the notion of Boolean algebra.[4] An interesting aspect of Boole's and Stone's work is that it also shows how too heavy reliance on formal similarities between theories can lead to developments that, with hindsight, appear as detours.

2 Boole's algebraization of logic

In the first half of the 19th century, through the work of Peacock, Gregory, De Morgan, and Hamilton an understanding of algebra had developed in Britain, according to which different meanings can be given to the symbols of algebra and where inferences should be independent from these interpretations.[5] George Boole (1815–1864), who was personally acquainted with some of these authors and was certainly familiar with their work, developed a calculus of logic in *The mathematical Analysis of Logic* (Boole 1847) and presented essentially the same calculus in his more famous *An Investigation of the Laws of Thought* (Boole 1854). In the earlier book the symbols of the calculus are interpreted primarily as choice operators ("elective symbols") on a class of things, later as classes of things. Other interpretations that are discussed by Boole are propositions,

[3]See (Schlimm 2008a) and (Schlimm 2008c); see also (Schlimm 2008b) for a more general discussion of the use of axiomatics for characterizing analogies.

[4]A *Boolean algebra* $\langle A, \cup, \cap, 0, 1, ' \rangle$ is a set A of elements, which has two binary operations ($\cup$ and $\cap$; sometimes they are also symbolized by $+$ and $\times$) that are commutative, associative, and for which the absorption and distributive laws hold. It has unique zero and one elements, and one unary operation of complementarity. Examples of Boolean algebras are fields of sets with union and intersection, the binary truth values of sentential classical logic, and probability events. See (Sikorski 1960).

[5]See (Peckhaus 1997) for an account of these developments.

probabilities, and a two-element domain containing only the numbers 0 and 1.[6]

According to Boole, his investigations were motivated by the realization of a certain similarity between the use of conjunction and disjunction of concepts in everyday language and addition and multiplication of numbers. This similarity had been noted much earlier by Leibniz, but "he did not find it easy to formulate the resemblance precisely" (Kneale and Kneale 1964, 404). Influenced by the aforementioned tradition in algebra, Boole employed algebraic formulas to articulate logical relationships, which allowed him to show how the analogy between logic and the algebra of numbers can be expressed by a set of common underlying formulas.

However, Boole was not only interested in expressing this analogy, but, together with an analysis of language, it also guided the construction of his calculus, since he had "the desire to retain as much as possible of the normal algebraic formalism in his new calculus of logic" (Kneale and Kneale 1964, 408). In a similar vein, Hailperin describes Boole as "hewing closely to 'common' algebra" in order to be able to use familiar techniques and procedures (Hailperin 1981, 77). This strategy is clearly reflected in Boole's choice of terminology for the logical operations and in the frequent references to this analogy in his presentations. A passage on pp. 36–37 of (Boole 1854) provides a telling example. Here, Boole dramatically describes a situation in which the analogy seems to break down. First, he argues that the inference from $x = y$ to $zx = zy$ holds both in algebra and for classes, if the variables are interpreted by classes and the multiplication symbol as intersection. But, with regard to the reverse inference, i. e., from $zx = zy$ to $x = y$, he notes that "the analogy of the present system with that of algebra, as commonly stated, appears to stop" (Boole 1854, 36), since this inference is not generally valid for classes, and explains: "In other words, the axiom of the algebraists, that both sides of an equation may be divided by the same quantity, has no formal equivalent here" (Boole 1854, 36–37). Luckily, further inspection reveals that in the case of $z = 0$ this

[6] In the terminology introduced in (Schlimm 2008b), these are mostly *object-rich* domains.

'axiom' is violated in algebra, too. Therefore, Boole can assert with relief that "the analogy before exemplified remains at least unbroken" (Boole 1854, 37).

Nonetheless, Boole did notice a genuine negative analogy: $x = x^2$ holds if the variables are interpreted as classes, but not if they are interpreted by natural numbers.[7] In fact, this equality only holds for $x = 0$ and $x = 1$, which leads Boole to consider a new universe of discourse that consists of only the two numbers 0 and 1.[8] He writes:

> Hence, instead of determining the measure of formal agreement of the symbols of Logic and those of Number generally, it is more immediately suggested to us to compare them with symbols of quantity *admitting only of the values 0 and 1.* Let us conceive, then, of an Algebra in which the symbols x, y, z, &c. admit indifferently of the values 0 and 1, and of these values alone. The laws, the axioms, and the processes of such an Algebra will be identical in their whole extent with the laws, the axioms, and the processes of an Algebra of Logic. Difference of Interpretation will alone divide them. Upon this principle the method of the following work is established. (Boole 1854, 37–38; emphasis in original)

Boole's strong emphasis on the formal similarities between algebra and logic may also account for a peculiarity of his system, a "defect of elegance," according to (Kneale and Kneale 1964, 408). What I am referring to is Boole's restriction of $x + y$ (denoting the class that contains elements from either x or y) to the case where $xy = 0$, which expresses that the intersection of x and y is empty.[9] As a consequence, the equation $x + x = x$ is barred from Boole's system, which prevents the possibility of formulating De Morgan's rules of distributivity, which express the duality of addition and multiplication, or union and intersection, respectively. Kneale and Kneale speculate that Boole's interpretation of $+$ was linked to his desire to be able to derive $x - y = z$ from $x = y + z$, since this inference is only valid if $yz = 0$. Another possible reason for Boole's choice, however, is

[7] Later Boole says that '$x = x^2$' expresses "the fundamental law of thought" (Boole 1854, 49).

[8] In (Boole 1847), the symbols 0 and 1 were used to represent the empty class and the entire universe of discourse.

[9] Note that this is not equivalent to interpreting $x + y$ as *disjoint union*, since the latter has a definite meaning also when the intersection of x and y is not empty. Exceptions that render expressions meaningless are not unknown in mathematics, e. g., in ordinary algebra the term 'x/y' is only meaningful if $y \neq 0$.

based on the analogy between the rules of algebra and logic. Admitting the symbol $x+y$ without Boole's restriction would also make '$1+1$' a meaningful term. The modern reader might have no objection to admitting $1+1=1$ in the system, but this clearly constitutes another negative analogy to the common algebra of natural numbers.[10] As such, it would certainly be undesirable from Boole's point of view. Later logicians, like Jevons, Peirce, and Schröder, valued the duality of addition and multiplication higher than the analogy to algebra, and thus "corrected" Boole's system (only Venn sided with Boole on this issue).[11] In fact, it is *leaving* Boole's "close analogy to mathematical notation" that Peckhaus considers as one of the characteristics of Jevons' logic, which he regards as "a considerable step forward in Boolean logic" (Peckhaus 2000, 279).

To summarize, the analogy between algebra and logic that is explicated by common formulas proved to be a powerful motivation for Boole's formulation of a calculus of classes and his introduction of a new suitable model for it. However, clinging too strongly to this analogy prevented him from developing a system that satisfied other general desiderata, like those of symmetry and duality.

3 Stone's unification of algebra and topology

In connection with the preparation of his book on *Linear Transformations in Hilbert Space* (1932), the young American mathematician Marshall Stone (1903–1989) developed an interest in Boolean algebras.[12] In his first announcement and summary of his results, Stone reports that he perceived an analogy between Boolean algebras, as presented by the axiomatizations of Huntington and others, and the theory of rings, as presented in the newly published textbook *Moderne Algebra* by van der Waerden.[13] In the latter,

[10] For a discussion of various notions of disjunctions and their relation to natural language, see (Jennings 1994), in particular pp. 70–77, where Boole's approach is discussed.

[11] See, e. g., (Schröder 1890–1905, I, 263). We see here how mathematicians can make different choices regarding the development of theories that depend upon their general views on the nature and the aims of mathematics, or, in other words, on their 'image' of mathematics (Corry 1996).

[12] See (Mehrtens 1979, 260–261).

[13] See (Huntington 1904) and (van der Waerden 1930, Ch. 3).

rings are defined axiomatically as systems of double composition[14] with the operations of addition (associative and commutative) and multiplication (associative and both left- and right-distributive with respect to addition). Furthermore, rings have unique additive inverses and a unique neutral element 0.[15] On the basis of the formal similarity between the definitions of Boolean algebras and rings, Stone was able to transfer the central notions of *ideals* and *homomorphisms* from the theory of rings to the theory of Boolean algebras (Stone 1934, 197).

That it were indeed the axiomatic presentations of Boolean algebras and rings that were the basis of the analogy that Stone perceived, is supported by the fact that Stone carefully studied these axiomatizations and even developed a new axiomatization for Boolean algebras in order to bring out the analogy more forcefully. The result was his "Postulates for Boolean algebras and generalized Boolean algebras" (1935), wherein he explains that he chose the postulates

> with the intention of emphasizing the analogy between Boolean algebras and abstract rings, the latter being systems which have already undergone extensive analysis. From this point of view our postulates appear to be as satisfactory as possible, so long as logical addition and multiplication are to be treated as the analogues of ring addition and multiplication. (Stone 1935a, 703)

In this context Stone mentions three principles that guided his quest for axioms:

> in the first place, they shall embody known properties of the operations of forming finite unions and intersections of classes [i. e., one of the most important instances of Boolean algebras]; in the second place, they shall embody as far as possible only such properties as are valid of the operations of addition and multiplication in a ring; and, in the third place, they shall be independent. (Stone 1935a, 704)

[14] "By a *system of double composition* we shall mean a set of elements in which, for any two elements $a, b, \ldots$, a *sum* $a + b$ and a *product* $a \cdot b$ belonging to the set are uniquely defined" (van der Waerden 1930, 37; quoted from the translation of the second edition, p. 32).

[15] Alternatively, a ring $\langle A, +, \times, 0\rangle$ may be characterized as consisting of an additive group $\langle A, +\rangle$ with neutral element 0 and a monoid (like a group, but without inverses) $\langle A, \times\rangle$, that are connected by distributive laws. To guarantee the existence of additive inverses Stone postulates that the equation $x + a = b$ has a unique solution in A for arbitrary elements a and b of the ring (Stone 1936, 39).

It seems that his first principle put him on a wrong track initially. Since, in general classes do not have inverses with respect to union and to the empty class, which are the straightforward candidates for being interpretations of addition and zero in a ring. As Stone points out, for classes a and b, and union $\cup$, the equation $x \cup a = b$ has only solutions for x, if a is contained in b, and the solution is unique only if the intersection of x and a is empty. Thus, under this interpretation the algebra of classes is *not* a ring and so Stone calls its relation to rings only one of analogy.[16] However, in the course of his investigations Stone made a surprising discovery:

> We have recently observed, however, that Boolean algebras can be regarded as rings of special type when the operation of forming the symmetric consequence is taken as ring addition. In consequence the most natural approach to a mathematical theory of Boolean algebras is not one based upon material in this paper. (Stone 1935a, 703)[17]

In other words, rather than correlating ring addition with union (i. e., addition in Boolean algebras) and ring multiplication with intersection (i. e., multiplication in Boolean algebras), which violates the the ring axiom requiring additive inverses, Stone noticed that all the ring axioms can be satisfied if ring addition is interpreted by a different operation on classes, namely that of *symmetric difference*.[18] With this new insight he went on to rewrite the paper on the connection between Boolean algebra and topology that he was also working on at the time. In a summary that was published earlier than the full-length paper, one reads, after a short reference to the analogy between Boolean algebras and abstract rings, the following:

> In the present note we shall show that Boolean algebras are actually special instances, rather than analogs, of the general algebraic systems known as rings. In establishing this result we must choose the fundamental operations in a Boolean algebra in an appropriate manner, indicated below. The algebraic theory developed in the previous communication

[16] Note, that Stone's notion of 'analogy' is weaker than that employed by Boole.

[17] We have here an example for the fact that a mathematician's understanding of 'natural' is not always the same as 'being obvious' or 'immediate'.

[18] The *symmetric difference* of two classes is defined as $(a \cap b)' \cup (a' \cap b')$. It contains all elements that are either in a or in b, but not in both.

> is thus a particular instance of the theory of rings and can be deduced in part from known theorems concerning rings. These facts were discovered just as the detailed exposition of our independent theory of Boolean algebras was on the point of completion; and they now compel a radical revision which will necessarily delay publication of the complete theory. (Stone 1935b, 103)

In the subsequently published paper on "The theory of representation for Boolean algebras" (1936), Stone shows that any Boolean algebra is isomorphic to, i. e., can be represented by, a field of sets,[19] and he motivates his algebraic approach to logic by the fact that it allows to connect many different areas of mathematics:

> Indeed, if one reflects upon various algebraic phenomena occurring in group theory, in ideal theory, and even in analysis, one is easily convinced that a systematic investigation of Boolean algebras, together with still more general systems, is probably essential to further progress in these theories. [...] The writer's interest in the subject, for example, arose in connection with the spectral theory of symmetric transformations in Hilbert space and certain related properties of abstract integrals. In the actual development of the proposed theory of Boolean algebras, there emerged some extremely close connections with general topology which led at once to results of sufficient importance to confirm our a priori views of the probable value of such a theory. (Stone 1936, 37)

Stone goes on to prove that Huntington's axiomatization of Boolean algebras is equivalent (i. e., mutually interpretable) with the axiomatization of commutative rings with unit element in which every element is idempotent (called *Boolean rings*), i. e., that "Boolean algebras are identical with those rings" (Stone 1936, 38). According to Stone, these results reveal

> the essential nature of all Boolean rings. In particular, it shows that the operation of addition in a Boolean ring corresponds abstractly to the operation of forming the symmetric difference or union (modulo 2) of classes, as indicated by the relation (6); and

[19] A *field of sets* is a family of sets that are closed under finite union and intersection, whose elements are closed under complements. On the importance of this theorem, Stone remarks: "Such a result is a precise analogue of the theorem that every abstract group is represented by an isomorphic group of permutations" (Stone 1936, 38).

> it shows similarly that the operation of multiplication corresponds to the operation of forming the intersection of classes, as indicated in the relation (7). (Stone 1936, 44)[20]

It is clear that Stone's aim for these investigations is part of a larger research project. He wants to apply the mathematical machinery developed in the theory of rings to other domains:

> What is of primary importance here is the identification of the abstract algebras arising from logic and the theory of classes with systems amenable to the methods developed by modern algebraists, namely, with those special rings which we have termed Boolean rings [...]. (Stone 1936, 43)

In a further paper Stone was able to connect the theory of Boolean rings also to topology by proving that "the theory of Boolean rings is mathematically equivalent to the theory of locally-bicompact totally-disconnected topological spaces" (Stone 1937b, 375).[21] Stone called the latter *Boolean spaces*, but they have subsequently become known as *Stone spaces* in his honor. This identification, also referred to as "the fundamental representation theorem" and "the basic theorem for the whole theory of Boolean algebras" (Sikorski 1960, 158), informs us about the models of the axioms and it allows for the transfer of topological techniques and results to the study of Boolean algebras, and vice versa. It has become known as the *Stone duality* and it has been extremely fruitful in later developments in logic, topology, algebra, and algebraic geometry.

Grosholz (1985; 2007, Ch. 10) has discussed this duality as an example of the growth of mathematical knowledge through the unification of related fields by exploiting structural analogies between them. The above presentation highlights that these structural analogies were investigated by Stone on the basis of axiomatic characterizations of the domains in question.

[20] The relations alluded to in the quotation are the definitions of the ring operations $+$ and $\times$ in terms of the primitives $\cup$ and $'$ of Boolean algebras (Stone uses $\vee$ for $\cup$):
(6) $a + b = ab' \cup a'b = (a' \cup b'')' \cup (a'' \cup b')'$, (7) $ab = (a' \cup b')'$.

[21] See also (Stone 1934, 198). In modern terminology, these spaces are compact, totally disconnected Hausdorff spaces.

4 Tarski's calculus of deductive systems

In the 1920s Alfred Tarski (1902–1983) studied logic and set theory in Warsaw and developed a meta-mathematical approach whose main ideas are presented in (Tarski 1930a) and (Tarski 1930b). Herein he considers formal mathematical theories as sets of sentences with a single primitive relation, namely that of being the set of consequences, $Cn(X)$, of a set X of sentences, which is defined axiomatically. A set of sentences, belonging to the set S of all sentences, that is closed under the relation of consequence is called a *deductive system*, and two such systems are said to be *equivalent* if they have the same set of consequences. These and other meta-mathematical notions that Tarski defines are developed into a "calculus of systems" in (Tarski 1935a) and (Tarski 1936), where negation, implication, and the set of all logically valid sentences are taken as primitives and the consequence relation is defined in terms of them. In particular, Tarski defines the *logical sum* $X \dot{+} Y$ of two deductive systems X and Y as $Cn(X + Y)$.[22] This allows him to show that the deductive systems satisfy the axioms of Boolean algebra, except that $X \dot{+} \overline{X} = S$ does not hold in general for deductive systems. Interpreted in the calculus of sentential logic this equation corresponds to the law of excluded middle, which holds in classical, but not in intuitionistic logic. This caught Tarski's attention, and he notes that "[t]he formal resemblance of the calculus of systems to the intuitionistic sentential calculus of Heyting is striking" (Tarski 1935a, 352). The *axiomatizable systems*, i. e., those sets of sentences for which there is a finite system of axioms, satisfy all axioms for Boolean algebra, as Tarski shows, and he is also able to prove that the axiomatizable systems are in fact the only ones that satisfy $X \dot{+} \overline{X} = S$ (Tarski 1935a, 356).

Around this time Tarski also studied axiomatizations of Boolean algebra (Tarski 1935b) and, possibly influenced by the results of Stone, was also led to consider fields of

[22] In Tarski's notation, $+$ stands for set-theoretic union, which was symbolized above by $\cup$, and $\overline{X}$ denotes the complement of X, symbolized by X', above.

sets as a different model of the axioms of Boolean algebra.[23] Tarski gives the following account of the development of his own theory:

> The speaker has recently developed a theory of deductive systems. Considered formally, this theory forms an interpretation of so-called Boolean algebra. [...] With respect to this it became possible to first extend the concepts and results of the theory of deductive systems to general Boolean algebra, and then to a different interpretation of this algebra, namely the theory of fields of sets [...]. In doing so, it turned out that there is an exact correspondence between the main concepts of the theory of deductive systems and those concepts that are transferred to the theory of Boolean fields from general abstract algebra [Reference to (Stone 1934)]: the deductive systems are coextensive with ideals, axiomatizable systems with principal ideals, complete systems with prime ideals, etc. The calculus of deductive systems is thus changed into a general calculus of ideals. (Tarski 1937, 186; translated from Mehrtens 1979, 270)

Thus, we are presented here with a clear example of the transfer of notions and methods between one domain (deductive systems) and another (ideal theory) that is mediated by a common system of axioms (for Boolean algebra). In other words, the axiomatization of Boolean algebra, together with the realization of different models thereof, is what allowed Tarski to relate his theory of deductive systems to the theory of ideals. Comparing the main concepts of both theories, Tarski noticed that they can be put into correspondence, such that the results from ideal theory can be carried over the axiomatic bridge to the theory of deductive systems.

Once such a connection has been established, it is straightforward to transfer results produced in one theory to the other. For example, Tarski's student Mostowski proved that the set of prime ideals of a field is either countable or has the cardinality of the continuum, and, due to the correspondence between the prime ideals in a field and the complete deductive systems of a theory, he could translate this result into a result about deductive systems: "*The cardinality of the set of complete systems of an arbitrary deductive theory is either* $\leq \aleph_0$ *or* $2^{\aleph_0}$" (Mostowski 1937, 46).

[23] See footnote 19, above.

Mostowski worked out in detail the connection between the calculus of deductive systems and the theory of rings, and, following the work of Stone, extended it to topology. This made it also possible to use topological considerations for finding solutions to certain problems formulated by Tarski for his calculus of systems. The advantage of this way of proceeding is that the topological arguments were easier and shorter than the direct proofs, which Mostowski had formulated earlier. He reports:

> The entire ballast of my earlier proofs is removed in the present formulation and, for the reader who is familiar with the elements of topology, the considerations presented here will have the character of immediate consequences of the investigations carried out by *Stone*. My reflections should therefore not be understood as genuine discoveries, but rather as vivid descriptions of the interesting and completely unexpected connection that exists between so seemingly distant areas like meta-mathematics and topology. (Mostowski 1937, 35)

After learning of Tarski's work, Stone also immediately realized the intimate relation between the theory of deductive systems and topology. Since the more direct approaches to the study of deductive systems had not been satisfactory, Stone even went so far as to suggest that "the profounder aspects of the theory of deductive systems must be studied by general methods of topology" (Stone 1937a, 225). Again, we find that the path that was opened up by axiomatics proved to be an extremely fruitful one.

5 Concluding remarks

The above presentation of three episodes from the history of mathematics could touch only upon a few aspects of the development of theories of Boole, Stone, and Tarski, obviously leaving aside a wide variety of other conceptual innovations and many technical details. The goal of this discussion was to bring to the fore the importance of the fact that the theories under investigation were presented axiomatically, and that this was indeed a key factor which allowed the researchers to draw connections between theories that were previously regarded as unrelated. Thus, axiomatization was neither just the conclusion of a previous development, nor did it discourage or suppress creative thinking, as was suggested by Felix Klein (Klein 1926, 335–6). On the contrary, the connections that Boole, Stone, and Tarski opened up allowed for the transfer of results and methods across different mathematical theories and thereby engendered many successful further developments.

The present paper supports Grosholz's discussion of the unification of logic and topology based on "the hypothesis of a partial structural analogy" (Grosholz 1985, 147) and her conclusion that the "correlation of mathematical structures leads to the expansion of mathematical knowledge precisely because it is initially unfounded, risky and corrigible" (Grosholz 1985, 152). In addition, it supplements this discussion by emphasizing the role of axiomatics in the discovery of the analogies in the first place. As we have seen in the cases of Boole and Stone, the initial attempts were indeed in need of correction and their initial hypotheses arose from careful considerations that took the axiomatic presentations of the domains in question as their starting points. The mathematical domains of algebra, Boolean algebra, ring theory, etc., were presented axiomatically and the formal similarities between the axiomatizations were the basis for Boole's, Stone's, and Tarski's advancements. Of course, I do not claim that *all* analogies in mathematics are discovered in this manner, but that, in the practice of mathematics, *some* have been found in this way. Thus, axiomatics can lead to a clarification of perceived analogies and to the discovery of new ones, and so it can be — and has been — used as a methodological tool at the service of mathematical creativity.

By being a vehicle for bridging mathematical theories, as illustrated in the case studies presented above, axioms can play an important part in the innovation of mathematics and thus contribute to the growth of mathematical knowledge in a way that goes far beyond the deductive generation of new theorems. As a consequence, any view that confines the role of axiomatics to what has been called the 'context of justification' does not do justice to their actual use in mathematical practice.

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